Handelshøyskolen BI

GRA 19703 Master Thesis

Thesis Master of Science 100% - W

Predefinert informasjon

Startdato:	09-01-2023 09:00 CET	**Termin:**	202310
Sluttdato:	03-07-2023 12:00 CEST	**Vurderingsform:**	Norsk 6-trinns skala (A-F)
Eksamensform:	T		
Flowkode:	202310||11184||IN00||W||T		
Intern sensor:	(Anonymisert)		

Deltaker

Navn:	Mikael Kenneth Nilsen og Ole Anders Bolle

Informasjon fra deltaker

Tittel *:	Exploring Risk-Based Portfolio Construction: Empirical Evaluation of Equal Risk Contribution and Inverse Volatility Portfolios
Navn på veileder *:	Alessandro Graniero

Inneholder besvarelsen konfidensielt materiale?:	Nei	**Kan besvarelsen offentliggjøres?:**	Ja

Gruppe

Gruppenavn:	(Anonymisert)
Gruppenummer:	152
Andre medlemmer i gruppen:	

Exploring Risk-Based Portfolio Construction: Empirical Evaluation of Equal Risk Contribution and Inverse Volatility Portfolios

Master Thesis

by
Mikael Kenneth Nilsen and Ole Anders Bolle
QTEM MSc in Business with Major in Finance
Supervised by Alessandro Graniero

Oslo, July 1, 2023

ABSTRACT

We examine the effectiveness of risk-based portfolio construction methods, specifically Equal Risk Contribution (ERC) and Inverse Volatility (IVP), using the same data set and performance measures as in DeMiguel, Garlappi, and Uppal's study from 2009 on the mean-variance model and its extensions. We create portfolios and evaluate their out-of-sample performance against the Equal Weighted (EW) strategy in terms of Sharpe Ratio, Certainty Equivalent, and turnover. Findings suggest that while ERC and IVP do not consistently outperform the EW strategy, they demonstrate significant potential, often matching or surpassing EW's performance.

Acknowledgements

We would like to express our sincere gratitude to our supervisor, Professor Alessandro Graniero, for his guidance throughout the process of completing this thesis. His insightful feedback and direction have been important in shaping the final outcome.

We are also grateful to the researchers Victor de Miguel, Lorenzo Garlappi, and Raman Uppal for inspiring our research question and for making the data they used in their research (2009) available online.

Contents

List of Acronyms

CAPM	Capital Asset Pricing Model
CEQ	Certainty Equivalent
DGU	DeMiguel, Garlappi, and Uppal
ERC	Equal Risk Contribution
EW	Equal Weight
G-MINVC	Generalized Minimum Variance Constrained
HML	High-Minus-Low
IVP	Inverse Volatility Portfolio
JB	Jarque-Bera
KS	Kolmogorov-Smirnov
MKT	Market (factor)
MINV	Minimum Variance
MINVC	Constrained Minimum Variance
MRC	Marginal Risk Contribution
MVC	Constrained Mean-Variance
MV	Mean-Variance
SMB	Small-Minus-Big
SR	Sharpe Ratio
TRC	Total Risk Contribution
UMD	Up-Minus-Down
VW	Value Weighted

List of Figures

List of Tables

List of Symbols

The description of the symbols in this list is a general rule. Some cases throughout the thesis have slightly different meanings; this is explicitly mentioned for the specific case.

μ_i	Expected returns excess of the risk-free rate for asset i
σ_i	Volatility of returns for asset i
$\sigma_{i,j}$	Covariance between asset i and j
Σ	Variance-Covariance matrix of returns
γ	Risk aversion coefficient for a power utility investor
x	Absolute weights in risky assets
w	Used as relative weights in risky assets
$\mathbf{1}_N$	Column vector of N ones

1 Introduction and motivation

Investment management and portfolio construction have been areas of intense interest and research for many decades, with one of the seminal works in the field being Modern Portfolio Theory, introduced by Harry Markowitz in the 1950s (Markowitz, 1952). Markowitz's work marked a significant turning point in the area, as it introduced the concept of mean-variance optimization as a framework for portfolio construction and asset allocation. Investors and portfolio managers have widely adopted this approach, as it provides a systematic method for balancing portfolio risk and returns.

However, the mean-variance approach has weaknesses, and one of the main criticisms is that portfolio optimization is highly sensitive to estimation errors. This was first pointed out by Michaud (1989), who noted that even small errors in the estimation of expected returns and covariance matrices could have a significant impact on the optimization results. This sensitivity to estimation errors has led many researchers to question the mean-variance approach's validity and seek alternative methods for portfolio construction that are less sensitive to estimation errors.

In response to this challenge, DeMiguel, Garlappi, and Uppal (2009) (DGU) conducted a study on the topic "Optimal versus Naive Diversification," where they explored various methods for reducing estimation errors in portfolio optimization. The authors conducted a comprehensive analysis of several portfolio construction methods, including the Equal Weighted (EW) portfolio strategy and the mean-variance approach. They concluded that more work must be put into estimating moments to obtain reliable and accurate results. This study provides valuable insights into the challenges and limitations of the mean-variance approach to portfolio optimization.

Building upon the work of DGU, our thesis aims to expand the examination of portfolio construction methods. While their study highlights the importance of careful moment estimation, other approaches to portfolio construction focus on managing risk. Instead of looking at ways to reduce estimation error, we propose portfolios that do not need as much estimation and thus partly eliminate the problem.

Specifically, we aim to examine the effectiveness of the risk-based portfolio construction techniques Equal Risk Contribution (ERC) (Risk Parity) and Inverse Volatility (IVP), as discussed by Maillard et al. (2010), Lee (2011), Clarke et al. (2013), and Roncalli (2016). To do this, we evaluate the portfolios' out-of-sample performance to that of the EW. In addition, we point to possible explanations for this performance.

To ensure an unbiased comparison of the effectiveness of our two proposed portfolio strategies, we have used the same data as DGU and recreated some of their portfolios with success. The data we have used are listed in Table 2 and further explained in Appendix A; the portfolios we created are listed in Table 1 and explained in Section 4. We evaluate the performance using the same three measures as DGU: the Sharpe ratio, Certainty Equivalent return, and turnover.

Our main finding is that IVP and ERC do not consistently beat the EW strategy in terms of Sharpe Ratio, Certainty Equivalent, or turnover. However, they still show considerable promise. Their performance either matches or significantly exceeds the EW strategy.

We believe the portfolios could have performed even better if the datasets analyzed were single stocks rather than portfolios of assets. We reason that value-weighted portfolios of assets are already diversified and have less idiosyncratic risk, which affects the relative performances in two ways. Firstly, the IVP and ERC could have limited some of this inherent risk in any asset by construction. Secondly, the EW makes fewer errors in allocation and suffers less from risk concentration when investing equally in broad indices.

Risk-based portfolios disregard returns in their optimization but still perform well. We point to three overarching reasons: (1) The relatively good performance of risk-based models can be the poor performance of mean-variance strategies because of estimation errors. (2) Risk-based portfolios reward and overweight assets with low risk and may thus include assets that empirically have been shown to provide a higher return than the risk would suggest due to what some researchers coin the "low-risk anomaly." (3) Risk-based portfolios are inherently constrained to be long-only and invest in all investible assets. Even if constraints limit the investible universe and thus possible reward, empirical evidence shows that constrained portfolios often perform well because they avoid extreme positive and negative weights. They effectively use "shrinkage" on the parameter estimations, leading to less estimation error.

The rest of our thesis is organized as follows: Section 2 reviews the literature on quantitative portfolio theory, specifically emphasizing DGU's paper and risk parity portfolios. We use existing theory and literature to explain an investor's portfolio choice and the expected performance in section 3. We devoted an entire section to explaining our portfolios' mathematical build-up and relationships in Section 4. Section 5 presents the methodology used to find, evaluate, and stress test the out-of-sample performances. We present the data used and some specific characteristics of the data that can help explain portfolio performance in Section 6. Section 7 analyzes our results, and section 8 is left for concluding remarks.

2 Literature Review

Our research draws inspiration from the study "Optimal versus Naive Diversification: How Inefficient Is the 1/N portfolio strategy?" by DeMiguel, Garlappi, and Uppal (2009) (DGU). DGU studied various methods for minimizing estimation errors in portfolio optimization. This in-depth analysis of several portfolio creation methods, such as the Equal Weighted (EW) portfolio strategy and mean-variance approach, concluded that achieving reliable results requires more refined moment estimation. DGU's evaluation spanned seven datasets, including sector and Fama-French factor portfolios, finding that no model consistently outperformed the EW strategy in terms of Sharpe Ratio, Certainty Equivalent, or turnover.

As a reaction, some researchers sought to either validate or contradict the superiority of the EW portfolio. Others investigated its impressive performance and extended the concept of diversification to portfolio construction methods beyond mean-variance. This section will summarize some of the main findings of these articles, starting with the direct extensions and comments on the article by DGU and continuing with other portfolio construction methods, mainly about risk parity. This summary will be the basis for expanding the original paper with additional ways to construct portfolios.

One point of critique against DGU's study is that they only used historical data to estimate expected returns and standard deviations, while other methods of forecasting or incorporating additional available information on stock returns could be employed. Allen et al. (2019) argue that with some forecasting ability, mean-variance optimized portfolios could outperform the EW strategy in out-of-sample tests, which they demonstrate through simulations and empirical analysis. Their key difference from DGU is that they use forward-looking estimates as input variables rather than relying strictly on historical data.

Furthermore, the portfolio performances can vary depending on the dataset used (Anderson et al., 2012), and the diversification benefits of the Equal Weight portfolio can be limited when the available assets are concentrated in a few separate markets or industries (Lee, 2011).

Another criticism of the DGU paper is that the constant 120-month rolling windows used are too short, and their way of estimating returns and covariances leads to more estimation errors than necessary Kritzman et al. (2010). To address this, the authors propose using more robust parameter estimation methods using naive but plausible estimates of expected returns, volatilities, and correlations. They provide empirical evidence that the optimization approach outperforms the EW portfolio.

Kirby and Ostdiek (2012) argue that the mean-variance approach outperforms the EW portfolio, but excessive turnover caused by estimation errors erodes this advantage. To overcome this, they propose volatility and reward-to-risk timing to deliver portfolios with lower turnover.

Although the findings of DGU suggest that the EW strategy may have some merit, other portfolio construction strategies have been proposed which may be more effective in certain circumstances. Following the era when the industry aimed to correct the shortcomings of mean-variance optimization, practitioners pivoted towards alternative portfolio construction techniques that did not rely on estimating expected returns. One such method was risk budgeting. Although the idea of asset risk contribution has existed for quite some time Litterman (1997), the strategy didn't take center stage in portfolio choice until many years later. We will focus more on a particular case of Risk Budgeting called Risk Parity.

The term 'Risk Parity' was coined by Edward Qian, a fund manager at PanAgora Asset Management Qian (2005). An early example is the All Weather fund portfolio developed by Bridgewater, designed to create a portfolio that was robust against various macroeconomic conditions. The strategy aimed to equalize assets that would thrive under different inflation and GDP conditions.

Later versions of risk parity shifted focus from resilience against macroeconomic conditions to emphasizing the total risk contribution of each component. This approach gained popularity after Maillard et al. (2010) published an article exploring the properties of the risk parity portfolio. Asness et al. (2012) compared the risk parity, "60-40", and market portfolios and concluded that risk parity portfolios outperform the market across countries and asset classes.

Some limitations of the papers mentioned on Risk Parity are that they either use limited datasets or do not thoroughly analyze other models to compare their relative performance. We can contribute to the ongoing discussion on constructing the best portfolio by expanding on an already thorough analysis of the mean-variance strategy. We plan to introduce risk-based approaches to the same data set and evaluate their performance using identical metrics. This integration will allow for a more comprehensive comparison and present an opportunity to observe the behavior and effectiveness of these strategies under similar conditions. Our contribution could provide valuable insights and further the "optimal portfolio" conversation.

3 Theory

In this section, we break down the theory of portfolio optimization and the specific traits among the portfolios we cover in our thesis. The purpose is to explain portfolio performance and see how specifications can alter the results.

3.1 Portfolio construction theory

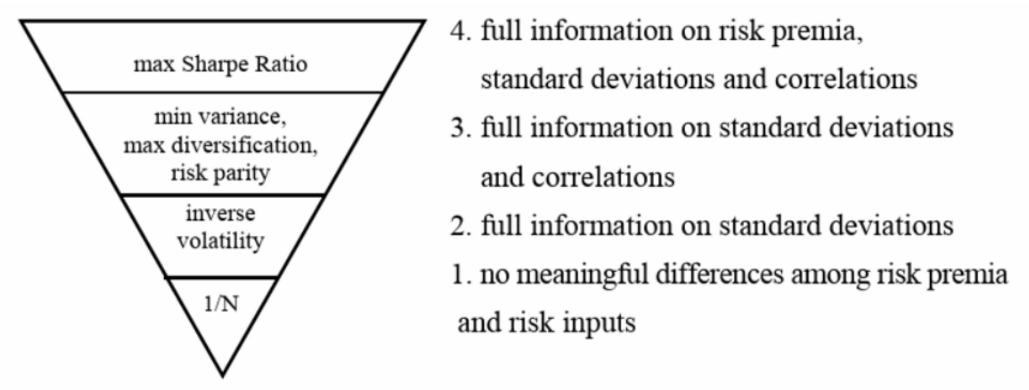
3.1.1 Choice of model and input parameters

Portfolio construction involves strategically selecting and weighing assets to meet a particular investment objective. All methods require an investor to make assumptions about future market behavior. The selection of an appropriate strategy largely hinges on the information available - or our confidence in that information.

Hallerbach (2013) captures this in his Portfolio Decision Pyramid (Figure 1), which lays out an investor's strategy choice based on the extent of their knowledge. An equal-weight strategy would be suitable if the investor knows nothing. An inverse volatility portfolio could be considered if they know the standard deviations of returns. A minimum variance or risk parity strategy that depends on the variance-covariance matrix may be optimal if correlations are known. Finally, if an investor has insight into expected returns, standard deviations, and correlations, a mean-variance approach would be the best fit.

However, an important caveat is the trade-off between the number of input parameters a model needs and the potential for estimation error. The more inputs a model requires, the higher the potential for inaccuracies. This notion is reinforced by Kan and Zhou (2007), who examined the performance of the mean-variance tangency portfolio. They report the loss in performance because of estimation error when either the mean, variance-covariance matrix, or both are unknown. Estimation errors in expected returns could significantly

Figure 1: Portfolio Decision Pyramid



This figure shows Hallerbach's (2013) Portfolio Decision Pyramid. It lays out the portfolio decision options an investor has based on the information available and the confidence the investor has in this information, moving from no information at the bottom of the inverse pyramid to all available information at the top.

degrade performance. Similarly, errors in the covariance matrix could also have a negative impact, albeit to a lesser extent. Most important, they found that the interaction term of estimation error - when both mean and covariance are unknown - is significantly detrimental to portfolio performance.

The potential for error in estimating mean returns is typically greater than in estimating the covariance matrix due to the inherent nature of these calculations. The mean return is a single number, while the covariance matrix is a collection of numbers representing the relationships between all possible pairs of assets in a portfolio. As a result, the covariance matrix estimation benefits from a larger sample size, which can reduce estimation error. Moreover, as the covariance matrix represents relationships between all asset pairs in a portfolio, it can provide valuable information about risk reduction from diversification. Conversely, mean returns do not offer this diversification effect and thus won't add any additional benefit to the model. As Jagannathan and Ma (2003) put it: "...the estimation error in the sample mean is so large that nothing much is lost in ignoring the mean altogether."

3.1.2 Mean-Variance Optimal Portfolio

The mean-variance optimal portfolio is a well-known construction method developed by Markowitz (1952). Given a set of assets, one can construct the most efficient portfolio with the lowest variance for a given expected return. The most optimal portfolio among these is the tangency portfolio. This portfolio maximizes the Sharpe Ratio (SR), giving the maximal excess return per unit of standard deviation.

Despite being groundbreaking during its inception, the strategy's sensitivity to input errors can result in volatile, inconsistent weightings[1]. Referring to Figure 1, the mean-variance strategy requires information about both the mean returns and the variance-covariance matrix to be effective.

3.1.3 Minimum Variance Portfolio

The minimum variance portfolio (MINV) is among the portfolios constructed by mean-variance optimization. The constraint on target return is removed, and the optimization problem instead finds the minimum achievable total risk. By construction, this portfolio has a lower expected return than the tangency portfolio.

By the two-fund theorem (Tobin, 1958), a combination of the risk-free asset and the tangency portfolio can achieve any risk level while still achieving the maximal SR. The question is why an investor would choose the minimum variance portfolio when the same risk can be achieved with a higher return. Surprisingly, the performance ex-post is often better than for other mean-variance approaches. As noted in section 3.1.1, estimation error in the expected

[1] As exemplified in DeMiguel et al. (2009): *"Consider the following extreme two-asset example. Suppose that the true per annum mean and volatility of returns for both assets are the same, 8 % and 20 %, respectively, and that the correlation is 0.99. In this case, because the two assets are identical, the optimal mean-variance weights for the two assets would be 50 %. If, on the other hand, the mean return on the first asset is not known and is estimated to be 9 % instead of 8 %, then the mean-variance model would recommend a weight of 635 % in the first asset and -535% in the second."*

mean of returns can contribute to this. Another explanation could be the cross-sectional equity risk anomaly documented by Ang et al. (2006), where high-risk assets gave low returns, implying that a portfolio based on minimizing risk could achieve higher returns.

3.1.4 Equal Weight Portfolio

The Equal Weight (EW) portfolio (”naive diversification”) distributes investments equally across all investible assets. Despite its apparent simplicity, the method has several benefits. Notably, it is straightforward to implement, requiring neither an intricate understanding of finance nor significant effort.

Further, according to DeMiguel et al. (2009), this strategy often performs better than more sophisticated portfolio-building methods out-of-sample. They attribute part of the extraordinary performance to the fact that the ”allocation mistakes” in EW are smaller than those made using input parameters with estimation errors. They suggest that roughly 3000 observations for 25 assets are required to construct mean-variance portfolios that, on average, outperform the EW approach.

One key aspect of this strategy is its tendency to promote diversification, reducing idiosyncratic risks by avoiding overemphasizing single markets. It invests more in small-cap stocks than the traditional market portfolio, leveraging the small-cap anomaly or ”size” factor. In addition, EW requires periodic rebalancing to uphold its distribution, effectively implementing a ”buy low, sell high” tactic known as a ”volatility pumping strategy.” Nevertheless, the selection of assets largely dictates the portfolio's performance, and more volatile assets increase the portfolio's overall risk.

3.1.5 Portfolios with short-sale constraints

By introducing constraints to the mean-variance model, we can avoid extreme weight distributions and thus enhance the model's performance. Moreover, not all investors can or desire to engage in short selling as part of their strategies, nor are all assets readily available for short selling.

Introducing a positive weight constraint can mathematically reduce estimation error because of a shrinkage effect. The Lagrangian for the unconstrained mean-variance model is[2]:

$$\mathcal{L} = x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x,$$

where x refers to the vector of asset weights, μ the vector of expected returns, γ the coefficient of risk aversion, and Σ the variance-covariance matrix. Adding the positive weight constraint changes this to:

$$\mathcal{L} = x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x + x^\top \lambda$$

where λ is the Lagrange multiplier. The notable difference between the constrained and unconstrained portfolio weights lies in the adjustment of the mean vector with the Lagrange multiplier for the constraint, represented as $\tilde{\mu} = \mu + \lambda$. The constraint becomes binding when an asset's expected return is low, implying that $\lambda > 0$ and that the expected returns of the constrained model exceed those of the unconstrained model. Therefore, adding the constraint is equivalent to "shrinking" the expected return toward the average (DeMiguel et al., 2009, p.1925). Applying a no-short-sale constraint on the minimum variance portfolio similarly shrinks the elements of the variance-covariance matrix (Jagannathan and Ma, 2003).

[2]See explanation in Section 4 and Appendix B.1

3.1.6 Risk budgeting

When considering portfolio risk, we typically consider a singular aspect of the entire portfolio, such as portfolio standard deviation. However, we can also look at portfolio risk as the sum of risk contributions from the assets that constitute the portfolio.

The risk budgeting portfolio allocates weights to assets so that they contribute the budgeted amount of total risk to the portfolio. Like with the minimum-variance and equal weight strategies, there is no apparent reason for the risk budgeting approach to perform well in terms of returns, as "returns" never enter the optimization problem. The portfolio is less prone to return loss from estimation error because it does not consider expected means.

Unlike the minimum-variance portfolio, which often results in asset concentration, the risk budgeting approach ensures investment in all investible assets and strictly maintains long positions, as emphasized by Clarke et al. (2013). One notable variant of risk budgeting is when the risk contribution of each asset in the portfolio is equalized. This specific approach to risk budgeting is often referred to as risk parity.

3.1.7 Risk Parity

A portfolio achieves risk parity when each asset contributes an equal percentage of risk across all assets (Lee, 2011). One way of achieving this is multiplying the weight of an asset, w_i, by its volatility, σ_i, and equalizing over all assets: $w_i\sigma_i = w_j\sigma_j,\ \forall\, i, j$. This is called the inverse volatility portfolio (IVP) because the weight of any asset is proportional to the inverse of its volatility. This portfolio will include more of assets with low volatility and less of assets with high volatility.

The IVP does, however, not account for the possible correlation of returns among assets and may thus lead to an asset contributing considerably more or less risk than other assets. It may also lead to a high concentration of portfolio weights in highly correlated markets. The equal risk contribution portfolio (ERC) alleviates this problem by considering correlation.

It can be shown that the weights in an ERC portfolio are proportional to the inverse of their beta (β_i) to the portfolio[3], $w_i \propto 1/\beta_{ip}$. The beta indicates the asset's sensitivity to the systematic risk of its portfolio. Consequently, assets with a higher beta are assigned less weight; conversely, assets with a lower beta are given more weight in the ERC. As a result, assets with high volatility or significant correlation with other assets will be penalized.

3.2 Further discussion of risk budgeting

The existence of risk budgeting portfolios

The optimal risk budgeting portfolio has a unique solution, thanks to its convex nature (Maillard et al., 2010). As highlighted earlier, the risk budgeting portfolio is constructed to include all investible assets in the model. If an asset is assigned a 0 budget, it can lead to multiple "optimal" solutions (Roncalli, 2016, p. 110). It is, therefore, more beneficial to remove the asset from the model altogether. By eliminating assets one does not intend to include, the number of parameters to compute is reduced, thus further mitigating estimation errors. The same applies to assets allocated with a negative risk contribution.

If an asset has a negative budget, the model indicates a greater concentration of risk in the portfolio's other assets. By assigning negative risk budgets, the portfolio could either have multiple solutions or potentially no solutions (Roncalli, 2016, p. 113).

[3] see Maillard et al. (2010) and section 4.6

Limitations of risk budgeting portfolios

The construction of risk parity does not consider the structure of the investible universe, particularly when several asset classes are involved. To illustrate, consider a portfolio consisting of five equity and five bond indices; the portfolio would balance equity and bond risk. However, with seven equity indices and three bond indices, despite equalized risk contribution across each index, we would have an imbalance between equity and bond risk. To rectify this issue, one could employ a risk budgeting method and manually adjust the risk budgets or use risk parity based on risk factors Roncalli (2016).

While proponents of risk parity claim its strength lies in not requiring the consideration of expected returns of assets, the choice of assets is still biased on assets' previous returns. The investible universe usually consists of assets that have performed well recently. Despite their historical significance, choices like including cryptocurrencies or excluding commodities ultimately fall to the investor. Thus, while risk parity provides a reliable asset allocation method, it cannot eliminate the need for decision-making.

The computational load is heavy when we include many assets because we must employ an optimization algorithm to find the unique solution. One way to manage this issue would be to group assets with similar risk characteristics or factors. However, this introduces selection bias.

3.3 Performance of risk-based portfolios

An important question when constructing a portfolio that does not consider asset returns is; how can we expect good performance in risk-adjusted returns? Some researchers point to the "low-risk anomaly" (Scherer, 2011) or "cross-sectional equity risk anomaly" (Ang et al., 2006) as a contributing factor.

The low-risk anomaly refers to the empirical observation that lower-risk stocks have historically provided higher risk-adjusted returns than higher-risk stocks.

Lochstoer and Muir (2022) found volatility to have a weak or negative risk premium. This observation contradicts common understanding that increased risk should lead to increased return. The IVP and ERC invest more in stocks with relatively lower risk and may thus exploit this phenomenon.

Scherer (2011) looked at the performance of the minimum-variance portfolio and found that 83% of the variation could be attributed to Fama-French factors. This means that its superior performance is not directly a function of lower risk but rather a side effect of selecting stocks that exploit these factor anomalies.

Asness et al. (2012) suggest that the low-risk anomaly stems from two effects: Leverage aversion and lottery-type stocks. Leverage aversion refers to the hesitation or unwillingness of investors or fund managers to use leverage in their portfolios. An investor can lever up a low-risk stock with low expected returns to achieve higher total returns. However, leverage aversion leads them to choose an unlevered high-risk asset with a high expected return instead. This could cause high-risk stocks to become overpriced and low-risk stocks to become underpriced, leading to higher risk-adjusted returns for low-risk stocks.

The "lottery ticket effect" is a term often used to describe the behavior of investors drawn to low-priced, high-risk stocks in the hopes of achieving outsized gains, much like buying a lottery ticket. This behavior can cause these high-risk stocks to be overpriced if many investors are "buying lottery tickets."

Another reason we pointed to earlier is that a portfolio based solely on the variance-covariance matrix as input is less prone to estimation error. The performance may, therefore, not point to something extraordinary with these models but rather a fault in the models that require more parameter inputs.

4 Portfolio Construction

Building upon the foundational theories of portfolio construction discussed in Section 3, this section delves into the mathematical representation of the portfolios we deploy and outlines their implementation process. The portfolios are listed in Table 1. We conclude this section by drawing connections in the construction methodology of these portfolios, highlighting their similarities.

Table 1: Asset-Allocation Models

#	Model	Abbreviation
0	Equal Weight with rebalancing (benchmark strategy)	EW or 1/N
1	Value-weighted market portfolio	VW
Classical mean-variance		
2	Mean-Variance based on sample moments	MV
Risk-Based Portfolios		
3	Minimum-variance	MINV
4	Inverse volatility portfolio	IVP
5	Equal risk contribution	ERC
Constrained Portfolios		
6	Mean-variance with short-sale constraints	MVC
7	Minimum-variance with short-sale constraints	MINVC
8	Minimum-variance with generalized constraints	G-MINVC

This table presents an overview of the portfolio construction models we have analyzed in our thesis. The last column gives the abbreviation we have used to refer to the strategies in the text and the result tables.

Following DeMiguel, Garlappi, and Uppal's (DGU) notation, R_t represents the N-vector of excess returns over the risk-free asset for the N risky investment options available at time t. The expected returns on these risky assets beyond the risk-free rate are symbolized as an N-dimensional vector, μ_t, while Σ_t stands for the related $N \times N$ variance-covariance matrix. Their sample equivalents are denoted as $\hat{\mu}_t$ and $\hat{\Sigma}_t$, respectively. The term M signifies the span over which these moments are computed, and T designates the entire data series length. The N-dimensional vector of ones is defined as $\mathbf{1}_N$. Lastly, x_t refers to the vector of portfolio weights allotted to the N risky assets, with the remainder, $1 - \mathbf{1}_N^\top x_t$, invested in the risk-free asset.

To calculate optimal[4] weights, we assume a universe with a risk-free asset when first constructing portfolios. This will affect the efficient frontier of risky assets, ensure a tangency portfolio for the mean-variance strategy, and allow for leverage in the trading strategies where this is relevant. As the benchmark case in DGU's study contains portfolios of only risky assets, we find the *relative* weights invested in risky assets as:

$$w_t = \frac{x_t}{|\mathbf{1}_N^\top x_t|} \tag{1}$$

We normalize by the absolute value of the sum of weights, $|\mathbf{1}_N^\top x_t|$ to ensure that the relative sign of asset position is correct in the instances where the sum of weights in risky assets is negative.

4.1 Mean-Variance

For comparability of results, we consider an investor whose utility is fully described by the mean and variance of a chosen portfolio. The certainty equivalent (CEQ) of a risky choice is the risk-free rate that an investor is willing to accept rather than investing in a risky portfolio strategy. It can be shown by a Taylor Expansion[5] that the CEQ is approximated as:

$$CEQ \approx W\left[\bar{R} - \frac{\gamma}{2}\sigma_R^2\right],$$

where W represents initial wealth, $\bar{R}$ is the expected return, γ is the investor's risk aversion, and σ_R^2 is the variance of the return. As utility is an increasing function for a risk-averse investor, maximizing utility is equivalent to maximizing the CEQ. For each period t, the investor seeks to maximize utility by

$$\underset{x_t}{max} \quad x_t^\top \hat{\boldsymbol{\mu}}_t - \frac{\gamma}{2} x_t^\top \hat{\boldsymbol{\Sigma}}_t x \tag{2}$$

[4]By "optimal," we mean the mathematical solution to portfolio problems and not the weights of the mean-variance *optimal* portfolio

[5]see Appendix B.1

The solution to the above problem is $x_t = (1/\gamma)\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t$, and the *relative* weights invested in risky assets is the tangent portfolio:

$$w_t = \frac{\hat{\boldsymbol{\Sigma}}_t^{-1}\hat{\boldsymbol{\mu}}_t}{\mathbf{1}_N^\top \hat{\boldsymbol{\Sigma}}_t^{-1}\hat{\boldsymbol{\mu}}_t} \tag{3}$$

We have assumed, like DGU, that the mean and variance fully describe the investor's preferences. If the returns were normally distributed, the mean and variance would capture all necessary information about the return distribution, and the approach would be most effective. However, returns often exhibit skewness and kurtosis. In section 6, we test for normality.

4.2 Minimum Variance

The minimum-variance portfolio (MINV) aims to minimize the variance of returns. This is mathematically represented by the optimization problem:

$$\min_{x_t} \; x_t' \hat{\boldsymbol{\Sigma}}_t x_t \quad \text{u.c. } \mathbf{1}_N' x_t = 1 \tag{4}$$

with solution

$$w_t = \frac{\hat{\boldsymbol{\Sigma}}_t \mathbf{1}_N}{\mathbf{1}_N^\top \hat{\boldsymbol{\Sigma}}_t \mathbf{1}_N} \tag{5}$$

To implement this approach, we rely solely on the sample covariance matrix and disregard the estimates of expected returns. Though this strategy does not conform to the standard structure of mean-variance expected utility, its weight allocations can be considered an extreme case of mean-variance, Equation (3). This could be the case if a mean-variance investor either disregards expected returns or, equivalently, imposes restrictions such that expected returns are identical across all assets, i.e., $\mu_t \propto \mathbf{1}_N$.

4.3 Equal Weight

The equal weight (EW) or "naive diversification" strategy involves allocating an equal portfolio weight of $1/N$ to each of the N risky assets. This strategy sidesteps any optimization or estimation process and entirely disregards the data. Mathematically:

$$w_t = \frac{\mathbf{1}_N}{N} \tag{6}$$

When all assets have the same correlation coefficient, identical means, and variances, the EW portfolio is the unique portfolio on the efficient frontier (Maillard et al., 2010). In comparison with the weights outlined in Equation (3) on mean-variance, the EW portfolio can also be interpreted as a strategy that does estimate the moments μ_t and Σ_t but enforces the condition that $\mu_t \propto \Sigma_t \mathbf{1}_N$ for all time periods t (DeMiguel et al., 2009). This implies that expected returns are associated with total risk instead of just systematic risk.

4.4 Value-weighted market portfolio

In a CAPM world, the market portfolio is the optimal investment strategy. This is a value-weighted portfolio of all available assets in the universe and is thus as good as impossible to invest in. Broad market indices can be used as market portfolio proxies, and in each dataset, there is an index that represents the appropriate "market portfolio." The value-weighted strategy involves holding this asset for the entire investment period. The turnover of such a strategy is 0.

4.5 Introducing Constraints

By introducing constraints on portfolios, we limit investment opportunities. Intuitively, this should also limit the potential for extraordinary performance. However, for strategies like the mean-variance, where we often see extreme weights in both short- and long directions, a weight constraint will limit these unnecessary risky positions and lead to more stable returns. Constraining the weights will also limit the turnover in these strategies considerably. We impose a "no short-sale" constraint on the mean-variance and minimum-variance strategies. We also limit the maximum leverage in any asset to 1 and the sum of invested wealth to 1, meaning we can't borrow to increase our leverage. The added constraints to the mean-variance and minimum-variance strategies are[6]:

$$x_t \geq \mathbf{0}$$
$$\mathbf{1}_N^\top x_t \leq 1$$

In addition, inspired by DGU, we created a portfolio with a set lower bound on weights in each asset to ensure diversification, which they call Generalized Minimum Variance Constrained (G-MINVC). The portfolio can be seen as an expansion on the minimum-variance strategy with the added constraint: $\mathbf{w}_i \geq a\mathbf{1}_N$, with $a \in [0, 1/N]$. By instituting a minimum weight allowance for each asset, it addresses the concentration issue often found in the minimum variance portfolio.

In the boundary case $a = 0$, the portfolio is the MINV, whereas in the other extreme, $a = 1/N$, we get the EW portfolio. The equal risk contribution portfolio (ERC) weights are also in this narrow interval (Maillard et al., 2010). Therefore, constructing the G-MINVC is an interesting test to see whether some form of optimization (ERC) can beat a heuristic allocation scheme (G-MINVC). We set a at the midpoint of the boundaries, $a = \frac{1}{2} \cdot \frac{1}{N}$.

To implement these non-linear optimization problems, we use MatLab's optimization software and function "quadprog."

[6]In the investment case of risky assets only, the second constraint becomes the binding constraint $\mathbf{1}_N^\top x_t = 1$

4.6 Risk Parity

To explain how to find the risk parity portfolios, we first present some measures and definitions based on Roncalli (2016) and Maillard et al. (2010). We use the stock return volatility as the risk measure for evaluating the risk parity portfolio for three reasons.

First, we don't want to assume an investor's preference toward Value at Risk, Expected Shortfall, or Maximum Drawdown. Second, we measure portfolio performances using the Sharpe Ratio and CEQ. These measures directly incorporate return volatility, easing the calculations for the risk parity portfolios. Third, as emphasized by Artzner et al. (1999), return volatility has desirable properties for risk measures, such as convexity.

We consider a portfolio $x = [x_1, x_2, ..., x_N]^\top$ of N risky assets. The return of asset i is r_i, the variance of asset i is denoted σ_i^2, and the covariance between asset i and asset j is σ_{ij}. The variance, σ_p^2, of portfolio x is:

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} = x^\top \Sigma x \tag{7}$$

The marginal risk contribution of asset i is defined as the impact on portfolio risk caused by a marginal change in the weight of asset i:

$$MRC_i = \frac{\partial \sigma_p^2}{\partial x_i} = \sum_{j=1}^{N} x_j \sigma_{ij} = (\Sigma x)_i, \tag{8}$$

where Σx is the vector of covariances of each asset i with the portfolio x, and $(\Sigma x)_i$ refers to the i-th element of this vector. The total risk contribution of an asset i is found by multiplying its marginal contribution with the asset weight:

$$TRC_i = x_i \frac{\partial \sigma_p^2}{\partial x_i} = x_i \sum_{j=1}^{N} x_j \sigma_{ij} = x_i (\Sigma x)_i \tag{9}$$

By definition, the sum of all assets' total risk contribution should amount to the portfolio risk:

$$\sum_{i=1}^{N} TRC_i = \sum_{i=1}^{N} x_i(\Sigma x)_i = x^\top \Sigma x \tag{10}$$

As a side note, the minimum variance portfolio is the portfolio where the marginal risk contributions are equalized across assets. This makes intuitive sense; if an asset had less marginal risk contribution than all other assets, it would benefit the total portfolio risk to invest more in this asset and less in all others.

Equal Risk Contribution Portfolio

The idea of the ERC portfolio is that each asset in the portfolio contributes the same amount of risk, or:

$$TRC_i = x_i \frac{\partial \sigma_p^2}{\partial x_i} = x_j \frac{\partial \sigma_p^2}{\partial x_j} = TRC_j \qquad \forall i, j \tag{11}$$

$$TRC_i = \lambda$$

From Equations (9) and (11), we can deduce that ERC fulfills:

$$\Sigma x = \lambda \frac{1}{x}, \tag{12}$$

where $\frac{1}{x}$ denotes the vector $[\frac{1}{x_1}, \frac{1}{x_2}, ... \frac{1}{x_N}]^\top$.

The beta, β_i, of asset i with respect to portfolio x indicates the asset's sensitivity to the portfolio's systematic risk. It can be expressed as

$$\beta_i = \frac{cov(r_i, r_p)}{var(r_p)} = \frac{(\Sigma x)_i}{x^\top \Sigma x} \tag{13}$$

By dividing Equation (12) by the portfolio variance, $x^\top \Sigma x$, and rearranging the expression, we get for each asset i:

$$x_i = \frac{\lambda^*}{\beta_i},$$

where $\lambda^* = \lambda \cdot 1/x^\top \Sigma x$, a scaled measure of risk contribution. Using the fact that $\sum_{i=1}^{N} x_i = 1$ we can see that the weight in asset i is

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{N} \beta_j^{-1}} \tag{14}$$

This gives a financial interpretation of the asset weights: The weight attributed to asset i is inversely proportional to its beta. A higher volatility or correlation with other assets will lead to less weight.

The expression in Equation (14) does not have a closed-form solution. It is endogenous because the β_i on the right-hand side is the asset's beta with the ERC portfolio, which will change with the calculated weights on the left-hand side. We could solve this set of non-linear equations by a recursive process with a defined stopping criterion (Chaves et al., 2012): Find the betas from an initial guess of asset weights; calculate new asset weights from these betas; restart the process with new asset weights as input; continue the process until weights are within a set tolerance. This method works well when the asset universe is small, but there is no guarantee that it will converge to a unique solution (Roncalli, 2016, p. 308).

A better way to find the ERC portfolio is to define a convex optimization problem. One example of an optimization problem was proposed by Maillard et al. (2010):

$$\begin{aligned} x^* &= \arg\min f(x) \\ u.c. &\begin{cases} \mathbf{1}^\top x = 1 \\ \mathbf{0} \leq x \leq \mathbf{1} \end{cases} \end{aligned} \tag{15}$$

with

$$f(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} (x_i(\Sigma x)_i - x_j(\Sigma x)_j)^2 \tag{16}$$

This problem can be solved as a Sequential Quadratic Program (SQP) with optimization software (Roncalli, 2016, p.102). However, optimization programs can be slow when the number of assets is large, so we use an algorithm that calculates analytical constituents to increase computational time. This algorithm was also introduced by Chaves et al. (2012). Unlike the iterative process with Equation (14), this method has a mathematical proof and more often converges. See Appendix B.2 for further details.

Maillard et al. (2010) established that the ERC portfolio is the tangency portfolio when assets have constant correlations and the same Sharpe ratio. They also show that the ERC can be viewed as a middle ground between EW and MINV, incorporating characteristics from both. One way of expressing the optimal weights from the ERC portfolio problem is:

$$x(c) = \arg\min \sqrt{x^\top \Sigma x}$$
$$\text{u.c.} \begin{cases} \sum_{i=1}^{N} ln(x_i) \geq c \\ \mathbf{1}^\top x = \mathbf{1} \\ x \geq \mathbf{0} \end{cases} \tag{17}$$

The constant c can be considered the minimum level of diversification among assets required to achieve the ERC portfolio. We will get different portfolios if we set c to any other value. One extreme value of c is $-\infty$. The remaining problem results in the minimum variance portfolio. The maximum value c can take is $-n \, ln \, n$ because $\sum_{i=1}^{N} ln(x_i)$ subject to $\sum_{i=1}^{N}$ is maximized for $x_i = 1/N$. If $x_i = 1/N$, we arrive at the equal weight portfolio.

Not only do weights for the ERC lie between EW and MINV, but the authors also show that the volatilities are ordered the same way: $\sigma_{\text{minv}} \leq \sigma_{\text{erc}} \leq \sigma_{\text{ew}}$.

Inverse Volatility Portfolio

The inverse volatility portfolio could be seen as a special case of the ERC when correlations among assets are equal. We can also see it as the ERC portfolio, assuming all asset correlations equal 0. This will make all weights proportional to their inverse volatility, $x_i \propto \frac{1}{\sigma_i}$. The weight proportions are found by

$$x_t = (\text{diag}(\hat{\mathbf{\Sigma}}_t))^{-1}\mathbf{1}_N \tag{18}$$

where diag(Σ) is a matrix with only the diagonal elements of the sample variance-covariance matrix, and its inverse is the diagonal matrix where all elements are the reciprocals of the corresponding elements in diag(Σ). These weight proportions are normalized to 1 by Equation (1).

5 Methodology

To ensure the comparability of results, we apply methods that closely align with those used by DeMiguel, Garlappi, and Uppal (DGU). This section will first go through our general method for assessing the performance of portfolios before moving on to the robustness tests we implement. For programming purposes and to address numerical problems, we use MatLab. We evaluate the performance of the k portfolios described in Section 4 on each of the datasets mentioned in Table 2 and described in Appendix A.

In brief, we employ a rolling window approach with 120-month estimation windows and monthly rebalancing. Specifically, for each dataset of length T months, we choose an estimation window of length $M = 120$ months. For the beginning of each month t, starting from $t = M + 1$, we use the excess returns from the previous M periods to estimate the parameters needed to implement a particular strategy. These parameters are used to determine the relative weight of each asset in every portfolio. The weights are used to compute the return in month $t + 1$. We then roll the estimation window forward one month by dropping the earliest return and adding the return for the next period in the dataset. This process is repeated until the end of the dataset is reached and will result in a vector of $T - M$ out-of-sample returns for the k portfolios.

From the vector of out-of-sample returns and matrix of portfolio weights, we calculate the performance metrics outlined in Subsection 5.1 and the robustness tests we describe in 5.2. For data management, we store the results from each dataset in mat-files which later are loaded in a separate MatLab program to make LaTeX tables.

5.1 Performance Metrics

We use the same performance metrics as DGU for a more thorough comparison. This means comparing out-of-sample Sharpe Ratio (SR), Certainty Equivalent (CEQ), and turnover.

5.1.1 Sharpe Ratio

The SR is a widely accepted performance measure. One basic assumption in financial theory is that rational investors prefer returns and are averse to risk. Hence, the objective is to maximize the expected return for a given level of risk. The SR allows us to evaluate how well a portfolio achieves this balance.

$$SR = \frac{\mu}{\sigma}, \tag{19}$$

where μ is the excess return and σ is the volatility of returns.

To ensure the SRs of the strategies are statistically different, we compute the p-value of the difference from the equal weight (EW) strategy using the approach suggested by Jobson and Korkie (1981)[7], like DGU.

5.1.2 Certainty Equivalent

The CEQ of a risky choice is the risk-free rate an investor is willing to accept rather than investing in a risky portfolio strategy. The CEQ for strategy k can be defined as follows:

$$\widehat{CEQ_k} = \hat{\mu_k} - \frac{\gamma}{2}\hat{\sigma}_k^2 \tag{20}$$

where $\hat{\mu}$ is the expected return from strategy k, $\hat{\sigma}^2$ is the variance of strategy k, and γ is the risk aversion coefficient of the investor. Both SR and CEQ use returns and risk to measure performance. The difference is that SR uses

[7] See Appendix B.3

the volatility of returns as the risk measure, while CEQ uses the variance. To ascertain whether the CEQs significantly differ across the strategies, we calculate the p-value of the difference[8] from the CEQ of EW.

5.1.3 Turnover and Return-loss

Turnover can indicate the frequency of transactions and potentially associated transaction costs. It is defined as:

$$Turnover = \frac{1}{T-M} \sum_{t=1}^{T-M} \sum_{j=1}^{N} (|\hat{w}_{k,j,t+1} - \hat{w}_{k,j,t^+}|), \tag{21}$$

where $\hat{w}_{k,j,t+1}$ represents the optimal weights in period $t+1$ for strategy k and asset j, and $\hat{w}_{k,j,t^+}$ are the optimal weights in period t multiplied with the cumulative asset returns from t to $t+1$ making them the weights before rebalancing. By dividing by the number of subsamples $T-M$, we get the average sum of all the absolute deviations in weights over the time period. Understanding the turnover of the strategies is of great importance, as transaction costs can significantly erode the profits from an investment strategy.

We report the absolute turnover for the EW portfolio, our benchmark portfolio. We report turnover relative to that of the EW for the other strategies. Furthermore, we consider how turnover affects the performance of the strategies by calculating the return loss with respect to the EW strategy by the same methodology as DGU.

By DGU's assumptions, we set the proportional transactional cost, c, to 50 basis points. The return net of transaction costs can be calculated as follows:

$$\text{Net return} = (1 + R_{k,p})(1 - c \cdot \sum_{j=1}^{N} |\hat{w}_{k,j,t+1} - \hat{w}_{k,j,t^+}|), \tag{22}$$

[8]See Appendix B.4

where $R_{k,p}$ is the return before rebalancing:

$$R_{k,p} = \sum_{j=1}^{N} R_{j,t+1}\hat{w}_{k,j,t} \tag{23}$$

The return loss for strategy k compared to the EW is then calculated as:

$$\text{return-loss}_k = \frac{\mu_{ew}}{\sigma_{ew}} \cdot \sigma_k - \mu_k, \tag{24}$$

where μ_{ew} and σ_{ew} are the monthly out-of-sample mean and volatility of the net returns from EW, and μ_k and σ_k are the corresponding quantities for strategy k. The return-loss can be interpreted as the additional return needed for a strategy k to perform as well as the EW in terms of SR.

It is important to note that we only report turnover to indicate what it would cost to implement a particular strategy. In the presence of actual transaction costs, the optimization problems would be altered.

5.2 Robustness tests

Like DGU, our benchmark case uses a 120-month estimation window, invests in risky assets only, rebalances every month, and compares turnover to the EW portfolio. We relax these assumptions one by one to see how it affects the performance of the portfolios. The rest of this section will explain what we expect from these robustness tests and how we implement them. The complete overview of the results is in Appendix C.

5.2.1 Different length of estimation window

In addition to 120-month rolling estimation windows, we test 60-month rolling estimation windows and increasing estimation windows. DGU finds that longer estimation windows are necessary for the mean-variance strategy to perform well. This makes sense; assuming there is a true mean, variance, and covariance for assets, the more data you have on returns, the closer you are to these "true" parameters. As mentioned in Section 3, Kan and Zhou (2007) finds that shorter estimation windows and more assets lead to more considerable losses because of estimation errors for the mean-variance strategy. They attribute losses to estimation errors in the mean and the variance-covariance matrix and find that the error in means contributes considerably more. The interaction term from when both mean and variance are unknown is even more significant.

Based on their findings, we suspect a shorter estimation window considerably affects the mean-variance strategies' performance. In contrast, the strategies that rely on no information (EW) or solely the variance-covariance matrix will be less affected because we make fewer errors in estimating the variance-covariance matrix. To supplement this robustness test, we look at the variation in sample means and the sample volatilities over time in Section 6. If the variation in sample means is greater, it suggests that portfolio strategies sensitive to the mean are more susceptible to changes in composition and overall errors.

If a shorter estimation window does not affect performance in portfolios that rely on the estimation of the variance-covariance matrix, it could give some pointers on investing if we have assets with a short history.

60-month rolling estimation windows

To incorporate 60-month rolling estimation windows, we conduct all analyses using the same methodology as the 120-month rolling window approach. To ensure investment horizons for both approaches are equally long, we adjust the starting point for investing by shifting it forward by 60 months. As $M = 60$, the number of subsamples is now $T - (M + 60)$.

Increasing windows instead of rolling windows

The difference from the 120-month rolling estimation windows is that we do **not** drop the earliest return when rolling the window forward. The estimation window increases with one month for each period. Even though a longer estimation window should affect the strategies positively, we expect this effect to be minimal, as the results from Kan and Zhou (2007) show only a slight decrease in loss to estimation error for large increases in estimation windows.

5.2.2 Different holding period

We test a holding period of 12 months to exploit possible short-term trending patterns in returns. Moment estimation is based on ten years of monthly return data. When we roll forward one month, we drop the earliest return and introduce one new return. It's unlikely that this new return will significantly affect the sample mean and variance-covariance matrix. However, we must remember that even minor adjustments in expected returns can drastically impact the optimal weights in the unconstrained mean-variance approach. The optimal weights based on sample moments will not change significantly for most other strategies.

After a period of negative (positive) return for an asset, the relative weight of this asset compared to the other assets in the portfolio will have decreased (increased). Because optimal portfolio weights do not change as much, we will buy more of the assets with negative returns and sell the assets with positive returns to get back to the optimal weights. We effectively buy "losing" assets and sell "winning" assets in terms of return. Rebalancing is a good strategy if the returns show mean-reverting properties, that a positive (negative) return is followed by a subsequent negative (positive) return. If, however, returns have trending properties - a positive (negative) return is followed by a subsequent positive (negative) return - we should hold the assets.

The optimal frequency of rebalancing is thus dependent on whether returns show trending or mean-reverting patterns. Ilmanen and Maloney (2015) find that a holding period of 12 months or 24 months seems to get the "best of both worlds." In their data, returns show a trending pattern over 3-12 months and a mean-reverting pattern over 3-5 years. We, therefore, test a strategy where we rebalance every 12 months rather than every month.

The one-lag auto-correlation of returns may indicate what we can expect from a differing holding period. In Section 6, we calculate the auto-correlation coefficients for 1-month and compounded 12-month returns and report their mean, max, and min values for each dataset. Positive auto-correlation implies trending properties and that holding is more beneficial, whereas negative auto-correlation implies mean-reversion and that we should rebalance. One limitation of this approach is that the optimal weights may change more over a year than one month, and thus we may not "rebalance" as much as we optimally could for all strategies.

To incorporate a 12-month holding period, we create a matrix of optimal weights where each entry corresponds to every twelfth entry in the matrix of optimal weights in the benchmark case. We also compound returns to have a series of annual returns. The SR for this robustness test is annual. To compare with the benchmark case, we must annualize the monthly SRs.

5.2.3 Including risk-free asset

Market timing becomes an aspect when we include a risk-free asset in the investible universe. In periods of high volatility, a mean-variance strategy may exit the market and place more wealth in the risk-free alternative because of risk aversion. The benchmark case evaluates risky assets only to focus on the asset allocation properties of each portfolio.

The minimum variance portfolios are all portfolios on efficient frontiers of risky assets. Unlike trading strategies, they will consist of only risky assets by construction. We have normalized the weights of the IVP and ERC to add up to 1, so to add the risk-free asset in these portfolios, we have to lever the weights. For comparability, we decided to lever the volatility of the portfolios to that of the equal weight strategy. A favorable consequence is that a higher SR for the ERC and IVP portfolios also means higher total returns because the denominators of the SRs are equal.

As we originally calculated the weights of the mean-variance strategy with a risk-free asset, we use these weights for this test. For the equal weight strategy, we add a risk-free asset, and the actual weights in risky assets will instead be:

$$w_t = \frac{1}{N+1}\mathbf{1}_N$$

5.2.4 Benchmark is buy-and-hold equal weight

A reason for an investor to implement the EW strategy is its simplicity. However, this strategy forces the investor to rebalance the portfolio regularly. Thus an even more enticing strategy would be to invest in the EW and never rebalance, a so-called buy-and-hold strategy.

This strategy has more risk than a rebalancing strategy, as assets that perform well over extended periods will constitute a higher fraction of the portfolio, and the eventual diversification benefits received from distributing wealth over many assets diminish[9].

To find the buy-and-hold returns, we first find the evolution of wealth (W) by making a matrix of cumulative returns (CR) for all assets and multiplying it by the vector of base weights:

$$\boldsymbol{W} = (\mathbf{CR})\, x_1 \tag{25}$$

The portfolio return in each period is then found by dividing wealth by its lagged self:

$$R_{t+1} = \frac{W_{t+1}}{W_t} - 1 \tag{26}$$

[9]See examples in Ilmanen and Maloney (2015)

6 Data Characteristics

The data we use in our analysis are presented in Table 2 and more thoroughly explained in Appendix A. We got the data from the webpage of Victor de Miguel, one of the authors of DeMiguel et al. (2009). We have nevertheless listed the sources for completeness. In this section, we present a few characteristics of the monthly return data that may help explain some of the performance for portfolio strategies.

Table 2: Data Overview

#	**Dataset and Source**	**N**	**Time period**	**Abbreviation**
1	Ten sector portfolios of the S&P 500 and the US equity market portfolio Source: Roberto Wessels	10 + 1	01/1981-12/2002	S&P Sectors
2	Ten industry portfolios and the US equity market portfolio Source: Ken French	10 + 1	07/1963-11/2004	Industry
3	Eight country indexes and the World Index Source: MSCI	8 + 1	01/1970-07/2001	International
4	SMB and HML portfolios and the US equity market portfolio Source: Ken French	2 + 1	07/1963-11/2004	MKT/SMB/HML
5	Twenty size- and book-to-market portfolios and the US equity MKT Source: Ken French	20 + 1	07/1963-11/2004	FF 1-factor
6	Twenty size- and book-to-market portfolios Source: Ken French	20 + 4	07/1963-11/2004	FF 4-factor

This table presents an overview of the different datasets used in our analysis, including their original sources. Each dataset contains monthly excess returns over the 90-day nominal US T-bill, sourced from Ken French's website. N denotes the number of risky assets, where the number after the "+" indicates the number of factor portfolios included. For datasets 1-4, the factor portfolio is specified in the table; for dataset 5, the factor portfolio is the US equity market portfolio (MKT); and for dataset 6, the factor portfolios are MKT, SMB, HML, and UMD. The final column shows abbreviated references for the datasets, which are used in the performance evaluation tables. Following the approach of DeMiguel et al. (2009), we have excluded five portfolios containing the largest firms from the 25 size- and book-to-market-sorted portfolios. This is due to the market, SMB, and HML factors nearly forming a linear combination of the 25 Fama-French portfolios.

To understand the characteristics of our data, we performed three tests. First, we examined the one-period autocorrelation of monthly and annual returns to identify any trending or mean-reverting properties. Second, we performed normality tests to emphasize that returns typically do not follow a normal distribution. Lastly, we analyzed the variation in sample means and volatilities to show how portfolio weights based on volatilities would be more stable.

The autocorrelation coefficients are outlined in Table 3. We can not draw definitive conclusions for most datasets. However, for the industries, FF1, and FF4 datasets, the results align with expectations: we observe positive monthly autocorrelation and negative annual autocorrelation. This suggests a trend-like pattern in monthly returns and mean-reverting properties for annual returns.

Table 3: Return Auto-correlation

	S&P Sectors	Industry Portf.	Inter'l Portf.	Mkt/ SMB/HML	FF 1-factor	FF 4-factor
1-month mean	-0.0368	0.0430	0.0183	0.1056	0.1142	0.1107
1-month max	0.1214	0.1217	0.0610	0.1571	0.2180	0.2180
1-month min	-0.1336	-0.0270	-0.0794	0.0396	0.0220	-0.0189
12-month mean	0.0225	-0.1197	0.0307	0.0578	-0.3576	-0.3065
12-month max	0.4301	0.2471	0.3031	0.3516	-0.1573	0.3516
12-month min	-0.5294	-0.3408	-0.3170	-0.1573	-0.4954	-0.4954

This table shows auto-correlation statistics for each dataset used in our study. The "mean" auto-correlation is an arithmetic average of all asset returns' auto-correlations in the dataset. Max shows the highest calculated auto-correlation, while min shows the lowest. "1-month" results are 1-lag auto-correlations on monthly returns, whereas "12-month" are for annual returns.

To test for normality, we employed the Jarque-Bera test and the Kolmogorov-Smirnov test adjusted by Lilliefors[10]. Results are reported in Table 4. Both tests show a clear pattern that returns in our datasets do not follow normal distributions. The Jarque-Bera test offers the most evident results, implying that returns express skewness and excess kurtosis. These results have implications for the effectiveness of the mean-variance strategy.

[10]See appendices B.5 and B.6

Table 4: Tests of normality

	S&P Sectors	Industry Portf.	Inter'l Portf.	Mkt/ SMB/HML	FF 1-factor	FF 4-factor
	Kolmogorov-Smirnov (Lilliefors) test					
Assets reject	4	7	3	3	20	23
Total assets	11	11	9	3	21	24
	Jarque-Bera test					
Assets reject	10	11	9	3	21	24
Total assets	11	11	9	3	21	24

This table shows the results from the Kolmogorov-Smirnov and Jarque-Bera tests on normality for a 5 % significance level. The first row of each test list how many assets reject the null hypothesis that returns come from a family of normal distributions. The second rows list the total number of assets in the dataset.

We estimate a sample mean and variance-covariance matrix for each time t in our rolling window approach. For each asset, we end up with $T - M$ sample means, m_t, and volatilities s_t per asset. Suppose expected returns and volatilities were constant, then $m_t = \bar{m}$ and $s_t = \bar{s}$ for all t. However, this is not the case. Based on Kan and Zhou (2007), we hypothesized that the sample means would show a more significant variation than the sample volatilities. To compare the variation, we express the standard deviation, $\sigma_l \quad l \in (m, s)$, as a fraction of its average observation:

$$variation_m = \frac{\sigma_m}{\overline{m}}$$
$$variation_s = \frac{\sigma_s}{\overline{s}}$$

Results are reported in Table 5. The variation in sample means is greater than the variation in volatility for all datasets. Even the asset with the smallest variability in its sample mean exhibits a variation that is approximately equal to the highest volatility variation found in any other asset. These results imply that portfolios that forego the mean return as parameter input suffer less from variation in optimal portfolio allocation and will have less turnover.

Table 5: Variation in mean and volatility of returns

	S&P Sectors	Industry Portf.	Inter'l Portf.	Mkt/ SMB/HML	FF 1-factor	FF 4-factor
Avg mean variation	25.66 %	71.05 %	68.73 %	124.92 %	72.18 %	80.40 %
Max mean variation	66.88 %	107.89 %	135.10 %	364.46 %	521.94 %	521.94 %
Min mean variation	14.17 %	24.55 %	22.10 %	24.55 %	24.55 %	18.84 %
Avg vol variation	10.27 %	13.91 %	13.36 %	20.59 %	14.48 %	15.14 %
Max vol variation	24.25 %	33.90 %	30.10 %	33.90 %	33.90 %	33.90 %
Min vol variation	3.39 %	7.03 %	5.27 %	9.24 %	8.75 %	8.75 %

This table shows the coefficient of variation for the mean and volatility of asset returns in the datasets we analyzed. The measures are the standard deviation in percentage of their average observation. The "Avg" rows refer to the average variation of all assets, max to the maximum variation of an asset in the dataset, and min to the minimum variation.

Less variation in volatility also implies that estimation error is smaller for the variance-covariance matrix

7 Results and analysis

This section reviews the out-of-sample performance of the portfolios listed in Table 1. First, we report performance metrics results and point to possible explanations for the comparative performance among portfolios. Second, we do a more theoretical analysis of specific portfolio performance before discussing the results from robustness tests.

We find similar out-of-sample performance to DeMiguel, Garlappi, and Uppal (DGU) for the corresponding portfolio strategies, confirming the reliability of our results for the Equal Risk Contribution (ERC) and Inverse Volatility Portfolios (IVP).

7.1 Sharpe Ratio

Table 6 displays the Sharpe Ratios (SR). The results for the strategies tested by DGU align with their findings.

The IVP outperforms the Equal Weight portfolio (EW) in five out of six datasets. However, of these, only three SRs significantly differ from EW. The three remaining SRs are insignificantly different, implying the same performance. The ERC has a significantly higher SR than EW in four out of six datasets at a 10 % level. The difference in SR is insignificant for the other two datasets. DGU concluded that no strategy consistently surpasses EW in terms of SR. We can expand on their conclusion by saying that neither the IVP nor ERC can consistently outperform EW. However, they are promising alternatives to portfolio optimization as their performance is equally good or better than that of EW.

The similar performance may come from structural similarities of the portfolios. The datasets employed contain portfolios of assets rather than single assets. This means that investible assets are already diversified and have little

Table 6: Sharpe Ratios

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.1876	0.1353	0.1277	0.2240	0.1623	0.1753
MV	0.0794	0.0679	-0.0332	0.2186	0.0128	0.1841
	(0.12)	(0.17)	(0.03)	(0.46)	(0.02)	(0.45)
MINV	0.0820	0.1554	0.1490	0.2493	0.2778	-0.0183
	(0.05)	(0.30)	(0.21)	(0.23)	(0.01)	(0.01)
IVP	0.1877	0.1398	0.1323	0.2227	0.1690	0.1895
	(0.50)	(0.04)	(0.16)	(0.46)	(0.00)	(0.00)
ERC	0.1804	0.1414	0.1319	0.2545	0.1689	0.3055
	(0.15)	(0.07)	(0.14)	(0.09)	(0.00)	(0.00)
VW	0.1444	0.1138	0.1239	0.1138	0.1138	0.1138
	(0.09)	(0.01)	(0.43)	(0.00)	(0.01)	(0.00)
MVC	0.0892	0.0678	0.0848	0.1084	0.1977	0.2024
	(0.09)	(0.03)	(0.17)	(0.02)	(0.02)	(0.27)
MINVC	0.0835	0.1425	0.1501	0.2493	0.1546	0.3581
	(0.01)	(0.41)	(0.16)	(0.23)	(0.35)	(0.00)
G-MINVC	0.1371	0.1451	0.1429	0.2468	0.1615	0.3028
	(0.08)	(0.31)	(0.19)	(0.25)	(0.47)	(0.00)

For each strategy, we display the monthly out-of-sample Sharpe ratio across our datasets. In the brackets, we have displayed the p-value, which shows the statistical difference between the Sharpe ratio of the strategy and that of the 1/N strategy.

idiosyncratic risk. An optimal IVP or ERC portfolio would, thus, invest about equally in each asset to ensure the risk contributions were equalized. With individual assets, there will be varying amounts of idiosyncratic risk for each asset. This could lead to IVP underweighting stocks with higher volatility. For EW, however, the weight will remain constant, while the ERC will fall between the two. Although ERC underweights riskier portfolios, it incorporates more than IVP as it also considers the correlation among the assets.

One interesting result that we would like to discuss is the performance of the ERC strategy. Its SR in the FF-4 dataset is 0.3055, almost double the SR in FF-1. The difference between FF-4 and FF-1 is the addition of the zero-investment portfolios SMB, HML, and UMD[11]. Intuitively, these portfolios should not be highly correlated with the market[12]. Low correlation, and especially negative correlation, offers great diversification opportunities for the ERC. This may make the weight distributions considerably different from IVP and EW as opposed to the case where all correlations are positive and similar.

As discussed in section 4, the weight distributions of ERC will be somewhere between the distributions of EW and the constrained minimum variance portfolio (MINVC). The similar performance of ERC and MINVC in FF-4 suggests a similar asset allocation.

7.2 CEQ

The results from our calculations of the Certainty Equivalent (CEQ) confirm much of what we see for the SR in section 7.1. We see from Table 7 that IVP and ERC perform similarly to EW and, on average, perform better than the other portfolios except on a few occasions. The Mean-Variance (MV) strategy performs exceptionally poorly, with often negative CEQ.

7.3 Turnover

From Table 8, we see that turnover for IVP and ERC fluctuates around 1, meaning that the turnover is about as small as for EW. This implies that the optimal weights for the portfolios are stable over time. This is expected from our analysis of the variation in expected means and expected volatilities variation in Section 6. Our analysis is further supported by the observation

[11] These acronyms refer to the Fama-French long-short portfolios Small-Minus-Big (SMB), High-Minus-Low (HML), and Up-Minus-Down (UMD) or momentum

[12] As a control, we calculated the unconditional variance-covariance matrix of returns for the FF-4 dataset and found: (1) the correlations between SMB and assets in a range of $1/2$ and $1/10$ of the correlation among regular assets; (2) the correlation between the factors HML and UMD, and assets to be negative and of absolute size $1/5$ to $1/10$ of the correlations among other assets. In the interest of space, we do not report the variance-covariance matrix as this seems like an isolated event.

Table 7: Certainty Equivalent

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0069	0.0050	0.0046	0.0039	0.0073	0.0072
MV	0.0031	-0.7816	-0.1365	0.0045	-2.7142	-0.0829
	(0.28)	(0.00)	(0.00)	(0.31)	(0.00)	(0.01)
MINV	0.0024	0.0052	0.0054	0.0039	0.0100	-0.0002
	(0.03)	(0.45)	(0.23)	(0.45)	(0.12)	(0.00)
IVP	0.0066	0.0051	0.0047	0.0036	0.0075	0.0071
	(0.18)	(0.23)	(0.28)	(0.08)	(0.08)	(0.42)
ERC	0.0064	0.0051	0.0047	0.0039	0.0075	0.0067
	(0.04)	(0.33)	(0.23)	(0.47)	(0.09)	(0.38)
VW	0.0053	0.0042	0.0044	0.0042	0.0042	0.0042
	(0.05)	(0.00)	(0.43)	(0.00)	(0.46)	(0.01)
MVC	0.0040	0.0023	0.0032	0.0030	0.0090	0.0075
	(0.29)	(0.10)	(0.29)	(0.28)	(0.03)	(0.42)
MINVC	0.0024	0.0047	0.0054	0.0039	0.0060	0.0051
	(0.01)	(0.40)	(0.21)	(0.45)	(0.12)	(0.17)
G-MINVC	0.0044	0.0048	0.0051	0.0038	0.0067	0.0070
	(0.04)	(0.42)	(0.28)	(0.40)	(0.17)	(0.45)

We display the monthly Certainty equivalent return for each strategy across our dataset. In the brackets, we have displayed the p-value, which shows the statistical difference between the certainty equal return of the strategy and that of the 1/N strategy.

that the mean-variance portfolios exhibit significantly higher turnover rates due to their sensitivity to changes in expected mean.

Again, ERC stands out in dataset FF-4. The turnover is comparable to that of the MINVC, further indicating that the weight distribution is dissimilar to EW and IVP and closer to MINVC and G-MINVC.

A consistent theme across all results is that unconstrained portfolios - those permitting short sales - have considerably higher turnover. This outcome, naturally, aligns with the inherent dynamics of such portfolios.

Panel B further expresses the inadequacy of MV as it shows a considerably higher return-loss than other portfolios. The IVP and ERC exhibit mostly

Table 8: Turnover

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0305	0.0216	0.0293	0.0237	0.0162	0.0198
Panel A: Relative turnover of each strategy						
MV	38.99	604206.31	3030.79	2.83	6304.93	2114.12
MINV	6.54	21.65	7.30	1.11	45.47	6.83
IVP	0.98	1.01	0.97	1.00	0.99	1.12
ERC	1.04	1.08	1.03	1.03	0.99	1.64
VW	0.00	0.00	0.00	0.00	0.00	0.00
MVC	4.53	7.17	7.23	4.12	17.53	13.81
MINVC	2.46	2.57	2.27	1.11	3.94	1.81
G-MINVC	1.30	1.52	1.47	1.09	1.77	1.70
Panel B: Return loss relative to 1/N (per month)						
MV	0.0145	231.8909	0.8140	0.0003	6.2283	0.9030
MINV	0.0048	0.0015	0.0000	-0.0004	-0.0008	0.0024
IVP	0.0000	-0.0002	-0.0002	0.0000	-0.0004	-0.0006
ERC	0.0003	-0.0002	-0.0002	-0.0005	-0.0003	-0.0028
VW	0.0017	0.0009	0.0000	0.0048	0.0022	0.0028
MVC	0.0085	0.0048	0.0034	0.0041	-0.0005	0.0002
MINVC	0.0042	-0.0001	-0.0007	-0.0004	0.0006	-0.0025
G-MINVC	0.0019	-0.0003	-0.0006	-0.0003	0.0001	-0.0029

We display the monthly turnover for the 1/N strategy for each strategy across our datasets. In panel A, we show the turnover for each strategy relative to the turnover of equal weight (1/N) in decimals. Panel B reports the return-loss for each strategy. A negative return-loss indicates a higher Sharpe Ratio when incorporating transaction costs.

negative return-loss. This indicates that even when considering transaction costs, the portfolios perform well.

One notable outlier for all performance metrics, and turnover specifically, is the MKT/SMB/HML dataset. With fewer assets, there is naturally less rebalancing and a smaller room for estimation error and, therefore, better relative performance among portfolios and less return-loss for all portfolios. This is especially clear with the mean-variance portfolio.

7.4 Portfolio performance ranking

In Table 9, we summarize the performance results for all portfolios and give them a rank based on how they compare.

IVP and ERC rank consistently among the best portfolios. As turnover is very similar to that of EW, and the CEQ[13] is not statistically different, we deem the results very promising for IVP and ERC as investment strategies. The IVP is even an easy strategy to implement, which could be an argument for its place as a future benchmark for portfolio construction.

As discussed in Section 4, the G-MINVC and ERC are in the same relative space of weight distributions between EW and MINVC. The turnover ranking reflects this: ERC and G-MINVC rank fourth and fifth, respectively, while EW and MINVC rank third and sixth.

7.5 Performance analysis

So far, we have pointed out structural differences among portfolios that may explain the differences in performance. This subsection will offer possible theoretical explanations for the specific portfolios' performance.

For all performance metrics, the MV ranks last. This illustrates that optimal risk ex-ante does not mean optimal risk ex-post and that estimation errors are too substantial for the model to be effective. A second reason for the poor performance is the model's sensitivity to input parameters and the excessive variation in these estimated parameters, as depicted in Section 6. This also leads to excessive turnover, as the optimal weights differ much from period to period. A third reason is that the mean and the variance do not capture all information about returns, as illustrated by the tests on normality in Section 6.

[13] The CEQ would also be different had we chosen a higher risk aversion coefficient than 1

Table 9: Portfolio performance ranking

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Total of Ranks	Final Rank
	Panel A: Rank based on Sharpe ratio							
1/N	2	6	6	5	5	7	31	6
MV	9	8	9	7	9	6	48	9
MINV	8	1	2	3	1	9	24	5
IVP	1	5	4	6	3	5	24	4
ERC	3	4	5	1	4	2	19	1
VW	4	7	7	8	8	8	42	8
MVC	6	9	8	9	2	4	38	7
MINVC	7	3	1	2	7	1	21	2
G-MINVC	5	2	3	4	6	3	23	3
	Panel B: Rank based on Certainty Equivalent							
1/N	1	4	6	3	5	2	21	1
MV	7	9	9	1	9	9	44	9
MINV	9	1	1	6	1	8	26	4
IVP	2	2	5	8	3	3	23	3
ERC	3	3	4	4	4	5	23	2
VW	4	7	7	2	8	7	35	8
MVC	6	8	8	9	2	1	34	6
MINVC	8	6	2	5	7	6	34	7
G-MINVC	5	5	3	7	6	4	30	5
	Panel C: Rank based on Turnover							
1/N	3	2	3	3	4	2	17	3
MV	9	9	9	8	9	9	53	9
MINV	8	8	8	7	8	7	46	8
IVP	2	3	2	2	2	3	14	2
ERC	4	4	4	4	3	4	23	4
VW	1	1	1	1	1	1	6	1
MVC	7	7	7	9	7	8	45	7
MINVC	6	6	6	6	6	6	36	6
G-MINVC	5	5	5	5	5	5	30	5

This table displays each strategy's ranking in terms of Sharpe Ratio, Certainty Equivalent, and turnover. 1 refers to the best rank, while 9 is the worst. We compute a final rank by summing the rank across datasets per strategy.

The risk parity portfolios IVP and ERC perform as well as or better than EW. They perform considerably better than MV, like the other three risk-based portfolios, MINV, MINVC, and G-MINVC. The common feature among these portfolios is that they disregard the expected mean when finding optimal weights and thus are subject to less estimation error, as we describe in Section 3. For the constrained versions, specifically, the low turnover points to more stable weights. This can be attributed to a more constant estimation of risk and the mathematics behind their construction. By constraining the interval of possible asset weights, we effectively use shrinkage on the estimated parameters and move them closer to an "average," as we show in Section 3.1.5.

Interestingly, portfolios that do not prioritize return optimization demonstrate reasonably strong performance in terms of returns. In addition to reduced estimation errors, we see two main explanations for this. First, risk-based portfolios invest more in assets with lower volatility. A reason for the outperformance may thus be the empirical finding that volatility has a weak or even negative risk premium and that assets with low volatilities, therefore, are rewarded with higher returns (Lochstoer and Muir, 2022). Scherer (2011) points out that this "anomaly" might come from exposure to other risk factors not captured by the market.

A second reason for the perceived positive performance is that we compare it to underperforming strategies, such as the MV. Thus, what may appear as a beneficial attribute of one portfolio could merely be highlighting the undesirable characteristics of its counterpart. However, the performance is still commendable when comparing the performance to the "theoretically optimal" market portfolio VW.

The IVP, surprisingly, often outperforms the ERC. It seems counter-intuitive that the potential diversification benefits of considering correlation are not reflected in superior returns for the investor. However, we only get diversification benefits from correlations if they are constant and predictable, a presumption that empirical evidence contradicts. Chua et al. (2009) demonstrate that correlations among assets are elevated during periods of a market downturn. While they were not pioneers in identifying correlation asymmetry, their work extended this finding across a broader range of assets. As a result, this asymmetry in correlation may lead to inaccurate risk contribution. It could result in ex-ante over-weighting of assets with low portfolio correlation, which may ex-post become high risk concentrations.

Despite the common intuition underlying IVP and ERC, they have distinct differences. Across most datasets, IVP's performance aligns closer with EW than ERC does. Given that IVP is the EW portfolio when all assets have the same volatility, the similarities in performance suggest that the volatilities of different assets in the datasets are relatively similar. In contrast, ERC takes pairwise correlation into account, offering a considerable improvement over EW in two of the datasets and a slight outperformance for all but one of the others.

Another interesting observation is the effect datasets have on the performance of portfolios. This is especially apparent with the S&P sectors dataset, where EW has a comparatively greater overperformance than in other datasets. A reason could be that each sector captures specific industry risk factors. An equal weight for each already diversified asset could thus evenly distribute the systematic risk. This notion is further supported by the similar SR of ERC and IVP, suggesting that EW benefits from the strengths of the risk-based portfolio for this dataset.

In concluding this analysis, we turn our attention to the performance of the value-weighted portfolio (VW). It is a standard recommendation for retail investors to allocate their funds across broad index funds. The VW in our analysis is a proxy for a value-weighted market portfolio akin to an index fund. Its performance, however, falls short; even when considering transaction costs, its SR fails to outpace that of the EW. Interestingly for this specific case, the EW portfolio could be interpreted as an investment in multiple index funds, suggesting that retail investors might reap benefits from diversifying across various index funds and rebalancing them regularly.

7.6 Robustness tests

For a thorough comparison, we have reported all the same performance metrics for each robustness test as we did for the benchmark case. We put all tables in Appendix C to conserve space. DGU did the same robustness tests and reported their findings in a separate appendix. Our results closely align with DGU's, with minor differences in relative turnover. The difference does not significantly impact our discussion, as the relative ranking and turnover magnitude are the same.

First, we changed the estimation window from 120 months to 60. This caused the SR for the mean-variance optimized portfolio to fall in all but one dataset. This poorer performance supports the idea that estimation error increases with smaller sample sizes. In contrast, the risk-based portfolios that use the estimated covariance exhibit similar performance to the benchmark scenario, suggesting that loss from estimation errors in returns exceed those in the covariance matrix. As expected, portfolios such as EW and VW remain unchanged, as their construction does not depend on estimation.

Since the CEQ is based on the sample mean and variances when comparing the same risk aversion coefficient, the observable changes in the CEQ are similar to that of the SR. As hypothesized in section 5.2, turnover for the risk-based portfolios, including IVP and ERC, is relatively lower than other portfolios. We attribute this to more stable weights for portfolios based on volatility, as the variation in expected volatility is smaller. It can also be attributed to the characteristic feature of risk-based portfolios that invest in all assets within the investible universe.

Second, we employed an increasing window estimation instead of a rolling window approach. This implementation did not enhance the performance of the mean-variance optimized portfolio noticeably, as expected. The results from Kan and Zhou (2007) show a slight decrease in loss to estimation error for large increases in estimation windows. The risk-based portfolios exhibited similar SR, further strengthening the idea that the covariance matrix is more resilient to estimation error due to a lack of data.

Third, we changed the holding period to 12 months from monthly rebalancing. Asset weights deviate more from the optimal allocation over longer holding periods, causing more unpredictable results. We tested monthly and annual returns for auto-correlation in Section 6. From these results, we could not say how we expected returns to be affected by a longer holding period except for in a few datasets. However, there are no apparent differences when we compare SR from the two cases. For instance, the annual Sharpe Ratio of EW in FF-4 for monthly rebalancing is 0.6073[14], which is marginally better than for the annual rebalancing strategy; opposite to what we expected.

From Table 20, we observe that turnover for the EW portfolio decreases, and relative turnover for most strategies remains stable, implying a lower turnover for them. A notable exception is the MV portfolio, which experiences considerably less turnover than the benchmark case, suggesting a substantial reduction in absolute turnover from the benchmark case. One possible explanation could be the influence of data seasonality on optimal weight. Any seasonal explanation for higher expected returns for an asset will influence the mean-variance optimal weights when rebalancing at the same period each year.

[14]We find the annual Sharpe Ratio by multiplying the monthly Sharpe Ratio by a factor of $\sqrt{12}$. This is based on the assumption that annual returns and variance of returns are annualized by multiplying by 12. $SR_{ann} = \mu_m \cdot 12/\sqrt{\sigma_m^2 \cdot 12} = SR_m \cdot \sqrt{12}$.

There is as good as no change in ranking between the two cases. This strengthens the conclusions we make in the benchmark case.

Fourth, we include a risk-free asset in the investible universe. The performance of minimum variance portfolios is unchanged, as the portfolios do not include the risk-free asset. The performance of the EW portfolio also remains unchanged. The reason is that the relative wealth invested in risky assets stays constant. This effectively multiplies the mean return and standard deviation with the same scalar, keeping the SR constant.

We applied leverage to the ERC and IVP to scale up the volatility to match that of EW. As the weight allocation in these three strategies is similar, the leverage will fluctuate around 1, leading to only a marginal change in SR. For the MV portfolio, the performance improves for all but one dataset, suggesting that mean-variance optimization that allows for leverage can time the market. Yet, this improvement is insufficient to surpass the performance of the risk-based portfolios or EW.

We switched the benchmark portfolio to a buy-and-hold EW strategy for our final robustness test. Here, the SR falls for the diversified equity portfolios while it rises for the factor portfolios. The concentration of assets, and thus risk, will increase over time. There is no apparent reason for the return to be greater; therefore, a smaller Sharpe Ratio is expected. For the factor portfolios, the relative outperformance may be due to some factors that provide higher risk-adjusted returns than others. As such, not rebalancing will lead to more weight allocation in the higher return factors.

Overall, out-of-sample performance and ranking change little when we adjust the assumptions for our investment strategy. This strengthens the explanations and conclusions we draw for the benchmark case.

8 Conclusion

Our thesis aims to assess the effectiveness of the risk parity portfolios Equal Risk Contribution (ERC) and Inverse Volatility Portfolio (IVP). Compared to several other portfolio construction methods over six datasets, the performance in terms of Sharpe Ratio (SR), Certainty Equivalent (CEQ), and turnover has yielded promising results.

Although we have to reach the same conclusion as DeMiguel, Garlappi, and Uppal (2009) (DGU), that no model consistently outperforms the Equal Weight strategy (EW), the potential of ERC and IVP as robust investment options should not be underestimated. After all, the ERC and IVP either outperformed or performed similarly to EW across all datasets. Even when we ease the base study's assumptions through robustness tests, the results do not change considerably. Our findings indicate that risk-based portfolios could serve as viable elements of an investment strategy for institutional investors.

We see three possible explanations for the effectiveness of IVP and ERC. Firstly, IVP and ERC only require the estimation of asset volatilities and the covariance matrix, respectively. By construction, they are long-only portfolios and apply, in effect, shrinkage on the estimated parameters. This makes them less susceptible to estimation errors. Instead of handling estimation error by imposing constraints on the mean-variance efficient portfolio, as DGU, the risk parity portfolios aim to minimize the estimation burden entirely.

Secondly, existing literature suggests that risk-based portfolios benefit from the low-risk anomaly, leading to higher returns that market risk can not explain. Thirdly, the relative out-performance may not be due to return-driven features in risk-based portfolios but rather undesirable features in other portfolios.

It is important to note that there are limitations in our research. Our datasets comprise exclusively value-weighted indices as investible assets. This some-what limits the advantages of portfolios constructed to diversify risk, as asset portfolios are already diversified. The portfolios could have performed even better than EW in a single-asset setting. A more thorough analysis of risk parity portfolios could also employ other risk measures and dynamic asset allocation with forecasting parameters.

APPENDICES

A Appendix: Data

This appendix further explains the data we have used in our research. The authors DeMiguel, Garlappi, and Uppal (2009) have made their data publicly available through their personal websites for replication purposes.

A.1 Sector portfolios

The "S&P Sectors" dataset contains the monthly excess returns on ten value-weighted industry portfolios using the Global Industry Classification Standard (GICS) developed by Standard & Poor's (S&P) and Morgan Stanley Capital International (MSCI). It covers the industries of Energy, Material, Industrials, Consumer-Discretionary, Consumer-Staples, Healthcare, Financials, Information-Technology, Telecommunications, and Utilities, spanning from January 1981 to December 2002. The original authors have expanded on this dataset by introducing the excess return on the US equity market portfolio (MKT) as a factor.

A.2 Industry portfolios

The "Industry" dataset records the monthly excess returns on ten industry portfolios in the United States. It includes Consumer-Discretionary, Consumer Staples, Manufacturing, Energy, High-Tech, Telecommunication, Wholesale and Retail, Health, Utilities, and Others. This dataset ranges from July 1963 to November 2004 and is retrieved from Kenneth French's website. Similarly to the Sectors portfolios dataset, the authors included the excess return on the US equity market portfolio (MKT) for this dataset.

A.3 International equity indexes

The "International" dataset incorporates eight international equity indices: Canada, France, Germany, Italy, Japan, Switzerland, the UK, and the US, as well as the World Index. Returns are calculated from the month-end US-dollar value of the country equity index from January 1970 to July 2001, sourced from MSCI.

A.4 MKT, SMB, and HML portfolios

The "MKT/SMB/HML" dataset is a revised version of the one employed by L'uboš Pástor for evaluating the Bayesian "Data-and-Model" approach to asset allocation. The assets encompass three broad portfolios: MKT, HML, and SMB, with monthly returns from July 1963 to November 2004 obtained from Kenneth French's website.

A.5 Size- and book-to-market-sorted portfolios

This dataset encompasses monthly returns on the 20 portfolios sorted by size and book-to-market, dating from July 1963 to December 2004, and is sourced from Kenneth French's website. We employ this dataset for two distinct experiments denoted as "FF-1-factor" and "FF-4-factor". The first dataset has the market portfolio (MKT) from dataset A.1 included, while the second also includes the SMB, HML, and UMD factors. These are Fama-French risk factors that try to capture risk premia on businesses with small market capitalization (SMB or "small minus big"), high book-to-market value (HML or "high minus low"), and momentum (UMD or "up minus down").

B Appendix: Some implementation details

B.1 Approximation of Certainty Equivalent

The mathematical proof for our approximation of the Certainty Equivalent for a power utility investor follows. We use this to find the mean-variance optimal portfolios. For this section, R denotes the uncertain return, $\bar{R}$ is the expected return, W refers to wealth, CEQ the certainty equivalent, σ is volatility and RRA (or γ) is the "relative risk aversion".

The certainty equivalent of a risky choice is the certain wealth that gives the same utility as the expected utility of the risky choice:

$$u(CE) = E[u(W \cdot R)] \tag{B.1}$$

We do a Taylor expansion around the expected return $\bar{R}$ on the right-hand side:

$$U(W \cdot R) \approx U(W \cdot \bar{R}) + U'(W \cdot \bar{R})W(R - \bar{R}) + \frac{1}{2}U''(W \cdot \bar{R})W^2(R - \bar{R})^2 + \dots \tag{B.2}$$

We take the expectation on both sides:

$$E[U(W \cdot R)] \approx E[U(W \cdot \bar{R}) + U'(W \cdot \bar{R})W(R - \bar{R}) + \frac{1}{2}U''(W \cdot \bar{R})W^2(R - \bar{R})^2 + \dots] \tag{B.3}$$

We switch out the left-hand side by the relation in (B.1) and number the equation and take out all constants from the expected value:

$$U(CE) \approx U(W \cdot \bar{R}) + U'(W \cdot \bar{R})W \cdot E[R - \bar{R}] + \frac{1}{2}U''(W \cdot \bar{R})W^2 \cdot E[R - \bar{R}]^2 + \dots \tag{B.4}$$

Acknowledge that $E[R - \bar{R}] = 0$ and that $E[R - \bar{R}]^2 = \sigma^2$. Next, we do a Taylor expansion on the left-hand side of (B.1):

$$U(CE) \approx U(E[W \cdot R]) + U'(E[W \cdot R])(CE - E(W \cdot R)) \tag{B.5}$$

Equalizing (B.5) and (B.1), we get:

$$
\begin{aligned}
U(E[W \cdot R]) + U'(E[W \cdot R])(CE - E(W \cdot R)) \approx U(W \cdot \bar{R}) + \frac{1}{2}U''(W \cdot \bar{R})W^2\sigma^2 \\
CE \approx W \cdot \bar{R} + \frac{1}{2}\frac{U''(W \cdot \bar{R})}{U'((W \cdot \bar{R})}W^2\sigma^2 \\
CE \approx W[\bar{R} + \frac{1}{2}\frac{U''(W \cdot \bar{R})}{U'((W \cdot \bar{R})}W \cdot \sigma^2] \\
CE \approx W[\bar{R} + \frac{1}{2}(-RRA)\sigma^2]
\end{aligned}
$$

The changes in the last line come from the fact that

$$RRA = -\frac{W \cdot U''(W)}{U'(W)}$$

B.2 Efficient algorithm for the Equal Risk Contribution Portfolio

Chaves et al. (2012)'s first algorithm is based on the Newton method, which is an iterative numerical method used to find the roots of a real-valued function. In our case, we find the roots of the system of nonlinear equations that constitutes the ERC problem:

$$F(y) = F(x, \lambda) = \begin{bmatrix} \Sigma \cdot x - \lambda \cdot \frac{1}{x} \\ \sum_{i=1}^{N} x_i - 1 \end{bmatrix} = \mathbf{0}, \tag{B.6}$$

where the first line corresponds to the N equations in the representation of the ERC portfolio in (12) and the second line is the added constraint that weights sum to 1. This results in an $(N+1)$ vector.

We write a linear approximation of the system around point c by Taylor expansion:

$$F(y) \approx F(c) + J(c) \cdot (y - c), \tag{B.7}$$

where $J(c)$ is the Jacobian matrix of $F(y)$ evaluated at point c. We get an approximation of y by setting $F(y) = 0$ and solving for y:

$$y = c - [J(c)]^{-1} \cdot F(y_{(n)})$$

The process is repeated, but this time with $c = y_{(n)}$:

$$y_{(n+1)} = y_{(n)} - [J(c)]^{-1} \cdot F(y_{(n)})$$

The idea is that the solution converges to y^* through repeated iterations. The Jacobian in (B.7) is a $(N+1) \times (N+1)$ matrix:

$$J(y) = J(x, \lambda) = \begin{bmatrix} \Sigma + \lambda \cdot \text{diag}\left(\frac{1}{x^2}\right) & -\frac{1}{x} \\ \mathbf{1} & 0 \end{bmatrix},$$

where diag $\left(\frac{1}{x^2}\right)$ is a diagonal matrix with elements equal to ${}^1/_{x_i^2}$. To implement the algorithm explained above, we

1. Assume initial weights x_0, a λ_0 between zero and one, and a stopping criterion ε
2. Calculate $F(y)$ and $J(y)$ as shown above
3. If $|y_{(n+1)} - y_{(n)}| < \varepsilon$, $y^* = y_{(n+1)}$ and you stop the process. Otherwise, go back to step 2.

Weights for the ERC portfolio are the N first elements of the y^* vector:

$$w_t = \begin{bmatrix} \mathbf{1}_N \\ 0 \end{bmatrix} \cdot y^* \tag{B.8}$$

B.3 Jobson Korkie z-statistic

The Jobson & Korkie z-statistic is asymptotically distributed as $N(0, 1)$. For two portfolios i and n, with estimated means, variances, and covariances $\hat{\mu}_i$,

$\hat{\mu_n}$, $\hat{\sigma_i}$, $\hat{\sigma_n}$, $\hat{\sigma_{in}}$ over a sample size $T-M$, we test $H_0 : \hat{\mu_i}/\hat{\sigma_i} - \hat{\mu_n}/\hat{\sigma_n} = 0$. The statistic is:

$$\hat{z}_{JK} = \frac{\hat{\sigma}_n \hat{\mu}_i - \hat{\sigma}_i \hat{\mu}_n}{\sqrt{\hat{\vartheta}}} \tag{B.9}$$

with

$$\hat{\vartheta} = \frac{1}{T-M}\left(2\hat{\sigma_i}^2\hat{\sigma_n}^2 - 2\hat{\sigma}_i\hat{\sigma}_n\hat{\sigma}_{in} + \frac{1}{2}\hat{\mu_i}^2\hat{\sigma_n}^2 + \frac{1}{2}\hat{\mu_n}^2\hat{\sigma_i}^2 - \frac{\hat{\mu_i}\hat{\mu_n}}{\hat{\sigma_i}\hat{\sigma_n}}\hat{\sigma}_{in}^2\right)$$

B.4 CEQ z-statistic

$\boldsymbol{v}$ denotes the vector of moments $\boldsymbol{v} = (\mu_i, \mu_n, \sigma_i^2, \sigma_n^2)$, $\hat{\boldsymbol{v}}$ its empirical counterpart obtained from a sample of size $T-M$, and $f(\boldsymbol{v}) = (\mu_i - \gamma^2\sigma_i^2) - (\mu_n - \gamma^2\sigma_n^2)$ is the difference in the certainty equivalent of two strategies i and n. The asymptotic distribution of $f(\boldsymbol{v})$ is:

$$\sqrt{T}(f(\hat{\boldsymbol{v}}) - f(\boldsymbol{v})) \sim N\left(0, \frac{\partial f}{\partial \boldsymbol{v}} \boldsymbol{\Theta} \frac{\partial f}{\partial \boldsymbol{v}}\right), \tag{B.10}$$

in which

$$\boldsymbol{\Theta} = \begin{pmatrix} \sigma_i^2 & \sigma_{i,n} & 0 & 0 \\ \sigma_{i,n} & \sigma_n^2 & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_{i,n}^2 \\ 0 & 0 & 2\sigma_{i,n}^2 & 2\sigma_n^4 \end{pmatrix}. \tag{B.11}$$

We find the z statistic, which is distributed as $N(0,1)$, by dividing by the square root of variance:

$$z = \frac{\sqrt{T}(f(\hat{\boldsymbol{v}}) - f(\boldsymbol{v}))}{\sqrt{\frac{\partial f}{\partial \boldsymbol{v}} \boldsymbol{\Theta} \frac{\partial f}{\partial \boldsymbol{v}}}} \sim N(0,1)$$

B.5 Kolmogorov-Smirnov (Lilliefors) test

The Kolmogorov-Smirnov test compares two cumulative distribution functions (CDF). It quantifies this comparison by calculating the maximum absolute difference between the two CDFs across all points in the series of observations. For an empirical CDF $G_T(x)$ and a theoretical normal distribution $F^*(x)$, the KS-statistic is defined as:

$$KS = \max_{t=1,\ldots,T} |G_T(x) - F^*(x)|$$

As we do not know the true mean and variance, we must employ the Kolmogorov-Smirnov test adjusted by Lilliefors (1967). The steps to performing the Lilliefors test is

1. Estimate the mean $\hat{\mu}$ and the variance $\hat{\sigma}^2$ of the data.

2. Sort the sample data by increasing order and denote the new sample $\{\tilde{r}_t\}_{t=1}^{n}$, with $\tilde{r}_1 \leq \cdots \leq \tilde{r}_n$. Then, by construction, we have $G_T(\tilde{r}_i) = \frac{t}{T}$.

3. Evaluate the assumed theoretical cdf $F^*(\tilde{r}_i)$ for all values $\{\tilde{r}_i\}_{t=1}^{T}$. In the case where the assumed distribution is normal, it is defined as $N(\hat{\mu}, \hat{\sigma}^2)$.

4. Compute the KS_L test statistic: $KS_L = \max_{t=1,\ldots,T} \left|F^*(\tilde{r}_t) - \frac{t}{T}\right|$.

The null hypothesis is that the observations come from a family of normal distributions. The critical value for a 5 % significance level is $0.886/\sqrt{T}$. We reject the null hypothesis if the test statistic is greater than the critical value.

B.6 Jarque-Bera test

The Jarque-Bera test is based on the fact that skewness (S) and kurtosis (K) are jointly zero under normality. The null hypothesis is that $S = 0$ and $K = 3$. Under normality, the sample skewness and kurtosis are mutually independent. The Jarque-Bera test statistic is defined as:

$$JB = T \left[\frac{\hat{S}^2}{6} + \frac{(\hat{K} - 3)^2}{24} \right],$$

6 and 24 come from the asymptotic variances of skewness and kurtosis, respectively. Under the null hypothesis, the statistic is distributed as a χ^2 with 2 degrees of freedom. If $JB \geq \chi^2_{1-\alpha}(2)$, we reject the null hypothesis at level α.

C Appendix: Tables for robustness checks

C.1 Table explanations

In this section we report the results from our robustness tests. As a reminder, we performed the following tests:

1. Changing the estimation period to 60 months instead of 120 months.
2. Changing to increasing estimation window instead of a rolling estimation window.
3. Changed to holding period to 12 month rather than 1 month.
4. Included a risk free asset in the investment universe instead of only having only risky assets.
5. Changed the benchmark to EW-Buy-and-Hold instead of EW with annual rebalancing.

In our report, we present tables featuring the Sharpe ratio, CEQ, turnover, Return Loss, and rankings across five distinct evaluations. The specific experiment under consideration can be discerned from the section header and the identifier listed beneath each table's title. Each identifier consists of various keys, with each key representing a specific choice associated with that particular experiment. The choices and their corresponding keys are as follows:

Choice	Original	Key	Experiment	Key
Estimation window length	120 months	M120	60 months	M60
Estimation window type	Rolling	Rolling	Increasing	Increasing
Holding period	1 month	hp1	12 months	hp12
Investable assets	Risky-only assets	RO1	Riskfree and risky assets	RO0
Benchmark strategy	EW with rebalancing	ewRebal	EW-Buy-and-Hold	ewBuyHold
Risk aversion	$\gamma = 1$	gamma1	Different γ	N/A

The following will be the tables for the robustness test we performed.

C.2 Tables for different estimation window length (M=60)

Table 10: Robustness: Sharpe Ratio, M=60

RO1_M60_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.1876	0.1353	0.1277	0.2240	0.1623	0.1753
MV	0.0070	0.0564	-0.1013	-0.0186	-0.0537	0.0701
	(0.04)	(0.13)	(0.00)	(0.00)	(0.00)	(0.08)
MINV	0.1267	0.1402	0.1316	0.2376	0.2326	-0.0261
	(0.18)	(0.46)	(0.45)	(0.34)	(0.09)	(0.00)
IVP	0.1890	0.1409	0.1317	0.2221	0.1710	0.1917
	(0.43)	(0.03)	(0.23)	(0.45)	(0.00)	(0.00)
ERC	0.1814	0.1426	0.1324	0.2525	0.1710	0.3073
	(0.20)	(0.06)	(0.15)	(0.11)	(0.00)	(0.00)
VW	0.1444	0.1138	0.1239	0.1138	0.1138	0.1138
	(0.09)	(0.01)	(0.43)	(0.00)	(0.01)	(0.00)
MVC	0.0899	0.1016	0.0719	0.1733	0.1896	0.1443
	(0.10)	(0.17)	(0.10)	(0.17)	(0.09)	(0.27)
MINVC	0.1005	0.1441	0.1306	0.2376	0.1541	0.3470
	(0.02)	(0.38)	(0.45)	(0.34)	(0.34)	(0.00)
G-MINVC	0.1447	0.1400	0.1304	0.2422	0.1585	0.3007
	(0.08)	(0.40)	(0.44)	(0.29)	(0.38)	(0.00)

This table shows the Sharpe Ratios for the case where the estimation window is 60 months instead of 120 months.

Table 11: Robustness: Certainty Equivalent, M=60

RO1_M60_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0069	0.0050	0.0046	0.0039	0.0073	0.0072
MV	-0.0101	-1.2805	-0.3236	-0.0171	-0.9358	-0.0440
	(0.08)	(0.00)	(0.00)	(0.01)	(0.00)	(0.01)
MINV	0.0040	0.0048	0.0047	0.0038	0.0089	-0.0003
	(0.14)	(0.46)	(0.47)	(0.37)	(0.26)	(0.00)
IVP	0.0067	0.0051	0.0047	0.0036	0.0076	0.0072
	(0.25)	(0.18)	(0.37)	(0.09)	(0.04)	(0.45)
ERC	0.0064	0.0051	0.0047	0.0039	0.0076	0.0067
	(0.06)	(0.29)	(0.26)	(0.48)	(0.05)	(0.38)
VW	0.0053	0.0042	0.0044	0.0042	0.0042	0.0042
	(0.05)	(0.00)	(0.43)	(0.00)	(0.46)	(0.01)
MVC	0.0038	0.0043	0.0025	0.0054	0.0088	0.0058
	(0.26)	(0.37)	(0.19)	(0.18)	(0.08)	(0.28)
MINVC	0.0031	0.0047	0.0046	0.0038	0.0061	0.0049
	(0.01)	(0.40)	(0.48)	(0.37)	(0.14)	(0.15)
G-MINVC	0.0047	0.0046	0.0045	0.0038	0.0066	0.0068
	(0.04)	(0.33)	(0.44)	(0.38)	(0.16)	(0.41)

This table shows the Certainty Equivalent for the case where the estimation window is 60 months instead of 120 months.

Table 12: Robustness: Turnover and return-loss, M=60

RO1_TC50_M60_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0305	0.0216	0.0293	0.0237	0.0162	0.0198
Panel A: Relative turnover of each strategy						
MV	181.72	54545.11	9428.42	60.20	4036.97	2348.44
MINV	11.41	50.74	17.14	1.46	99.80	11.02
IVP	1.03	1.07	1.04	1.07	1.05	1.20
ERC	1.15	1.21	1.14	1.13	1.06	1.95
VW	0.00	0.00	0.00	0.00	0.00	0.00
MVC	5.13	15.52	9.96	2.96	16.81	14.37
MINVC	3.82	4.27	3.91	1.46	8.04	2.49
G-MINVC	2.11	2.22	2.25	1.30	3.65	1.98
Panel B: Return loss relative to 1/N (per month)						
MV	0.0513	21.4113	3.2128	0.0213	2.7206	0.5503
MINV	0.0039	0.0053	0.0022	-0.0002	0.0052	0.0031
IVP	-0.0000	-0.0002	-0.0002	0.0001	-0.0005	-0.0007
ERC	0.0003	-0.0003	-0.0002	-0.0004	-0.0005	-0.0028
VW	0.0017	0.0009	0.0000	0.0048	0.0022	0.0028
MVC	0.0073	0.0036	0.0045	0.0019	-0.0002	0.0028
MINVC	0.0037	0.0000	0.0003	-0.0002	0.0010	-0.0023
G-MINVC	0.0018	-0.0000	0.0001	-0.0002	0.0004	-0.0028

This table shows the turnover and return-loss for the case where the estimation window is 60 months instead of 120 months.

Table 13: Robustness: Ranking, M=60

RO1_M60_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Total of Ranks	Final Rank
Panel A: Rank based on Sharpe ratio								
1/N	2	6	6	5	5	5	29	6
MV	9	9	9	9	9	8	53	9
MINV	6	4	3	4	1	9	27	5
IVP	1	3	2	6	4	4	20	2
ERC	3	2	1	1	3	2	12	1
VW	5	7	7	8	8	7	42	8
MVC	8	8	8	7	2	6	39	7
MINVC	7	1	4	3	7	1	23	3
G-MINVC	4	5	5	2	6	3	25	4
Panel B: Rank based on Certainty Equivalent								
1/N	1	3	4	3	5	2	18	3
MV	9	9	9	9	9	9	54	9
MINV	6	4	2	7	1	8	28	4
IVP	2	1	3	8	3	1	18	2
ERC	3	2	1	4	4	4	18	1
VW	4	8	7	2	8	7	36	7
MVC	7	7	8	1	2	5	30	5
MINVC	8	5	5	6	7	6	37	8
G-MINVC	5	6	6	5	6	3	31	6
Panel C: Rank based on Turnover								
1/N	2	2	2	2	2	2	12	2
MV	9	9	9	9	9	9	54	9
MINV	8	8	8	7	8	7	46	8
IVP	3	3	3	3	3	3	18	3
ERC	4	4	4	4	4	4	24	4
VW	1	1	1	1	1	1	6	1
MVC	7	7	7	8	7	8	44	7
MINVC	6	6	6	6	6	6	36	6
G-MINVC	5	5	5	5	5	5	30	5

This table shows the ranking of portfolios for the case where the estimation window is 60 months instead of 120 months.

C.3 Tables for increasing estimation window length

Table 14: Robustness: Sharpe Ratio, Increasing Window

RO1_M120_gamma1_hp1_Increasing_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.1876	0.1353	0.1277	0.2240	0.1623	0.1753
MV	0.0451	-0.0308	-0.0970	0.2507	0.0892	0.0824
	(0.07)	(0.01)	(0.00)	(0.30)	(0.15)	(0.11)
MINV	0.0763	0.1538	0.1460	0.2669	0.2294	-0.0534
	(0.05)	(0.30)	(0.21)	(0.14)	(0.08)	(0.00)
IVP	0.1876	0.1382	0.1303	0.2404	0.1670	0.1909
	(0.50)	(0.04)	(0.25)	(0.16)	(0.01)	(0.00)
ERC	0.1827	0.1395	0.1301	0.2679	0.1666	0.3132
	(0.25)	(0.06)	(0.22)	(0.04)	(0.01)	(0.00)
VW	0.1444	0.1138	0.1239	0.1138	0.1138	0.1138
	(0.09)	(0.01)	(0.43)	(0.00)	(0.01)	(0.00)
MVC	0.2221	0.0831	0.0693	0.1820	0.2107	0.2364
	(0.31)	(0.06)	(0.12)	(0.25)	(0.00)	(0.07)
MINVC	0.1144	0.1488	0.1500	0.2669	0.1299	0.3602
	(0.08)	(0.30)	(0.12)	(0.14)	(0.05)	(0.00)
G-MINVC	0.1374	0.1466	0.1477	0.2649	0.1488	0.3120
	(0.11)	(0.26)	(0.09)	(0.15)	(0.08)	(0.00)

This table shows the Sharpe Ratios for the case where the estimation window is increasing instead of rolling.

Table 15: Robustness: Certainty Equivalent, Increasing Window

RO1_M120_gamma1_hp1_Increasing_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0069	0.0050	0.0046	0.0039	0.0073	0.0072
MV	0.0009	-0.1654	-1.7130	0.0044	-0.0058	-0.7021
	(0.12)	(0.00)	(0.00)	(0.30)	(0.14)	(0.00)
MINV	0.0022	0.0052	0.0052	0.0042	0.0080	-0.0006
	(0.04)	(0.45)	(0.27)	(0.35)	(0.38)	(0.00)
IVP	0.0067	0.0050	0.0047	0.0038	0.0074	0.0071
	(0.20)	(0.27)	(0.40)	(0.31)	(0.15)	(0.40)
ERC	0.0065	0.0050	0.0047	0.0041	0.0074	0.0069
	(0.08)	(0.34)	(0.34)	(0.35)	(0.20)	(0.43)
VW	0.0053	0.0042	0.0044	0.0042	0.0042	0.0042
	(0.05)	(0.00)	(0.43)	(0.00)	(0.46)	(0.01)
MVC	0.0086	0.0030	0.0024	0.0039	0.0103	0.0086
	(0.28)	(0.12)	(0.23)	(0.48)	(0.00)	(0.22)
MINVC	0.0037	0.0050	0.0053	0.0042	0.0049	0.0053
	(0.06)	(0.49)	(0.19)	(0.35)	(0.01)	(0.20)
G-MINVC	0.0046	0.0050	0.0052	0.0041	0.0061	0.0071
	(0.08)	(0.49)	(0.16)	(0.38)	(0.01)	(0.48)

This table shows the Certainty Equivalent for the case where the estimation window is increasing instead of rolling.

Table 16: Robustness: Turnover and return-loss, Increasing Window

RO1_TC50_M120_gamma1_hp1_Increasing_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0305	0.0216	0.0293	0.0237	0.0162	0.0198
Panel A: Relative turnover of each strategy						
MV	17.82	2920.31	9882.42	1.24	1159.76	891.19
MINV	4.33	8.35	2.46	0.95	18.96	4.70
IVP	0.96	1.00	0.95	0.98	0.97	1.11
ERC	1.00	1.04	1.00	0.98	0.97	1.48
VW	0.00	0.00	0.00	0.00	0.00	0.00
MVC	0.27	5.37	1.06	2.69	6.19	4.82
MINVC	1.35	1.44	1.14	0.95	0.63	1.36
G-MINVC	0.98	1.29	1.02	0.93	0.89	1.51
Panel B: Return loss relative to 1/N (per month)						
MV	0.0114	2.0272	6.7960	-0.0005	0.1088	1.8186
MINV	0.0049	0.0001	-0.0005	-0.0007	-0.0010	0.0028
IVP	0.0000	-0.0001	-0.0001	-0.0003	-0.0002	-0.0006
ERC	0.0002	-0.0002	-0.0001	-0.0007	-0.0002	-0.0030
VW	0.0017	0.0009	0.0000	0.0048	0.0022	0.0028
MVC	-0.0016	0.0033	0.0038	0.0011	-0.0023	-0.0020
MINVC	0.0029	-0.0005	-0.0009	-0.0007	0.0015	-0.0027
G-MINVC	0.0020	-0.0004	-0.0008	-0.0007	0.0007	-0.0031

This table shows the turnover and return-loss for the case where the estimation window is increasing instead of rolling.

Table 17: Robustness: Ranking, Increasing Window

RO1_M120_gamma1_hp1_Increasing_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Total of Ranks	Final Rank
Panel A: Rank based on Sharpe ratio								
1/N	3	6	6	7	5	6	33	7
MV	9	9	9	5	9	8	49	9
MINV	8	1	3	3	1	9	25	5
IVP	2	5	4	6	3	5	25	4
ERC	4	4	5	1	4	2	20	1
VW	5	7	7	9	8	7	43	8
MVC	1	8	8	8	2	4	31	6
MINVC	7	2	1	2	7	1	20	2
G-MINVC	6	3	2	4	6	3	24	3
Panel B: Rank based on Certainty Equivalent								
1/N	2	4	6	7	5	2	26	3
MV	9	9	9	1	9	9	46	9
MINV	8	1	3	4	2	8	26	2
IVP	3	3	5	9	3	4	27	5
ERC	4	2	4	6	4	5	25	1
VW	5	7	7	2	8	7	36	8
MVC	1	8	8	8	1	1	27	4
MINVC	7	6	1	3	7	6	30	7
G-MINVC	6	5	2	5	6	3	27	6
Panel C: Rank based on Turnover								
1/N	6	3	4	7	6	2	28	5
MV	9	9	9	8	9	9	53	9
MINV	8	8	8	4	8	7	43	8
IVP	3	2	2	5	4	3	19	2
ERC	5	4	3	6	5	5	28	4
VW	1	1	1	1	1	1	6	1
MVC	2	7	6	9	7	8	39	7
MINVC	7	6	7	3	2	4	29	6
G-MINVC	4	5	5	2	3	6	25	3

This table shows the ranking of portfolios for the case where the estimation window is increasing instead of rolling.

C.4 Tables for different holding period (12 months)

Table 18: Robustness: Sharpe Ratio, 12M Holding Period

RO1_M120_gamma1_hp12_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.7427	0.4522	0.3649	0.7742	0.5641	0.6067
MV	0.3527	-0.0546	-0.1980	0.8632	0.4212	0.6366
	(0.09)	(0.02)	(0.05)	(0.36)	(0.28)	(0.45)
MINV	0.2699	0.5059	0.4835	0.7205	0.7061	-0.0342
	(0.01)	(0.36)	(0.02)	(0.36)	(0.24)	(0.02)
IVP	0.7170	0.4643	0.3860	0.7124	0.5832	0.6566
	(0.19)	(0.14)	(0.05)	(0.19)	(0.04)	(0.00)
ERC	0.6864	0.4702	0.3833	0.7622	0.5829	0.9575
	(0.04)	(0.10)	(0.02)	(0.45)	(0.04)	(0.00)
VW	0.4423	0.3864	0.4117	0.3864	0.3864	0.3864
	(0.02)	(0.03)	(0.30)	(0.03)	(0.05)	(0.03)
MVC	0.2253	0.2218	0.2891	0.3666	0.6232	0.8983
	(0.02)	(0.06)	(0.31)	(0.07)	(0.22)	(0.04)
MINVC	0.3109	0.4582	0.4770	0.7205	0.5324	1.0842
	(0.01)	(0.48)	(0.02)	(0.36)	(0.36)	(0.03)
G-MINVC	0.4677	0.4692	0.4496	0.7156	0.5530	0.9582
	(0.05)	(0.39)	(0.03)	(0.35)	(0.40)	(0.01)

This table shows the Sharpe Ratios for the case where the holding period is 12 months instead of 1 month.

Table 19: Robustness: Certainty Equivalent, 12M Holding Period

RO1_M120_gamma1_hp12_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0907	0.0618	0.0539	0.0503	0.0902	0.0889
MV	0.0483	-0.3712	-1.1184	0.0602	-1.7714	0.0803
	(0.17)	(0.00)	(0.00)	(0.25)	(0.01)	(0.49)
MINV	0.0259	0.0581	0.0738	0.0507	0.1242	-0.0017
	(0.00)	(0.43)	(0.03)	(0.48)	(0.17)	(0.00)
IVP	0.0872	0.0627	0.0565	0.0456	0.0926	0.0887
	(0.13)	(0.32)	(0.20)	(0.11)	(0.11)	(0.48)
ERC	0.0838	0.0629	0.0568	0.0507	0.0925	0.0854
	(0.02)	(0.33)	(0.08)	(0.47)	(0.12)	(0.42)
VW	0.0643	0.0497	0.0516	0.0497	0.0497	0.0497
	(0.02)	(0.31)	(0.03)	(0.01)	(0.42)	(0.20)
MVC	0.0249	0.0242	0.0416	0.0387	0.1153	0.1141
	(0.11)	(0.12)	(0.37)	(0.33)	(0.05)	(0.13)
MINVC	0.0356	0.0561	0.0713	0.0507	0.0753	0.0665
	(0.01)	(0.35)	(0.04)	(0.48)	(0.15)	(0.22)
G-MINVC	0.0595	0.0600	0.0652	0.0501	0.0824	0.0880
	(0.06)	(0.43)	(0.12)	(0.49)	(0.16)	(0.48)

This table shows the Certainty Equivalent for the case where the holding period is 12 months instead of 1 month.

Table 20: Robustness: Turnover and return-loss, 12M Holding Period

RO1_TC50_M120_gamma1_hp12_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0091	0.0066	0.0080	0.0072	0.0053	0.0061
Panel A: Relative turnover of each strategy						
MV	30.67	1692.32	1182.71	2.85	3292.65	1043.90
MINV	7.61	26.42	8.68	1.15	45.83	7.52
IVP	0.97	0.98	1.03	1.06	0.96	1.09
ERC	1.09	1.04	1.10	1.05	0.97	1.66
VW	0.00	0.00	0.00	0.00	0.00	0.00
MVC	4.76	8.02	7.40	4.32	12.06	8.82
MINVC	2.65	3.08	2.61	1.15	4.49	1.84
G-MINVC	1.41	2.01	1.69	1.11	2.15	1.77
Panel B: Return loss relative to 1/N (per month)						
MV	0.0889	1.2293	1.7352	-0.0059	4.3653	6.2800
MINV	0.0627	0.0032	-0.0188	0.0039	-0.0147	0.0239
IVP	0.0034	-0.0020	-0.0041	0.0041	-0.0036	-0.0076
ERC	0.0077	-0.0029	-0.0036	0.0008	-0.0036	-0.0326
VW	0.0542	0.0104	-0.0075	0.0624	0.0287	0.0356
MVC	0.1334	0.0594	0.0231	0.0538	-0.0098	-0.0371
MINVC	0.0663	-0.0000	-0.0200	0.0039	0.0065	-0.0297
G-MINVC	0.0420	-0.0022	-0.0150	0.0042	0.0024	-0.0336

This table shows the turnover and return-loss for the case where the holding period is 12 months instead of 1 month.

Table 21: Robustness: Ranking, 12M Holding Period

RO1_M120_gamma1_hp12_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Total of Ranks	Final Rank
Panel A: Rank based on Sharpe ratio								
1/N	1	6	7	2	5	7	28	6
MV	6	9	9	1	8	6	39	7
MINV	8	1	1	5	1	9	25	3
IVP	2	4	5	7	3	5	26	4
ERC	3	2	6	3	4	3	21	1
VW	5	7	4	8	9	8	41	9
MVC	9	8	8	9	2	4	40	8
MINVC	7	5	2	4	7	1	26	5
G-MINVC	4	3	3	6	6	2	24	2
Panel B: Rank based on Certainty Equivalent								
1/N	1	3	6	5	5	2	22	2
MV	6	9	9	1	9	6	40	8
MINV	8	5	1	4	1	9	28	5
IVP	2	2	5	8	3	3	23	3
ERC	3	1	4	2	4	5	19	1
VW	4	7	7	7	8	8	41	9
MVC	9	8	8	9	2	1	37	7
MINVC	7	6	2	3	7	7	32	6
G-MINVC	5	4	3	6	6	4	28	4
Panel C: Rank based on Turnover								
1/N	3	3	2	2	4	2	16	3
MV	9	9	9	8	9	9	53	9
MINV	8	8	8	7	8	7	46	8
IVP	2	2	3	4	2	3	16	2
ERC	4	4	4	3	3	4	22	4
VW	1	1	1	1	1	1	6	1
MVC	7	7	7	9	7	8	45	7
MINVC	6	6	6	6	6	6	36	6
G-MINVC	5	5	5	5	5	5	30	5

This table shows the ranking of portfolios for the case where the holding period is 12 months instead of 1 month.

C.5 Tables for portfolios that include risk-free asset

Table 22: Robustness: Sharpe Ratio, Including Riskfree Asset

RO0_M120_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.1876	0.1353	0.1277	0.2240	0.1623	0.1753
MV	0.0979	0.0274	-0.0027	0.1966	0.3678	0.4058
	(0.16)	(0.04)	(0.02)	(0.30)	(0.00)	(0.00)
MINV	0.0820	0.1554	0.1490	0.2493	0.2778	-0.0183
	(0.05)	(0.30)	(0.21)	(0.23)	(0.01)	(0.01)
IVP	0.1877	0.1405	0.1328	0.2216	0.1680	0.1899
	(0.50)	(0.02)	(0.15)	(0.44)	(0.01)	(0.00)
ERC	0.1799	0.1420	0.1322	0.2527	0.1678	0.3011
	(0.14)	(0.06)	(0.13)	(0.10)	(0.02)	(0.00)
VW	0.1444	0.1138	0.1239	0.1138	0.1138	0.1138
	(0.09)	(0.01)	(0.43)	(0.00)	(0.01)	(0.00)
MVC	0.0892	0.0628	0.0760	0.1084	0.1976	0.2024
	(0.09)	(0.02)	(0.13)	(0.02)	(0.02)	(0.27)
MINVC	0.0835	0.1425	0.1501	0.2493	0.1546	0.3581
	(0.01)	(0.41)	(0.16)	(0.23)	(0.35)	(0.00)
G-MINVC	0.1371	0.1451	0.1429	0.2468	0.1615	0.3028
	(0.08)	(0.31)	(0.19)	(0.25)	(0.47)	(0.00)

This table shows the Sharpe Ratios for the case where we include a risk-free asset as an investable asset.

Table 23: Robustness: Certainty Equivalent, Including Riskfree Asset

RO0_M120_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0064	0.0046	0.0042	0.0030	0.0070	0.0069
MV	-0.0837	-0.0760	-0.0384	0.0134	-0.1026	-0.1133
	(0.02)	(0.00)	(0.00)	(0.25)	(0.03)	(0.03)
MINV	0.0024	0.0052	0.0054	0.0039	0.0100	-0.0002
	(0.04)	(0.36)	(0.13)	(0.04)	(0.09)	(0.00)
IVP	0.0064	0.0048	0.0045	0.0030	0.0073	0.0076
	(0.40)	(0.02)	(0.09)	(0.48)	(0.01)	(0.00)
ERC	0.0062	0.0049	0.0045	0.0035	0.0073	0.0138
	(0.18)	(0.05)	(0.06)	(0.06)	(0.01)	(0.00)
VW	0.0053	0.0042	0.0044	0.0042	0.0042	0.0042
	(0.02)	(0.00)	(0.14)	(0.00)	(0.26)	(0.00)
MVC	0.0040	0.0020	0.0027	0.0030	0.0089	0.0075
	(0.33)	(0.10)	(0.27)	(0.49)	(0.02)	(0.37)
MINVC	0.0024	0.0047	0.0054	0.0039	0.0060	0.0051
	(0.01)	(0.50)	(0.11)	(0.04)	(0.16)	(0.19)
G-MINVC	0.0044	0.0048	0.0051	0.0038	0.0067	0.0070
	(0.06)	(0.42)	(0.13)	(0.05)	(0.28)	(0.48)

This table shows the Certainty Equivalent for the case where we include a risk-free asset as an investable asset.

Table 24: Robustness: Turnover and return-loss, Including Riskfree Asset

RO0_TC50_M120_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0305	0.0227	0.0295	0.0205	0.0173	0.0206
Panel A: Relative turnover of each strategy						
MV	1472.97	8567.69	749.29	1031.01	24280.69	13769.81
MINV	6.54	20.59	7.26	1.28	42.71	6.57
IVP	0.99	1.01	0.99	1.16	0.98	1.29
ERC	1.07	1.08	1.05	1.29	0.99	6.55
VW	0.00	0.00	0.00	0.00	0.00	0.00
MVC	4.52	7.30	7.26	4.76	16.63	13.28
MINVC	2.46	2.45	2.26	1.28	3.70	1.74
G-MINVC	1.29	1.44	1.46	1.26	1.67	1.63
Panel B: Return loss relative to 1/N (per month)						
MV	0.1390	0.4848	0.1056	0.0398	0.7659	0.9020
MINV	0.0048	0.0015	0.0000	-0.0004	-0.0008	0.0024
IVP	-0.0000	-0.0002	-0.0002	0.0000	-0.0003	-0.0006
ERC	0.0003	-0.0003	-0.0002	-0.0004	-0.0003	-0.0057
VW	0.0017	0.0009	-0.0000	0.0048	0.0022	0.0028
MVC	0.0084	0.0051	0.0038	0.0041	-0.0005	0.0002
MINVC	0.0041	-0.0001	-0.0008	-0.0004	0.0006	-0.0025
G-MINVC	0.0019	-0.0003	-0.0006	-0.0004	0.0001	-0.0029

This table shows the turnover and return-loss for the case where we include a risk-free asset as an investable asset.

Table 25: Robustness: Ranking, Including Riskfree Asset

RO0_M120_gamma1_hp1_Rolling_ewRebal.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Total of Ranks	Final Rank
Panel A: Rank based on Sharpe ratio								
1/N	2	6	6	5	6	7	32	6
MV	6	9	9	7	1	1	33	7
MINV	9	1	2	3	2	9	26	5
IVP	1	5	4	6	4	6	26	4
ERC	3	4	5	1	5	4	22	1
VW	4	7	7	8	9	8	43	9
MVC	7	8	8	9	3	5	40	8
MINVC	8	3	1	2	8	2	24	3
G-MINVC	5	2	3	4	7	3	24	2
Panel B: Rank based on Certainty Equivalent								
1/N	1	6	7	9	5	5	33	6
MV	9	9	9	1	9	9	46	9
MINV	8	1	1	4	1	8	23	3
IVP	2	3	4	8	3	2	22	2
ERC	3	2	5	6	4	1	21	1
VW	4	7	6	2	8	7	34	8
MVC	6	8	8	7	2	3	34	7
MINVC	7	5	2	3	7	6	30	5
G-MINVC	5	4	3	5	6	4	27	4
Panel C: Rank based on Turnover								
1/N	3	2	3	2	4	2	16	3
MV	9	9	9	9	9	9	54	9
MINV	8	8	8	6	8	7	45	8
IVP	2	3	2	3	2	3	15	2
ERC	4	4	4	7	3	6	28	4
VW	1	1	1	1	1	1	6	1
MVC	7	7	7	8	7	8	44	7
MINVC	6	6	6	5	6	5	34	6
G-MINVC	5	5	5	4	5	4	28	5

This table shows the ranking of portfolios for the case where we include a risk-free asset as an investable asset.

C.6 Tables for portfolios where the benchmark is equal-weight buy-and-hold

Table 26: Robustness: Sharpe Ratio, EW Buy-and-Hold

RO1_M120_gamma1_hp1_Rolling_ewBuyHold.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.1725	0.1289	0.1129	0.2271	0.1825	0.1934
MV	0.0794	0.0679	-0.0332	0.2186	0.0128	0.1841
	(0.12)	(0.17)	(0.03)	(0.46)	(0.02)	(0.45)
MINV	0.0820	0.1554	0.1490	0.2493	0.2778	-0.0183
	(0.05)	(0.30)	(0.21)	(0.23)	(0.01)	(0.01)
IVP	0.1877	0.1398	0.1323	0.2227	0.1690	0.1895
	(0.50)	(0.04)	(0.16)	(0.46)	(0.00)	(0.00)
ERC	0.1804	0.1414	0.1319	0.2545	0.1689	0.3055
	(0.15)	(0.07)	(0.14)	(0.09)	(0.00)	(0.00)
VW	0.1444	0.1138	0.1239	0.1138	0.1138	0.1138
	(0.09)	(0.01)	(0.43)	(0.00)	(0.01)	(0.00)
MVC	0.0892	0.0678	0.0848	0.1084	0.1977	0.2024
	(0.09)	(0.03)	(0.17)	(0.02)	(0.02)	(0.27)
MINVC	0.0835	0.1425	0.1501	0.2493	0.1546	0.3581
	(0.01)	(0.41)	(0.16)	(0.23)	(0.35)	(0.00)
G-MINVC	0.1371	0.1451	0.1429	0.2468	0.1615	0.3028
	(0.08)	(0.31)	(0.19)	(0.25)	(0.47)	(0.00)

This table shows the Sharpe Ratios for the case where we do not rebalance the EW portfolio.

Table 27: Robustness: Certainty Equivalent, EW Buy-and-Hold

RO1_M120_gamma1_hp1_Rolling_ewBuyHold.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0068	0.0047	0.0040	0.0038	0.0082	0.0080
MV	0.0031	-0.7816	-0.1365	0.0045	-2.7142	-0.0829
	(0.27)	(0.00)	(0.00)	(0.21)	(0.00)	(0.01)
MINV	0.0024	0.0052	0.0054	0.0039	0.0100	-0.0002
	(0.06)	(0.37)	(0.12)	(0.46)	(0.21)	(0.00)
IVP	0.0066	0.0051	0.0047	0.0036	0.0075	0.0071
	(0.45)	(0.02)	(0.07)	(0.24)	(0.00)	(0.00)
ERC	0.0064	0.0051	0.0047	0.0039	0.0075	0.0067
	(0.38)	(0.06)	(0.06)	(0.41)	(0.00)	(0.19)
VW	0.0053	0.0042	0.0044	0.0042	0.0042	0.0042
	(0.30)	(0.00)	(0.14)	(0.00)	(0.14)	(0.00)
MVC	0.0040	0.0023	0.0032	0.0030	0.0090	0.0075
	(0.26)	(0.12)	(0.37)	(0.29)	(0.16)	(0.41)
MINVC	0.0024	0.0047	0.0054	0.0039	0.0060	0.0051
	(0.04)	(0.50)	(0.11)	(0.46)	(0.01)	(0.08)
G-MINVC	0.0044	0.0048	0.0051	0.0038	0.0067	0.0070
	(0.14)	(0.44)	(0.13)	(0.49)	(0.00)	(0.25)

This table shows the Certainty Equivalent for the case where we do not rebalance the EW portfolio.

Table 28: Robustness: Turnover and return-loss, EW Buy-and-Hold

RO1_TC50_M120_gamma1_hp1_Rolling_ewBuyHold.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Panel A: Relative turnover of each strategy						
MV	Inf	Inf	Inf	Inf	Inf	Inf
MINV	Inf	Inf	Inf	Inf	Inf	Inf
IVP	Inf	Inf	Inf	Inf	Inf	Inf
ERC	Inf	Inf	Inf	Inf	Inf	Inf
VW	NaN	NaN	NaN	NaN	NaN	NaN
MVC	Inf	Inf	Inf	Inf	Inf	Inf
MINVC	Inf	Inf	Inf	Inf	Inf	Inf
G-MINVC	Inf	Inf	Inf	Inf	Inf	Inf
Panel B: Return loss relative to 1/N (per month)						
MV	0.0136	227.1418	0.7747	0.0005	7.0585	0.9611
MINV	0.0043	0.0014	-0.0005	-0.0002	0.0001	0.0026
IVP	-0.0005	-0.0004	-0.0007	0.0002	0.0008	0.0003
ERC	-0.0002	-0.0004	-0.0007	-0.0003	0.0008	-0.0024
VW	0.0012	0.0007	-0.0005	0.0053	0.0032	0.0037
MVC	0.0075	0.0046	0.0027	0.0044	0.0006	0.0010
MINVC	0.0037	-0.0002	-0.0012	-0.0002	0.0016	-0.0022
G-MINVC	0.0015	-0.0005	-0.0010	-0.0002	0.0012	-0.0025

This table shows the turnover and return-loss for the case where we do not rebalance the EW portfolio.

Table 29: Robustness: Ranking, EW Buy-and-Hold

RO1_M120_gamma1_hp1_Rolling_ewBuyHold.tex

Strategy	S&P Sectors $N = 11$	Industry Portf. $N = 11$	Inter'l Portf. $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Total of Ranks	Final Rank
Panel A: Rank based on Sharpe ratio								
1/N	3	6	7	5	3	5	29	6
MV	9	8	9	7	9	7	49	9
MINV	8	1	2	3	1	9	24	4
IVP	1	5	4	6	4	6	26	5
ERC	2	4	5	1	5	2	19	1
VW	4	7	6	8	8	8	41	8
MVC	6	9	8	9	2	4	38	7
MINVC	7	3	1	2	7	1	21	2
G-MINVC	5	2	3	4	6	3	23	3
Panel B: Rank based on Certainty Equivalent								
1/N	1	5	7	6	3	1	23	2
MV	7	9	9	1	9	9	44	9
MINV	9	1	1	5	1	8	25	4
IVP	2	2	5	8	4	3	24	3
ERC	3	3	4	3	5	5	23	1
VW	4	7	6	2	8	7	34	7
MVC	6	8	8	9	2	2	35	8
MINVC	8	6	2	4	7	6	33	6
G-MINVC	5	4	3	7	6	4	29	5
Panel C: Rank based on Turnover								
1/N	1	1	1	1	1	1	6	1
MV	9	9	9	8	9	9	53	9
MINV	8	8	8	7	8	7	46	8
IVP	3	3	3	3	3	3	18	3
ERC	4	4	4	4	4	4	24	4
VW	2	2	2	2	2	2	12	2
MVC	7	7	7	9	7	8	45	7
MINVC	6	6	6	6	6	6	36	6
G-MINVC	5	5	5	5	5	5	30	5

This table shows the ranking of portfolios for the case where we do not rebalance the EW portfolio.

REFERENCES

Allen, D., C. Lizieri, and S. Satchell. 2019. In Defense of Portfolio Optimization: What If We Can Forecast? *Financial analysts journal* 75:20–38. doi:10.1080/0015198X.2019.1600958.

Anderson, R. M., S. W. Bianchi, and L. R. Goldberg. 2012. Will My Risk Parity Strategy Outperform? *Financial analysts journal* 68:75–93. doi:10.2469/faj.v68.n6.7.

Ang, A., R. J. Hodrick, Y. Xing, and X. Zhang. 2006. The Cross-Section of Volatility and Expected Returns. *The Journal of finance (New York)* 61:259–299. doi:10.1111/j.1540-6261.2006.00836.x.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath. 1999. Coherent Measures of Risk. *Mathematical finance* 9:203–228. doi:10.1111/1467-9965.00068.

Asness, C. S., A. Frazzini, and L. H. Pedersen. 2012. Leverage Aversion and Risk Parity. *Financial analysts journal* 68:47–59. doi:10.2469/faj.v68.n1.1.

Chaves, D., J. Hsu, F. Li, and O. Shakernia. 2012. Efficient Algorithms for Computing Risk Parity Portfolio Weights. *Journal of Investing* 21:150.

Chua, D. B., M. Kritzman, and S. Page. 2009. The Myth of Diversification. *Journal of Portfolio Management* 36:26.

Clarke, R., H. de Silva, and S. Thorley. 2013. Risk Parity, Maximum Diversification,and Minimum Variance: An Analytic Perspective. *Journal of portfolio management* 39:39–53. doi:10.3905/jpm.2013.39.3.039.

DeMiguel, V., L. Garlappi, and R. Uppal. 2009. Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy? *The Review of financial studies* 22:1915–1953. doi:10.1093/rfs/hhm075.

Hallerbach, W. G. 2013. Advances in Portfolio Risk Control: Risk! Parity? doi:10.2139/ssrn.2259041.

Ilmanen, A., and T. Maloney. 2015. Portfolio Rebalancing Part 1 of 2: Strategic Asset Allocation. *AQR Portfolio Solutions Group* .

Jagannathan, R., and T. Ma. 2003. Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *The Journal of finance (New York)* 58:1651–1683. doi:10.1111/1540-6261.00580.

Jobson, J. D., and B. M. Korkie. 1981. Performance Hypothesis Testing with the Sharpe and Treynor Measures. *The Journal of finance (New York)* 36:889–908. doi:10.1111/j.1540-6261.1981.tb04891.x.

Kan, R., and G. Zhou. 2007. Optimal Portfolio Choice with Parameter Uncertainty. *Journal of financial and quantitative analysis* 42:621–656. doi:10.1017/S0022109000004129.

Kirby, C., and B. Ostdiek. 2012. It's All in the Timing: Simple Active Portfolio Strategies That Outperform Naïve Diversification. *Journal of Financial and Quantitative Analysis* 47:437–467. doi:10.1017/S0022109012000117.

Kritzman, M., S. Page, and D. Turkington. 2010. In Defense of Optimization: The Fallacy of 1/ N. *Financial analysts journal* 66:31–39. doi:10.2469/faj.v66.n2.6.

Lee, W. 2011. Risk-Based Asset Allocation: A New Answer to an Old Question? *Journal of portfolio management* 37:11–28. doi:10.3905/jpm.2011.37.4.011.

Lilliefors, H. W. 1967. On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown. *Journal of the American Statistical Association* 62:399–402. doi:10.1080/01621459.1967.10482916.

Litterman, R. 1997. Hot Spots™ and Hedges. *Journal of portfolio management* 23:52–75. doi:10.3905/jpm.1996.052.

Lochstoer, L. A., and T. Muir. 2022. Volatility Expectations and Returns. *The Journal of Finance* 77:1055–1096.

Maillard, S., T. Roncalli, and J. Teiletche. 2010. The Properties of Equally Weighted Risk Contribution Portfolios. *Journal of portfolio management* 36:60–70. doi:10.3905/jpm.2010.36.4.060.

Markowitz, H. 1952. PORTFOLIO SELECTION. *The Journal of finance (New York)* 7:77–91. doi:10.1111/j.1540-6261.1952.tb01525.x.

Michaud, R. O. 1989. The Markowitz Optimization Enigma: Is 'Optimized' Optimal? *Financial analysts journal* 45:31–42. doi:10.2469/faj.v45.n1.31.

Qian, E. E. 2005. On the Financial Interpretation of Risk Contribution: Risk Budgets Do Add Up. doi:10.2139/ssrn.684221.

Roncalli, T. 2016. *Introduction to Risk Parity and Budgeting*. Place of publication not identified: Chapman and Hall/CRC.

Scherer, B. 2011. A Note on the Returns from Minimum Variance Investing. *Journal of Empirical Finance* 18:652–660. doi:10.1016/j.jempfin.2011.06.001.

Tobin, J. 1958. Liquidity Preference as Behavior Towards Risk. *The Review of economic studies* 25:65–86. doi:10.2307/2296205.