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Mind the Gap! Stylized Dynamic Facts and Structural Models†

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We study what happens to identified shocks and to dynamic responses when the data generating process features q disturbances but $q_1 < q$ *variables are used in an empirical model. Identified shocks are linear combinations of current and past values of all structural disturbances and do not necessarily combine disturbances of the same type. Theory-based restrictions may be insufficient to obtain structural dynamics. We revisit the evidence regarding the transmission of house price and of uncertainty shocks. We provide suggestions on how to compare the dynamics of larger scale DSGEs models with smaller scale VARs.* (*JEL* E12, E13, E23, E31, E43, R31)

It is common in macroeconomics to collect stylized facts about the dynamic transmission of certain identified shocks using (small-scale) vector autoregressive (VAR) models and then build (larger scale) dynamic stochastic general equilibrium (DSGE) models to explain the patterns found (see, e.g., Galí 1999; Iacoviello 2005; Basu and Bundick 2017, among many others).

Several authors, including Ravenna (2007); Fernández-Villaverde et al. (2007); and Giacomini (2013) emphasized that such a matching exercise is imperfect as the linear solution of a DSGE model has a vector autoregressive-moving average (VARMA) format. To reduce the mismatch, the VAR should feature a large number of lags; but even a generous lag length may be insufficient in endemic cases. When long lags can not be used due to short data, the *non-invertibility* or *non-fundamentalness* problem is typically taken care by (i) simulating data from the linear decision rules of the same length as the actual data, (ii) running the same VAR on both actual and simulated data, and (iii) comparing the dynamics of the

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endogenous variables in the two systems after shocks are conventionally identified (see Chari, Kehoe, and McGrattan 2005).

In recent years, the term non-invertibility has been employed generically, to cover misspecification problems preventing researchers from getting information about theoretical quantities using a VAR. Thus, the presence of anticipated disturbances (Leeper, Walker, and Yang 2013); news (Forni, Gambetti, and Sala 2018); news and noise (Blanchard, L'Huillier, and Lorenzoni 2013); omitted variables (Kilian and Lütkepohl 2017); and latent variables have been listed as causing non-invertibility.

This paper studies a related mismatch problem, which may also prevent researchers from getting information about the objects of interest from a VAR and could be important for deciding which theory is consistent with the data. We call it *deformation*. It is an aggregation distortion and occurs when the data generating process (DGP) features q structural disturbances, but only $q_1 < q$ variables enter in the empirical model. We investigate two questions. Given that not all structural disturbances can be obtained, will the innovations provide information about "classes" of disturbances? Will they give information about a particular disturbance? In general, the answer is negative.

Deformation makes identified shocks mongrels with little economic interpretation for two reasons. Identified shocks are unlikely to combine structural disturbances of the same type, making it difficult to relate, say, identified technology shocks with the total factor productivity (TFP) disturbances present in a model. Furthermore, when the empirical model is too small, shock identification requires much more stringent conditions than usual, which limit the type of disturbances one can analyze in practice. Perhaps more importantly, the shocks one can identify will be, in general, linear combinations of current and *past* structural disturbances. Thus, they will display stronger propagation than the corresponding disturbances in the DGP.

The first problem (named *cross-sectional deformation*) emerges when the DGP is such that several structural disturbances contemporaneously affect the variables entering the small-scale empirical model. The second problem (named *time deformation*) instead occurs whenever the small-scale empirical model is specified without paying sufficient attention to the theory used to explain the data and it is exacerbated when the empirical model does not respect the theoretical relationship between control and state variables or alters the law of motion of the state variables. Cross-sectional deformation makes robust theoretical restrictions insufficient to obtain the structural disturbances. Time deformation alters the information flow of the structural disturbances.

The Plan of the Paper.—After an illustrative example in Section I, to enhance the intuition and to differentiate deformation from standard non-invertibility problems, Section II derives the formal results, assuming a linear state space representation for the DGP. Our focus is on general equilibrium models, but deformation has identical implications in partial equilibrium settings, since the linear solution of such models also has a state space representation. We provide sufficient conditions for the identification of a "class" or a particular disturbance, highlight the distortions when the mismatch is due to the omission of control or state variables, and give conditions for VAR-DSGE comparison exercises to be valid.

Section III provides a constructive approach to compare a larger scale DGP and a small-scale empirical model, when one has an idea of the process that may have generated the data. With a standard New Keynesian model as DGP, we show the problems occurring when the empirical model is too small; how time deformation can be reduced by more explicitly linking the empirical model to the theory; and which disturbances are more likely to be identified in different empirical systems.

Section IV reverts the viewpoint of Section III, starts from an arbitrarily small-scale empirical model, and examines how the matching exercise is affected by disturbances potentially omitted from the theory. We take the four-variable VAR used by Iacoviello (2005) as given and compare the dynamics induced by identified house price shocks and by preference disturbances in a model with either the original four disturbances or the original four plus a disturbance to the borrowing constraints of entrepreneurs, which is nowadays employed to explain the macrofinancial linkages present in the data, see, e.g., Lindé (2018). While the dynamics induced by identified house price shocks and preference disturbances are closely aligned in the baseline scenario, this is not the case when the theory features five disturbances, because the responses to identified house price shocks also reflect the dynamics induced by monetary policy and the borrowing constraint disturbances.

Section V extends the analysis to DGPs displaying nonlinear terms, such as those generated by higher order perturbed solutions of equilibrium models, currently used to analyze risk or uncertainty disturbances. We demonstrate that the results of Section II hold unchanged, that deformation biases are likely to be more severe, and use Basu and Bundick's (2017) model to show them.

Section VI concludes providing suggestions to users who want to avoid the deformation trap in practice. Given that deformation may be pervasive, the practice of comparing small-scale VAR and larger scale DSGE responses should be considerably refined. Showing that the pattern of responses to interesting impulses is similar is insufficient for a structural model to be considered successful. Carefully selected exercises, like those discussed in Sections III and IV, may provide information about the extent of deformation deficiencies and the quality of the DGSE-VAR match.

Apart from using small-scale VARs to validate the implication of a theory, it is popular to use them to cross off theories inconsistent with the data (see, e.g., Angeletos, Collard, and Dellas 2020), or to estimate structural parameters via response matching (see, e.g., Christiano, Eichenbaum, and Evans 2005). With deformation, the magnitude and persistence of the responses obtained from an identified VAR shock are generally unreliable. Thus, it is dangerous to exclude theories using, say, the magnitude of multipliers or the share of the variance explained, or to provide policy advice based on the structural estimates. For the exercises to be valid, one needs empirical facts that are insensitive to deformation.

Contribution to the Literature.—Our work is related to Canova and Hamidi Sahneh (2018), who analyze the effects of cross-sectional deformation on Granger causality tests, and to Miranda-Agrippino and Ricco (2019), who examine the conditions for shock identification in SVAR-IV under partial identificability. Early work by Blanchard and Quah (1989); Hansen and Sargent (1991); Marcet (1991); Lütkepohl (1984); Braun and Mittnik (1993); and Faust and Leeper (1997) is also relevant as it discusses similar issues but in different settings. Some of the results we present have similar flavor as Wolf (2020), but they are due to deformation rather than insufficient identification restrictions. Our analysis is also linked to the large literature investigating non-invertibility (recently studied in, e.g., Beaudry et al. 2019; Plagborg-Møller and Wolf forthcoming; Pagan and Robinson 2022; Chahrour and Jurado 2022). In particular, it is connected to Kilian and Lütkepohl (2017) and Forni, Gambetti, and Sala (2018), who have pointed out that rectangular systems, like those we analyze, always generate non-invertibility.

Our contribution is to formally derive the mapping between larger scale DGP and smaller scale empirical model when certain endogenous variables are absent from the empirical system; to bring to light cases where informational sufficiency conditions may fail; and to stress that deformation issues may arise even in ideal conditions when the DGP features no news or anticipated shocks, all theoretical quantities are observables, and the standard invertibility condition holds, but short samples or identification convenience make applied researchers work with small-scale empirical models.

Fernández-Villaverde, Rubio-Ramírez, and Sargent (2005), the working paper version of Fernández-Villaverde et al. (2007), derives a general mapping between reduced-form innovations and structural disturbances which is valid for the cases we consider. Relative to that paper, we make specific assumptions about the set of observable variables and this allows us to sharpen the general expressions they derive.

I. Some Intuition

Consider a simple consumption-saving problem where there are disturbances to TFP (Z_t), to the price of investment (V_t), and to preferences (B_t).[1] The representative agent maximizes

$$\max_{C_t} \sum_{t=1}^{\infty} \beta^t U(C_t)$$

subject to the constraints

$$\begin{aligned} C_t/B_t + I_t \;=\; O_t \;&=\; Z_t K_{t-1}^{\alpha}, \\ K_t \;&=\; (1-\delta) K_{t-1} + V_t I_t, \end{aligned}$$

where O_t is output, C_t is consumption, I_t investment and K_t is the capital stock, while α, β and δ are parameters. We assume that $0 < \alpha < 1, 0 < \beta < 1$ and that (Z_t, V_t, B_t) are i.i.d. with unitary means and standard deviation $\sigma_i, i = Z, V, B$. When $U(C_t) = \log C_t$ and $\delta = 1$, the solution is (see the Appendix)

$$\log O_t = \alpha \log K_{t-1} + \log Z_t \tag{1}$$

$$\log C_t = \log(1 - \alpha\beta) + \alpha \log K_{t-1} + \log B_t + \log Z_t \tag{2}$$

$$\log K_t = \log(\alpha\beta) + \alpha \log K_{t-1} + \log V_t + \log Z_t\,. \tag{3}$$

[1] We are grateful to Thomas Drechsel for suggesting a version of this example.

The theory has three endogenous variables and three disturbances (two supply (Z_t, V_t) and one demand B_t). In a VAR with $o_t = \log O_t, c_t = \log C_t, k_t = \log K_t$, all structural disturbances are identifiable from the innovations using theory-based recursive restriction ($z_t = \log Z_t$ can be obtained from the innovations in o_t; given z_t, the other two innovations determine $v_t = \log V_t$ and $b_t = \log B_t$).

Deformation.—Suppose a researcher employs an empirical model with only two observable variables. Given that at most two disturbances can be obtained, would she be able to identify a "demand" and a "supply" disturbance? Would she be able to trace out the dynamics due to the preference disturbance? The answers depends on the variables used.

Suppose (k_t, c_t) are employed. Integrating out o_t (a control) from the problem, the solution of the theory is

$$k_t = \log(\alpha\beta) + \alpha k_{t-1} + u_{1t} \tag{4}$$

$$c_t = \log(1 - \alpha\beta) + \alpha k_{t-1} + u_{2t} \tag{5}$$

where $u_{1t} = v_t + z_t$, $u_{2t} = b_t + z_t$. Note that u_{2t} mixes demand (b_t) and supply (z_t) disturbances and that recursivity is lost. Thus, a VAR featuring (k_t, c_t) exhibits cross-sectional deformation, because three structural shocks are mapped into two innovations. Here, current and past values of the observables do not provide enough information to extract a supply or the preference disturbance because the theoretical restrictions, which are valid in the original three-variable system, fail.

Suppose instead (o_t, c_t) enters the empirical model. Integrating out k_t (a state) from the problem, the solution of the theory is

$$c_t = b_c + \alpha c_{t-1} + u_{1t} \tag{6}$$

$$o_t = b_y + \alpha o_{t-1} + u_{2t} \tag{7}$$

where $u_{1t} = z_t + b_t - \alpha b_{t-1} + \alpha v_{t-1}$, $u_{2t} = z_t + \alpha v_{t-1}$, and b_c, b_y are constant. Omission of k_t causes two new states c_{t-1}, o_{t-1} to appear in the solution. In addition, recursivity is lost and u_{1t} mixes demand and supply disturbances, but now with different timing. Thus, a VAR with (c_t, o_t) displays both cross-sectional and time deformation. In such a system the (recursive) cross correlation between u_{jt} and current and lagged values of any of the structural disturbances does not go to 1, even when the number of lags goes to infinity. Thus, it is impossible to recover the disturbances of interest using current and lagged values of (c_t, o_t). Because adding future values does not help either, the recoverability condition of Chahrour and Jurado (2022) also fails. Note that, also in this case, theoretical motivated restrictions will not identify any structural disturbance.

Is there a two-variable system which allows the identification of a supply and a demand disturbance? If the two great ratios, $(k_t - o_t)$ and $(c_t - o_t)$ are used as observables, one can recover v_t, b_t from the innovations. Thus, while individual variables may not allow the identification of classes or of particular disturbances, linear combinations of observables of the original model might. This happens because each disturbance enters the decision rule of one linear combination only. Whether underidentification of z_t causes an inferential problem depends on the interest of the researcher. If one simply wants to derive a demand and a supply shock, as we do here, no further issue arises.

The example also tells us that, if some of the structural disturbances are not of crucial interest, there may be a carefully selected set of variables which may allow the identification of the remaining disturbances in a small-scale empirical system. As seen later, a sufficient condition for this to happen is that the disturbances of interest enter the reduced system in a block diagonal format.

Relationship with Non-invertibility.—For the readers familiar with the "invertibility" language, one may note that the systems (4)–(5) and (6)–(7) are non-invertible in the observables, although for different reasons. Furthermore, non-invertibility is not driven by the properties of the structural disturbances (there is no news or anticipated disturbances) nor by the intrinsic dynamics of the original system (here $\alpha < 1$), but by the scale of the empirical model. The system with great ratios is, instead, invertible because (the history of) each combination of variables carries unique information about one structural disturbance.

How different is deformation from traditional non-invertibility? We explicitly consider empirical systems featuring less observables than structural disturbances ("rectangular" systems), while the literature focuses on informational deficiencies present in systems with as many observable variables as structural disturbances ("square" systems). (1)–(3) could be one such square system; and it is easy to verify that with k_t, c_t, o_t as observables, the "poor man invertibility" condition (see Fernández-Villaverde et al. 2007) is satisfied and, as mentioned, all disturbances can be obtained from the innovations of a VAR.

Deformation and Omitted Variables.—It is useful to stress that omitting variables present in the theory does not necessarily generate deformation problems. What is crucial is that the omission causes a mismatch between the number of VAR variables and the number of structural disturbances (assuming all are of interest). To illustrate, consider the original consumption-saving model, but now assume that the TFP disturbance Z_t is an AR(1) with persistence ρ.[2] The solution is

$$\log O_t = \alpha \log K_{t-1} + \rho \log Z_{t-1} + \log e_t^z \tag{8}$$

$$\log C_t = \log(1 - \alpha\beta) + \alpha \log K_{t-1} + \log B_t + \rho \log Z_{t-1} + \log e_{t^z} \tag{9}$$

$$\log K_t = \log(\alpha\beta) + \alpha \log K_{t-1} + \rho \log Z_{t-1} + \log V_t + \log e_t^z \tag{10}$$

$$\log Z_t = \rho \log Z_{t-1} + \log e_t^z. \tag{11}$$

[2] We thank one of the referees for suggesting such an example.

In this system there are three disturbances and four endogenous variables. Suppose that a researcher uses a VAR with (o_t, c_t, k_t). It is easy to check that the "poor man invertibility" condition holds, despite the fact that the exogenous state z_t is omitted. Moreover, when the VAR features sufficient lags, it is possible to recover the three structural disturbances using theoretically motivated recursive restrictions.

To restate the concept differently, deformation occurs when the empirical system is not large enough relative to the vector of structural disturbances. Omission of theory relevant variables is neither a necessary nor a sufficient condition for deformation to emerge. Thus, as long as the system is square and the "poor man invertibility" condition holds, omission of state variables does not cause deformation problems.

Deformation and Measurement Errors.—Although in the theory all disturbances are structural, deformation would emerge unchanged if the theory, instead, is driven by a mixture of structural disturbances and measurement errors. Suppose, for instance, that v_t is a measurement error. Then, a VAR with (k_t, c_t) will still display cross-sectional deformation and a VAR with (c_t, o_t) will display both cross-sectional and time deformations. Finally, in the VAR with the two great ratios, a researcher will be able to identify the preference disturbance (but not the TFP disturbance).

To sum up, deformation may emerge even when traditional forms of non-invertibility are absent and it is produced by a dimensionality mismatch between the empirical model and the disturbances of the DGP. In this situation, the variables entering the empirical system determine the informational content of the reduced-form innovations and the dimensionality mismatch problem becomes more severe when endogenous state variables are omitted. In general, strict conditions are needed to recover a "class" or a particular disturbance and one needs to verify they hold for the vector of observables used. The next section formalizes these conclusions.

II. The Analytical Results

This section derives the mapping between structural disturbances and reduced-form innovations when the empirical model contains different combinations of endogenous states and controls (Propositions 1 and 2) and compares the dynamic responses in the theory with those obtained in various empirical systems (Proposition 3). We employ the generic term "empirical system" throughout the section because the implications we derive hold when a researcher estimates a VAR or a state space model. We assume that the DGP is of the form:

$$x_t = A(\theta)x_{t-1} + B(\theta)e_t \tag{12}$$

$$y_t = C(\theta)x_{t-1} + D(\theta)e_t \tag{13}$$

where x_t is a $k \times 1$ vector of endogenous and exogenous states, y_t is a $m \times 1$ vector of endogenous controls, $e_t \sim N(0, \Sigma(\theta))$ is a $q \times 1$ vector of disturbances, $\Sigma(\theta)$ a diagonal matrix and θ a vector of structural parameters; $A(\theta)$ is $k \times k$, $B(\theta)$ is $k \times q$, $C(\theta)$ is $m \times k$, $D(\theta)$ is $m \times q$. For convenience, we let the eigenvalues of $A(\theta)$ to be all less than 1 in absolute value. Thus, if there are disturbances with permanent effects, (12)–(13) represent a properly scaled version of the data process. Predictable disturbances or news about future disturbances are not considered to leave standard non-invertibility issues aside. While (12)–(13) are general, in our applications they are produced by the (log-)linear solution of the optimality conditions of a structural macroeconomic model.

In general, $m \geq q$ and some of the endogenous variables may be latent. Hence, the variables entering the empirical model are $z_t = S[x_t', y_t']'$, where S is a selection matrix. Fernández-Villaverde et al. (2007) assume $S = \text{diag}[0, I]$[3] and consider $m = q$; Ravenna (2007) and Pagan and Robinson (2022) assume that either $S = I$ and consider $m + k = q$, or $S = \text{diag}[0, I]$ and consider $m = q$. In general, S is chosen so that the dimension of z_t matches the number of structural disturbances.

The reduced-form (innovation representation) corresponding to (12)–(13) is

$$x_t = A(\theta) x_{t-1} + K_x(\theta) u_t \tag{14}$$

$$y_t = C(\theta) x_{t-1} + K_y(\theta) u_t \tag{15}$$

where $u_t = z_t - E[z_t \mid \Omega_{t-1}]$ is a $q \times 1$ vector of innovations, Ω_{t-1} includes (at least) lags of z_t, $K_x(\theta)$, and $K_y(\theta)$ are steady-state Kalman gain matrices, and for those x_t and y_t belonging to z_t, $K_i(\theta)$ has a row with zeros except in one position.

Given (14)–(15), the identification of the structural disturbances requires a mapping from u_t into e_t. When the empirical model is a VAR, Sims and Zha (2006) and Plagborg-Møller and Wolf (forthcoming) developed sufficient conditions to obtain e_t from current and past z_t; Chahrour and Jurado (2022) discuss sufficient conditions to recover e_t from current, past and future z_t. Here, when $S = I$, one needs to invert $\begin{pmatrix} B(\theta) \\ D(\theta) \end{pmatrix} e_t = u_t$; when $S = \text{diag}[0, I]$, one needs to invert $D(\theta) e_t = u_t$. In both cases, standard conditions apply, see Rubio-Ramírez, Waggoner, and Zha (2010).

In the identification exercise two assumptions are implicitly made. First, there is no misspecification in (12)–(13), at least, as far as sources of disturbances are concerned, so that $\dim(z_t) = \dim(e_t)$. If disturbances are left out, the identification exercises becomes problematic, even when excluded disturbances are orthogonal to included ones, and included disturbances account for a large portion of the variability of z_t. Second, when $z_t = y_t$, and $\dim(z_t) = \dim(e_t)$, Ω_{t-1} it is typically specified to include long lags of z_t to take care of omitted states. When disturbances are left out, having a rich Ω_{t-1} is generally insufficient to make the identification problem well behaved.

[3] $\text{diag}[a, b]$ is the operator that combines the matrices a and b into a block diagonal matrix of appropriate dimensions.

Three Small Empirical Systems.—In our analysis $\dim(z_t) < \dim(e_t)$. Thus, we focus on the situation when, say, a two-variable VAR is used to collect dynamic facts but the DGP features more than two disturbances. A researcher who wants to interpret the dynamics of the small-scale empirical system may employ a theoretical model that is less complex than the DGP and may specify only enough disturbances to match the number of empirical variables. We show that the dynamics produced by such model may not be relevant for the comparison and omitted disturbances may play a crucial role. To ease the notation, we omit the dependence of the reduced-form matrices $A, B, C, D, K_x, K_y, \Sigma$ on the structural parameters θ, unless it creates confusion. Let $z_{it} \equiv S_i[x_t', y_t']'$, where S_i is a $q_i \times (m+k)$ and $\dim(z_{it}) = q_i < \dim(e_t) = q, \forall i$. We consider three S_i matrices.

- **Case 1:** $S_1 = \text{diag}[I, S_{12}]$. This choice of S_i generates an empirical system which retains the states but integrates out part of the controls. The DGP in terms of $z_{1t} = [x_t', y_{1t}']'$, $y_{1t} \equiv S_{12} y_t$ is

$$x_t = Ax_{t-1} + Be_t \tag{16}$$

$$y_{1t} = C_1 x_{t-1} + D_1 e_t \tag{17}$$

or $z_{1t} = F_1 z_{1t-1} + G_1 e_t$, where $F_1 = \begin{pmatrix} A & 0 \\ C_1 & 0 \end{pmatrix}$ and $G_1 = \begin{pmatrix} B \\ D_1 \end{pmatrix}$. Let $F = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$, $G = \begin{pmatrix} B \\ D \end{pmatrix}$.

- **Case 2:** $S_2 = \text{diag}[S_{21}, S_{22}]$. This choice of S_i generates an empirical system which integrate out part of the states and part of the controls. Let $x_t' = (x_{1t}', x_{2t}')$, $y_t' = (y_{1t}', y_{2t}')$, where (x_{1t}, y_{1t}) are the variables excluded from the empirical system.

The DGP in terms of $z_{2t} = [x_{2t}', y_{2t}']'$, where $x_{2t} \equiv S_{21} x_t$, $y_{2t} \equiv S_{22} y_t$, is

$$x_{2t} = A_2 x_{2t-1} + B_2 e_t + w_{1t-1} \tag{18}$$

$$y_{2t} = C_2 x_{2t-1} + D_2 e_t + w_{2t-1} \tag{19}$$

or $z_{2t} = F_2 z_{2t-1} + G_2 e_t + w_{t-1}$, where $F_2 = \begin{pmatrix} A_2 & 0 \\ C_2 & 0 \end{pmatrix}$ and $G_2 = \begin{pmatrix} B_2 \\ D_2 \end{pmatrix}$, where $w_{1t-1} = H_2 x_{1t-1}$ and $H_2 = [A_{21} \;\; C_{21}]'$. Alternatively, using (12) to separate observable and non-observable states, and integrating x_{1t} out, the DGP for z_{2t} is

$$x_{2t} = \tilde{A}_{21} x_{2t-1} + \tilde{A}_{22} x_{2t-2} + \tilde{B}_{20} e_t + \tilde{B}_{21} e_{t-1} \tag{20}$$

$$y_{2t} = \tilde{C}_{21} x_{2t-1} + \tilde{C}_{22} x_{2t-2} + \tilde{D}_{20} e_t + \tilde{D}_{21} e_{t-1}. \tag{21}$$

(18)–(19) point out the misspecification present using a first-order VAR for z_{2t}. (20)–(21) shows that DGP for the observables is a VARMA(2,1).

- **Case 3:** $S_3 = \text{diag}\left[S_{31}, 0\right]$. This choice of S_i generates an empirical system which repackages the states and eliminates the controls. The DGP in terms of $z_{3t} = x_{3t} \equiv S_{31} x_t$ is

$$x_{3t} = A_3 x_{3t-1} + B_3 e_t + w_{3t-1} \tag{22}$$

where w_{3t-1} is a function of the repackaged states. Analogously with Case 2, one may write (22) as

$$z_{3t} = \bar{A}_{31} z_{3t-1} + \bar{A}_{32} z_{3t-2} + \bar{B}_{30} e_t + \bar{B}_{31} e_{3t-1}. \tag{23}$$

The processes for $z_{it}, i = 1, 2, 3$ are obtained integrating out the relevant variables from the decision rules. They can also be equivalently obtained substituting optimality conditions into others, prior to the computation of the decision rules. The matrices characterizing these solutions generally differ from those obtained solving the original model and crossing out the rows corresponding to the variables absent from z_{it}, because not all the original states are necessarily used in the computation of the decision rules. Section III provides examples of smaller scale empirical systems which correspond to (16)–(17), (20)–(21), and (23) for a specific DGP.

The innovation representation of (12)–(13), when z_{it} are observables is

$$x_{it} = A x_{it-1} + \hat{K}_{ix} u_{it} \tag{24}$$

$$y_{it} = C x_{it-1} + \hat{K}_{iy} u_{it} \tag{25}$$

where $u_{it} = z_{it} - E\left[z_{it} \mid \Omega_{it-1}\right]$is a $q_i \times 1$ vector, $\hat{K}_{ix}$, $\hat{K}_{iy}$ are steady-state Kalman gain matrices featuring some rows with zeros except in one position.

We study the mapping between u_{it} and e_t when $q_i < q$. Given that not all disturbances can be identified, we ask whether a researcher can recover a "class" of disturbances or a particular disturbance appearing in the DGP. We then study whether the dynamic induced by identified shocks match those in the DGP.

The Mapping between Innovations and Structural Disturbances When the Empirical System Eliminates Theoretical Controls.—We analyze the relationship between u_{1t} and e_t, when $E\left[z_{1t} \mid \Omega_{1t-1}\right] = \tilde{F}_1 z_{1t-1}$ and thus

$$u_{1t} = z_{1t} - \tilde{F}_1 z_{1t-1}. \tag{26}$$

PROPOSITION 1:

(*i*) *If* $\tilde{F}_1 = S_1 F S_1^* \equiv F_1$, *then* $u_{1t} = G_1 e_t$, *where* S_1^* *is the generalized inverse of* S_1, $G_1 = S_1 G$ *depends on* θ, *and is a* $q_1 \times q$ *matrix.*

(*ii*) *A block diagonal* G_1 *is sufficient to identify classes of disturbances.*

(*iii*) *If G_1 has at most one nonzero element in row k, one can obtain e_{jt}, for some k and j.*

(The proof of all the propositions is in the Appendix).

As point (i) indicates, when z_{1t} is used in the VAR, the innovations u_{1t} respect the timing protocol of the structural disturbances e_t, but cross-sectionally deform them because G_1 is a $q_1 < q$ matrix. Because G_1 is rectangular, one may ask when elements of the innovation vector carry enough information to recover some structural disturbances. Suppose that structural disturbances are order by classes, i.e., disturbances 1 to j_1 belong to class 1, disturbances $j_1 + 1$ to j_2 belong to class 2, etc. As point (ii) indicates, the k-th element of u_{1t} compresses a class of structural disturbances only if G_1 has a block diagonal structure. Finally, as point (iii) suggests, the k-th element of u_{1t} carries information about e_{jt} if G_1 has at most one nonzero element in row k in position j.

The restrictions in (ii) and (iii) are strong and unlikely to be satisfied in a large class of general equilibrium models. They require that the theory features many "conveniently" placed delay restrictions so that, contemporaneously, either a reduced number of disturbances of the same class affects the k-th variable of the empirical model or only one structural disturbance affects the k-th variable.

Proposition 1 determines the properties of u_{1t}, given e_t. Thus, u_{1t} will be a mean zero process and its autocovariance function will be restricted by

$$E(u_{1t}u'_{1t-s}) = E(G_1 e_t e'_{t-s} G'_1), s \geq 0. \tag{27}$$

When e_t are i.i.d., the variance of u_{1t} and e_t differ and the magnitude of the amplification depends on G_1. Thus, a e_{jt} disturbance with a small variance or small loadings G_{1j} will be hard to identify. Similarly, the serial correlation properties of u_{1t} depend on the structure and magnitude of G_1 and its row dimension. In general, cross-sectional distortions may make the autocovariance function of u_{1t} insufficient to recover the autocovariance of some e_{jt}, unless additional restrictions are imposed.

The Mapping between Innovations and Structural Disturbances When the States in the Empirical and the Theoretical Models Differ.—We analyze the relationship between u_{it} and e_t when $E[z_{it} | \Omega_{it-1}] = \tilde{F}_i z_{it-1}, i = 2,3$ so that

$$u_{it} = z_{it} - \tilde{F}_i z_{it-1}. \tag{28}$$

PROPOSITION 2:

(i) $u_{it} = \lambda_i(L) e_t$, *where λ_i depends on θ and is $q_i \times q$ for each L, $i = 2,3$.*

(ii) $u_{it} = \psi_i(L) u_{1t}, i = 2,3,$ *where $\psi(L)$ is a function of A, $\hat{K}$ and K, $\hat{x}_t$ and x_t.*

Point (i) states that when the empirical system eliminates state variables, u_{2t} will not respect the timing protocol of the structural disturbances e_t and cross-sectionally deform them. However, an empirical system including only the state variables of the

DGP does not solve time deformation problems since their law of motion may be altered. Thus, also u_{3t} will, in general carry too little information to recover one or more components of e_t. Note that $S_2 F S_2^* = \tilde{F}_2$, or $S_{31} A S_{31}^* = \tilde{F}_3$ are insufficient to avoid time deformation problems.

Point (ii) indicates that, in general, $u_{it} \neq u_{1t}, i = 2,3$ and the timing of information they contain differs even when $S_i F S_i^* = \tilde{F}_i, \forall i$. In other words, it matters which variables enter the empirical system. To clearly see this, let $\lambda_1(L)^*$ be the generalized inverse of $\lambda_1(L)$. Then,

$$u_{it} = \lambda_i(L)\lambda_1(L)^* u_{1t} \equiv \psi_i(L) u_{1t}. \tag{29}$$

By construction $\psi_{i0} = I$. Thus, an impulse in u_{1t} and $u_{it}, i = 2,3$ has identical effects on the variables present in both z_{1t} and $z_{it}, i = 2,3$ but will last longer when z_{it} are the observables—persistence is altered. Hence, the dynamics induced by identified shocks in small-scale empirical systems of the same dimension but featuring different variables will generally differ.

Equation (28) is misspecified when states are omitted or repackaged. What happens when u_{it} are constructed using a larger information set, e.g., $u_{it} = z_{it} - \tilde{F}_i(L) z_{it-1},\ L = 1,2,\ldots$? Because both z_{2t} and z_{3t} are VARMA processes, standard issues discussed in the literature apply. In principle, $\tilde{F}_i(L)$ must be nonzero for $L \to \infty$ for time deformation biases to disappear. Still, even when $L \to \infty$, cross-sectional deformations will remain.

Proposition 1 is related to the aggregation results of Faust and Leeper (1997). Because their DGP is a VAR, they can not analyze the consequences of omitting states or altering their law of motion. Proposition 2 has the same flavor as the main result in Fernández-Villaverde et al. (2007). The main difference is that here $u_{it}, i = 2,3$ are reduced ranked moving averages of e_t and the reason is time deformation rather than non-invertibility.

Dynamic Responses.—Consider z_{it} responses to an impulse in the shocks. In the DGP they are

$$\begin{aligned} z_{it} &= S_i \begin{pmatrix} B \\ D \end{pmatrix} e_t \\ z_{it+h} &= S_i \begin{pmatrix} A^h B \\ C A^{h-1} B \end{pmatrix} e_t \qquad i = 1,2,3; h = 1,2,\ldots \end{aligned} \tag{30}$$

In the empirical system with z_{1t} as observables, they are

$$\begin{aligned} z_{1t} &= u_{1t}, \\ z_{1t+h} &= \tilde{F}_1^h u_{1t}. \end{aligned} \tag{31}$$

The impact effect differs because $u_t = G_1 e_t$ and G_1 is not a square matrix. Thus, having the correct B, D matrices may be insufficient to recover some e_{jt}, unless G_1 only has one nonzero element in the j-th row. However, if $\tilde{F}_1 = \begin{pmatrix} A \\ S_{12} C \end{pmatrix}$ responses at longer horizons to a properly identified shock are proportional to those of the DGP. Thus, qualitatively, (31) provides a good approximation to (30), if some element of e_t can be recovered from u_{1t}.

The responses computed in systems with $z_{it}, i = 2, 3$ as observables are instead:

$$z_{it} = u_{1t}, \tag{32}$$

$$z_{it+h} = \left(\sum_{k=0}^{h} \tilde{F}_i^{h-k} \psi_{ik} \right) u_{1t}.$$

Here, the dynamic responses of z_{it} will be distorted, even in the (unlikely) case that some of element of e_t can be recovered from the u_{it} vector. Thus, both quantitatively and qualitatively, the dynamics of these systems may have nothing to do with those of the DGP. We summarize the discussion in a proposition.

PROPOSITION 3:

(*i*) *Identified impulse responses constructed in a z_{1t} system could match those of the structural model if $\tilde{F}_1 = \begin{pmatrix} A \\ S_{12} C \end{pmatrix}$ and G_1 has at most one nonzero element in one row.*

(*ii*) *Even if the conditions in* (*i*) *holds, the dynamic responses obtained from identified shocks in a z_{it} system, $i = 2, 3$, differ from those of the DGP.*

(31)–(32) provide an analytic approach to compute deformation biases in impulse responses. Braun and Mittnik (1993) derived a similar expression when the empirical model and the DGP are VARs.

Summary.—When $q_i < q$, the variables entering in the empirical model determine the quality of the (small) VAR- (large) DSGE matching exercises. Eliminating controls generally creates innovations that cross sectionally combine the structural disturbances, but eliminating states or repackaging their law of motion creates both cross-sectional and time distortions. However, an empirical model with all the theoretical states (and none of the controls) may not be enough for proper inference. When the VAR omits or repackages some of the states, long lags are needed for a VAR to reproduce the VARMA of the DGP and for identified shocks to have any relationship with the structural disturbances. When long lags can not be used because of short samples, careful variable selection may reduce time deformation. In general, the qualitative and quantitative dynamics produced by the identified shocks under deformation may have nothing to do with those of the structural disturbances.

III. Given a Theory, How Do I Choose the Variables of a Small-Scale VAR?

To illustrate the practical implications of the propositions and the problems that may emerge matching a larger DGP to a small-scale VAR model we use a standard New Keynesian model featuring five structural disturbances: a permanent a_t and a transitory ζ_t TFP shock, a preference χ_t shock, a cost push μ_t shock, and a monetary policy ε_t shock. The optimality conditions are (see Canova and Ferroni 2011 for details):

$$\chi_t = E_t \chi_{t+1} - \frac{1}{1-h} E_t g_{t+1} + \frac{h}{1-h} g_t + r_t - E_t \pi_{t+1} \tag{33}$$

$$\pi_t = E_t \pi_{t+1} \beta + k_p \Big[\frac{h}{1-h} g_t + (1+\sigma_n) n_t\Big] + k_p(\mu_t - \chi_t) \tag{34}$$

$$o_t = \zeta_t + (1-\alpha) n_t \tag{35}$$

$$r_t = \rho_r r_{t-1} + (1-\rho_r)(\phi_y g_t + \phi_p \pi_t) + \varepsilon_t \tag{36}$$

$$g_t = a_t + o_t - o_{t-1}, \tag{37}$$

where (33) is the Euler equation, (34) is the Phillips curve, (35) is the production function, (36) is the Taylor rule, and (37) is the definition of output growth. o_t is output and g_t its growth rate, n_t is hours worked, π_t is the inflation rate, r_t is the nominal interest rate, and c_t is consumption. h is the coefficient of (external) consumption habit, β the discount factor, σ_n the inverse of the Frisch elasticity of labor supply, κ_p the slope of the Phillips curve, α the labor share in production, ϕ_y, ϕ_π the coefficients of the Taylor rule. The disturbances evolve as AR(1) processes with persistence $0 < \rho_j < 1\, j = z, a, \chi, \mu, \varepsilon$ while ρ_ϵ is assumed to be zero.

We solve the model[4] using a first-order perturbation setting $\alpha = 0.33$; $\beta = 0.99; \sigma_n = 1.5; h = 0.9; k_p = 0.05; \phi_y = 0.1; \phi_p = 1.5; \rho_r = 0.8$; $\rho_z = 0.1; \rho_a = 0.5; \rho_\chi = 0.5; \rho_\mu = 0.1; \rho_\varepsilon = 0$. We obtain decision rules of the form (12)–(13). The minimal state vector is $x_{t-1} = [o_{t-1}, r_{t-1}, \zeta_{t-1}, a_{t-1}, \mu_{t-1}, \chi_{t-1}]'$, and the control vector is $y_t = [g_t, o_t, \pi_t, n_t, r_t]'$. Thus, $A(\theta)$ is 6×6, $B(\theta)$ is 6×5, $C(\theta)$ is 5×6 and $D(\theta)$ is 5×5. It is easy to verify that the "poor man invertibility" condition holds when $z_t = y_t$ and that all disturbances are identifiable from the VAR once a sufficient number of lags and proper identification restrictions are employed.

Smaller Scale VARs.—Given that the theory has five disturbances, we consider systems with less than five variables. We ask (i) which deformation distortions each system displays; (ii) which disturbance could be identified using theory-based restrictions; (iii) whether there is a minimum size of the VAR below which all identified shocks become mongrels.

[4] The structural models are solved using Dynare, see Adjemian et al. (2011).

The first system employs four observable variables, $z_t = (o_t, \pi_t, n_t, r_t)$. The theory corresponding to this system, can be obtained integrating out g_t from the solution. Alternatively, one can use (37) in (33)–(36) and solve the resulting set of equations (the optimality conditions of all smaller scale models discussed in this section are in the online Appendix). Since g_t is a control, the minimal state vector remains $x_{t-1} = [o_{t-1}, r_{t-1}, \zeta_{t-1}, a_{t-1}, \mu_{t-1}, \chi_{t-1}]'$. It is easy to verify that $A(\theta), B(\theta)$ are unaltered. This system corresponds to Case 1 of Section II. Because five structural disturbances are mapped into four innovations, Proposition 1 tells us that cross-sectional deformation will be present.

The second empirical system employs three variables, $z_t = (o_t, \pi_t, n_t)$. It is obtained integrating out g_t, r_t from the solution or substituting (37) in (33)–(36) and then (36) in the remaining equations. Here a control, g_t, and an endogenous state, r_{t-1}, are eliminated. Thus, this empirical system corresponds to Case 2 of Section II. When r_t is integrated out, the minimal state vector is $x^*_{t-1} = [o_{t-1}, o_{t-2}, \zeta_{t-1}, a_{t-1}, \mu_{t-1}, \chi_{t-1}]'$, because the Euler equation becomes a second difference equation. Proposition 2 tells us that the innovations of this system will mix $e_{t-s}, s \geq 0$, cross-sectionally, and Proposition 3 that dynamic biases will be larger than in the four-variables system.

The third system employs $z_t = (\pi_t, n_t, r_t)$ as observables. In this VAR an endogenous control, g_t, and an endogenous state, o_{t-1}, are integrated out. Here the minimal state vector is now $\tilde{x}_{t-1} = [n_{t-1}, r_{t-1}, \zeta_{t-1}, a_{t-1}, \mu_{t-1}, \chi_{t-1}]'$ because the optimality conditions remain a system of first-order difference equations. Since, given ζ_{t-1}, n_{t-1} proxies for o_{t-1}, states are simply repackaged. Thus, deformation problems should be less pronounced than in an empirical system with $z_t = (o_t, \pi_t, n_t)$.

Time Deformation.—To evaluate whether time deformation distortions are present, it is sufficient to check if the autocorrelation function of the innovations of the three systems, which we calculate analytically from the solution and the innovation representation (24) and (25), have any element significantly different from zero. Online Appendix Figures C1–C3 present the function for each system, together with a 95 percent asymptotic tunnel for the hypothesis that the autocorrelation at each horizon is zero—which would hold if time deformation is absent.

As expected, the (o_t, π_t, n_t) system has innovations displaying considerable serial correlation and numerous elements of the autocorrelation function are significant. The other two systems have serially uncorrelated innovations. Figure 1 provides evidence on the causes of time deformation in the (o_t, π_t, n_t) system. It presents the cross-correlation function between the innovations and the structural disturbances together with a 95 percent asymptotic tunnel for the hypothesis that they are all zero—absent time deformation, only the contemporaneous elements should be significant. The innovations correlate with several lags of the transitory TFP and monetary policy disturbances. Thus, the shocks that one may be able to identify in this system will be time contaminated.

Cross-Sectional Deformation.—Each of the three system displays cross-sectional deformation. To examine whether one will still be able to identify, say, a stationary technology or a monetary policy disturbances using theory-based restrictions, we

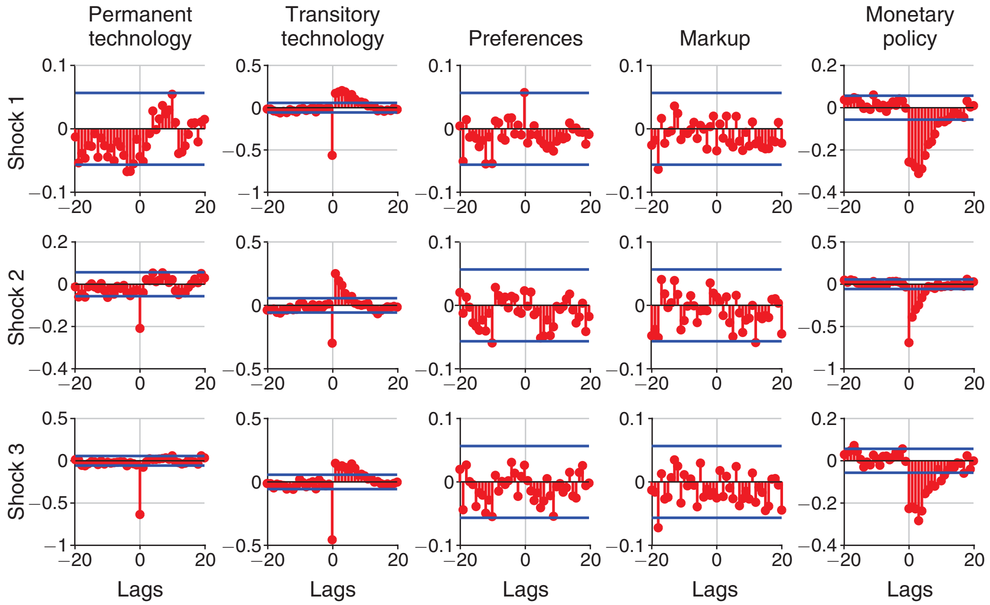

FIGURE 1. CROSS-CORRELATION FUNCTION, INNOVATIONS IN THE (o_t, π_t, n_t) SYSTEM AND STRUCTURAL SHOCKS

Note: Parallel lines delimit 95 percent asymptotic tunnel for the hypothesis of zero.

present in Table 1, the matrix with the contemporaneous mapping between innovations and structural shocks.

With four observables, the monetary policy disturbance remains identifiable as it will maintain, for example, a unique set of theory-based sign restrictions on the four observable variables. However, positive stationary TFP and negative preference disturbances will be confused when sign restrictions are used for identification, as they both produce an instantaneous fall in (o_t, π_t, n_t, r_t).

In the (o_t, π_t, n_t) system, distortions are magnified. Here sign restrictions can not separate any of the stationary structural disturbances. Intuitively, larger distortions occur for two reasons. First, the Euler equation defines a dynamic aggregate demand in output and inflation, while the Phillips curve and the production function define a dynamic aggregate supply equation in the same variables. Because they are both instantaneously moved by, e.g., TFP and preference disturbances, it will be impossible to separate them using output, inflation and hours data. Second, the Euler equation depends on a_{t-1}, ζ_{t-1} and, because o_{t-2} enters the equation, also on ζ_{t-2}. Thus, the aggregate demand equation evolves more persistently in response to disturbances than in the original model.

In the (π_t, n_t, r_t) system, the sign and the magnitude of the loadings of the structural disturbances are the same as in the four-variable system. As compared with the (o_t, π_t, n_t, r_t) system, we loose the possibility to distinguish stationary TFP, permanent TFP and preference shocks. However, there is no change in the ability to recover monetary policy disturbances. Hence, a careful choice of observables in a smaller scale system may minimize time deformation distortions and allow the identification of monetary policy disturbances using theory-based restrictions. In

Table 1—Entries of the λ_0 Matrix

Innovations	Structural shocks				
	a_t	ζ_t	χ_t	μ_t	ϵ_t
(o_t, π_t, n_t, r_t) system					
u_{1t}	0.02	−0.72	0.09	−0.01	−0.30
u_{2t}	−0.16	−0.31	0.04	0.04	−0.72
u_{3t}	−1.46	−1.08	0.13	−0.01	−0.45
u_{4t}	−0.05	−0.09	0.01	0.01	0.78
(o_t, π_t, n_t) system					
u_{1t}	−0.00	−0.72	0.08	0.01	−0.31
u_{2t}	−0.16	−0.30	0.04	0.06	−0.73
u_{3t}	−1.50	−1.08	0.13	0.02	−0.46
(π_t, n_t, r_t) system					
u_{1t}	−0.16	−0.31	0.04	0.04	−0.72
u_{2t}	−1.46	−1.08	0.13	−0.01	−0.45
u_{3t}	−0.05	−0.09	0.01	0.01	0.78

particular, since inclusion of the nominal rate in the system may drastically change the conclusions, VAR users should experiment with different information sets and present the results obtained in each of them in their papers.

Cholesky Factors.—Table 2 displays the Cholesky factors of the covariance matrix of the innovations of original model (assuming disturbances have unit variance and with the rows and columns corresponding to the variables solved out eliminated) and of the three smaller systems. While zero restrictions are not a feature of our model, applying the same recursive restrictions to the innovations of the original and of the reduced systems, as suggested in, e.g., Chari, Kehoe, and McGrattan (2005), provides useful summary statistics for comparing the two representations.

In the (o_t, π_t, n_t, r_t) system the signs of the Cholesky factor match those of the original model, but magnitudes are altered, sometimes substantially (see the (3,2) or (4,2) elements). A similar picture emerges for the (π, n_t, r_t) system. Thus, instantaneous responses to orthogonal shocks in these two systems qualitatively mimic those of the original model, but display magnitude distortions.

For the (o_t, π_t, n_t) system, biases are more significant as the signs and magnitudes are affected. For example, while in the original system an orthogonal unitary shock to n_t implies a roughly similar instantaneous effect on o_t and π_t, the same shock in the (o_t, π_t, n_t) system has a 15 times larger effect on o_t and a negative effect on π_t.

Impulse Responses.—We show dynamic deformation distortions when we identify shocks with contemporaneous sign restrictions.

Figure 2 presents the responses to a monetary policy shock in the (π_t, n_t, r_t) system when policy disturbances are identified assuming that an increase in r_t leads to a contemporaneous fall in π_t, n_t. Online Appendix Figure C.4 has the responses to a monetary shock in the (o_t, π_t, n_t, r_t) system. Dotted lines represent the identified sets consistent with the restrictions. Superimposed as continuous lines are the responses of the original five-variable model. The three-variable system encodes

Table 2—Cholesky Factors

Observables	Original system				Reduced system			
(o_t, π_t, n_t, r_t)	0.75				0.79			
	0.68	0.26			0.56	0.57		
	1.06	1.14	0.96		1.14	0.45	1.42	
	−0.43	−0.14	0.17	0.07	−0.22	−0.70	0.27	0.08
(o_t, π_t, n_t)	0.75				9.56			
	0.68	0.26			5.17	1.50		
	1.06	1.14	0.96		15.37	−0.02	1.52	
(π_t, n_t, r_t)	0.26				0.80			
	1.14	0.96			1.12	1.51		
	−0.14	0.17	0.07		−0.66	0.36	0.24	

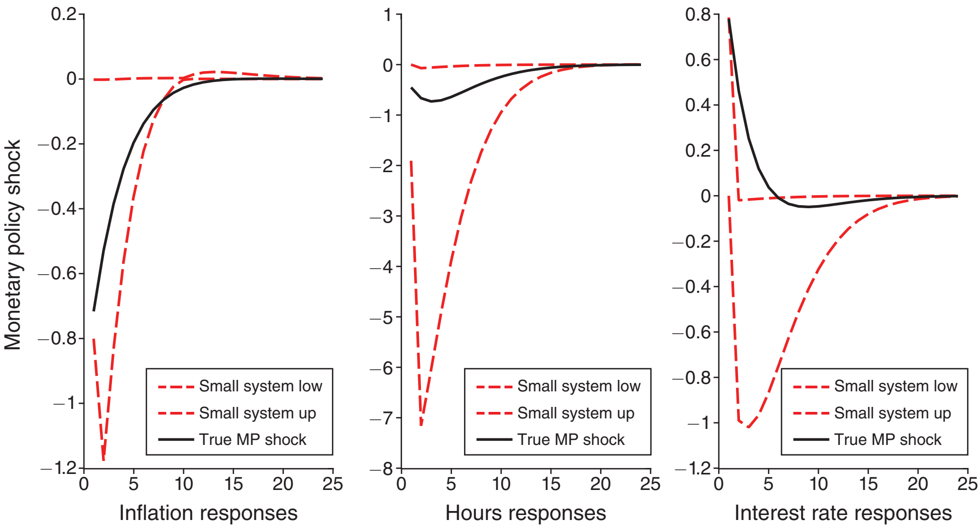

Figure 2. Responses to Identified Monetary Policy Shocks, (π_t, n_t, r_t) System

Notes: The dashed regions report the profile of the identified set. The solid line reports the responses in the DGP.

enough information to recover monetary policy disturbances and omitting output and its growth rate does not affect our ability to interpret the responses to identified monetary shocks, provided hours enter the empirical system. Given over 25 years of empirical literature investigating the dynamics induced by monetary policy disturbances, it is comforting to find that these shocks can be identified with conventional restrictions, even in trivariate VARs models.

Recall that Table 1 implies that positive stationary TFP and negative preference disturbances have the same contemporaneous sign implications in the four-variable system. Figure 3, which plots the responses to sign-identified stationary TFP disturbances, shows that indeed the size of estimated impact responses is significantly off; and that dynamic responses are more persistent in the smaller system. Hence, theory-based restrictions valid in the five-variable model only identify a linear

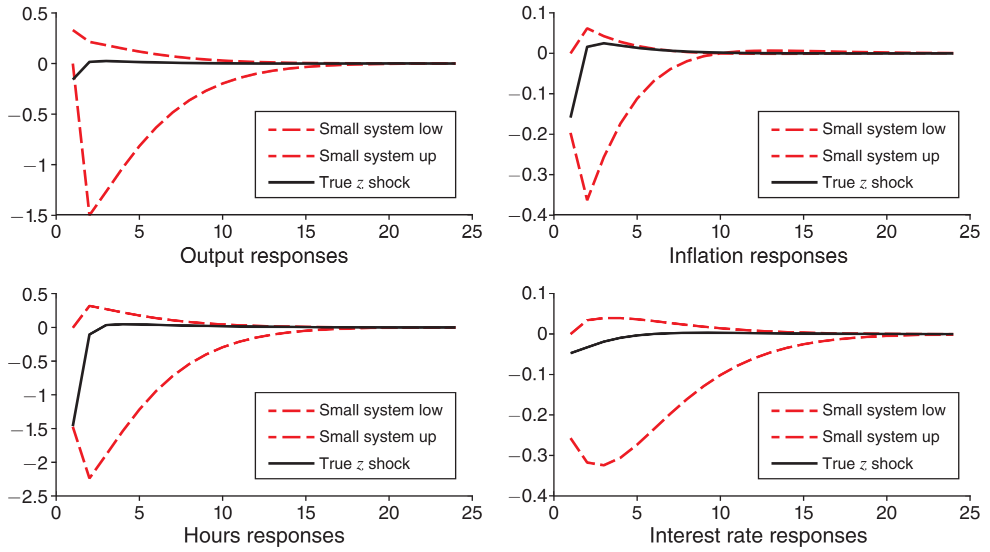

Figure 3. Responses to Identified Stationary TFP Shocks, (o_t, π_t, n_t, r_t) System

Notes: The dashed regions report the profile of the identified set. The solid line reports the responses in the DGP.

combination of the two disturbances, reminiscent of the masquerading effect discussed in Wolf (2020).

An Empirical Model with Only the Theoretical States.—Omission of the theoretical states or failure to proxy for them generates time deformation. However, an empirical system with only the states (and none of the controls) does not necessarily produce interpretable identified shocks.

Starting from the original five-variable system and integrating out all but $z_t = (o_t, r_t)$ produce a solution where the state vector is unchanged. However, the optimization problem is different because, for example, o_{t+2} and r_{t+1} now appear in the equilibrium conditions. Since $(\bar{A}(\theta), \bar{B}(\theta))$ differ from the original $(A(\theta), B(\theta))$ matrices, this system will also feature timing distortions and mongrel identified shocks. Figure 4, which plots the cross-correlation of the innovations with the five structural disturbances, confirms this fact: the innovations u_t are serially correlated and load on a number of lags of the monetary policy disturbance.

Cross-sectional deformation also matter. With $z_t = (o_t, r_t)$, one can at most identify a linear combination of the five disturbances via sign restrictions. However, no combination separates, say, a supply from a demand type disturbance. For example, identified monetary policy shock will combine markup and monetary policy disturbances. Hence, a two-variable VAR is too small to make economic sense of the shocks one recovers.

Permanent Technology Shocks and Hours Worked.—In the literature it is common to use a VAR with output growth (or labor productivity) and hours to identify permanent TFP shocks. The dynamics are then compared with the dynamics

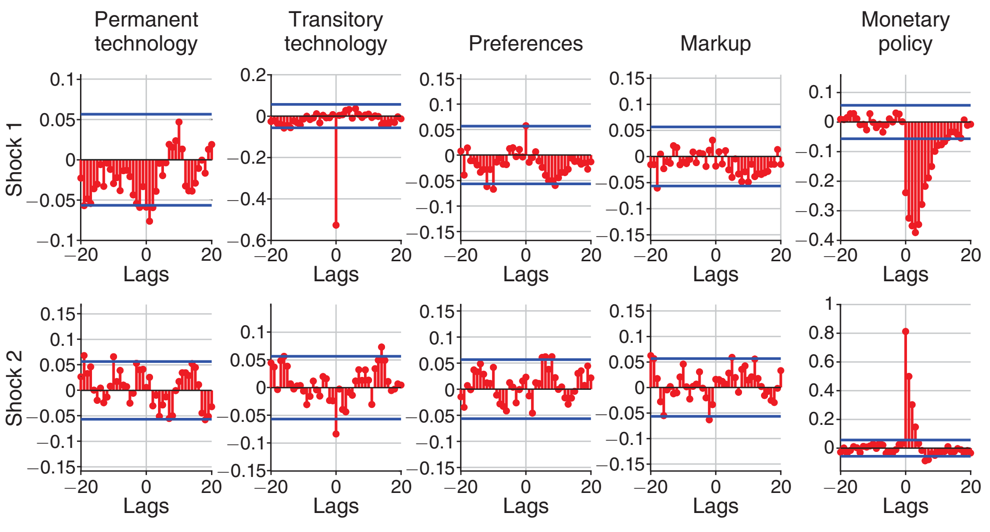

Figure 4. Cross-Correlation Function, Innovations in (o_t, r_t) System and Structural Shocks

permanent TFP disturbances produce in standard RBC or new Keynesian models, see, e.g., Galí (1999). While the comparison is meaningful when the DGP features, say, a permanent TFP and a monetary policy disturbances, it may be inappropriate when the model of this section generates the observed data.

When $z_t = (g_t, n_t)$, lagged output growth and lagged hours become state variables. Since the states and their law of motion are altered, the innovations of the (g_t, n_t) system are related to several lags and leads of the structural disturbances. For example, lags of the permanent TFP disturbances and of the preference disturbances load significantly on the second innovation (see online Appendix Figure C.5). Hence, in this system, there is no guarantee that the identified technology shock will only capture the permanent technology disturbance.

Figure 5 shows that if the DGP only has a permanent TFP and monetary policy disturbance, the responses obtained identifying a permanent supply shock in a VAR with $z_t = (g_t, n_t)$ replicate well the dynamics produced by permanent TFP disturbances (compare the dashed blue and the solid black lines). Instead, when the model of this section is the DGP, magnitude and persistence distortions are important (see the red dashed line). Here, the model can not be reduced to a bivariate system with output growth and hours and meaningful innovations. Once again, a two-variable VAR is too small for identified permanent TFP shocks to make sense. One needs at least a four-variable VAR for identified permanent technology shocks to bear any resemblance with the permanent TFP disturbances the theory features.

R^2 for Invertibility.—It is common in the literature to check whether a structural disturbance can be obtained from a particular vector of VAR variables using the R^2 of a regression of that disturbance on the reduced form innovations (alternatively, on the variables of the empirical system), see, e.g., Sims and Zha (2006) or Plagborg-Møller and Wolf (forthcoming). While the approach is appealing when the VAR includes as many variables as disturbances in the theory, it may give misleading

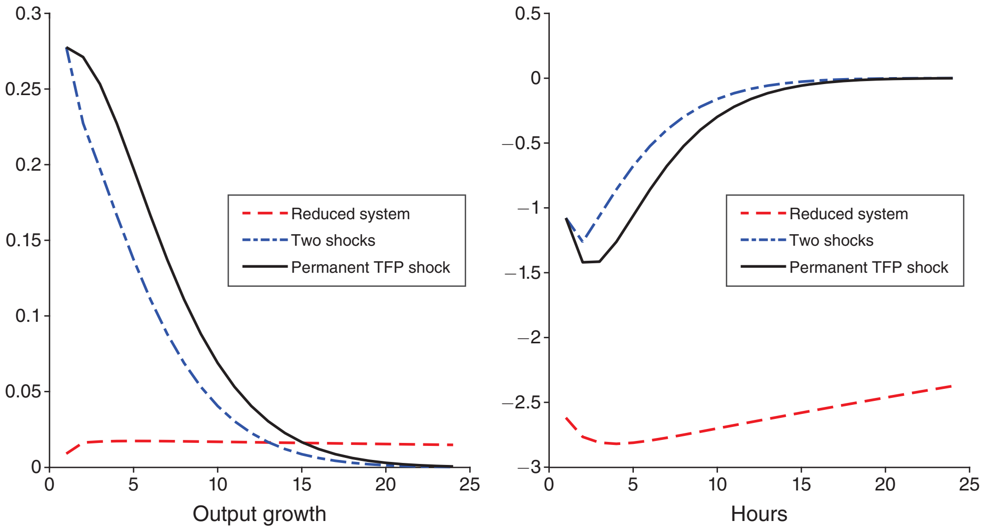

Figure 5. Responses to Identified Permanent TFP Shocks, (g_t, n_t) System

information in the cases we consider. The reason is that a R^2 tells whether there is enough information in the observables, but it does not tell us if certain theory-based identification restrictions are valid. To clarify the point, consider the $z_t = (o_t, \pi_t, n_t, r_t)$ system. The R^2 of a regression of the stationary TFP disturbance on z_t is 0.98, suggesting that there is enough information to recover the disturbance. However, as we have already discussed, stationary TFP and preference disturbances have the same sign implications on these four variables. Hence, if one imposes theory-based restrictions, she will end up with a mongrel mixing preference and stationary TFP disturbances (see Figure 3). A similar issue also emerges in smaller systems. For example, in the $z_t = (o_t, \pi_t, n_t)$ system, the monetary policy disturbance has an R^2 of 0.99 on z_t, but theory-based sign restrictions will confuse stationary technology, preference, markup, and monetary policy disturbances.

Thus, in rectangular systems, having a high R^2 is necessary to be able to identify a structural disturbance but it may not be enough when theory-based sign restrictions are employed for identification. As we have discussed, the sign and the magnitude of the entries of the contemporaneous mapping between innovations and disturbances, provide complementary information to understand which vector of observable variables allows the identification of the disturbance of interest.

IV. Given a Small-Scale VAR, Does a Theory Match the Facts?

The dynamics of output and inflation following house price disturbances have become of primary policy importance following the 2008 financial crisis. Starting with Iacoviello (2005) many authors have tried to understand whether the responses obtained in a SVAR can be rationalized with a structural model featuring housing services, leveraged agents, and standard macroeconomic frictions. Since house price disturbances are not necessarily a major source of macroeconomic fluctuations, at least in normal times, the theoretical models employed to interpret the data typically contain several other disturbances, see, e.g., Rabanal (2018) and Lindé (2018) for recent examples. However, apart from obvious core choices, it is not clear which other disturbances should be included.

Iacoviello (2005) sidesteps the problem by selecting the minimum number of disturbances needed to map the empirical evidence into a structural model. He uses a four-variable VAR to construct the dynamic responses to recursively identified house price shocks and a model with preferences, monetary policy, technology, and cost push disturbances to estimate the structural parameters; and then interprets the SVAR dynamics through the lenses of preference disturbances. Here we take the four-variable VAR and the identified house price shocks as given, and ask whether they would still be interpretable though the lenses of preference disturbances when the theory is enlarged to include LTV disturbances to the entrepreneurs' problem, which have been extensively used to study the dynamics of house prices since Iacoviello's seminal work. In other words, we ask whether deformation problems could prevent a researcher to map preference disturbances into identified house price shocks and, if this is the case, what identified house price shocks would capture.

To be clear about the scope of the exercise, in Section III we take a theory as given, and ask which small empirical model allows the identification of interesting disturbances and with what restrictions. Here we reverse that viewpoint, take a VAR and an identification scheme as given, and ask whether omitted disturbances alter our perception of the match between the theory and the VAR.

The Properties of the Enlarged Model.—The optimality conditions and the law of motion of the disturbances are in online Appendix Section D. The model economy features 8 endogenous states (lagged house holdings of impatient consumers and of entrepreneurs, lagged bond holdings of patients and impatient consumers, lagged capital shock, lagged output, lagged nominal interest rate, and lagged inflation) and 15 endogenous controls. When the VAR includes output, nominal rate, inflation, house prices, *and the stock of housing*, the "poor man invertibility" condition holds (all eigenvalues of $A - BD^{-1}C$ are less than 1 in absolute value). Furthermore, the R^2 of a regression of each disturbances on the simulated data is 1. Thus, when at least these five variables enter the VAR, there are no informational deficiencies and all structural disturbances are potentially recoverable.

We take data for real GDP (O_t), the nominal interest rate (R_t), inflation rate (π_t), and real house prices (q_t) from the FRED data base for the period 1975:1–2018:3 and identify house price shocks using the same lag setting (2 lags), the same data transformation (HP filtering of GDP and house prices)[5], and the same recursive identification scheme of Iacoviello (2005).

The Evidence.—The first row of Figure 6 plots the posterior 68 percent response intervals to an identified house price shock in the data and the responses to preference disturbances in the theory with four disturbances. Iacoviello's main result holds

[5] While this choice alters the timing of house price shocks and the responses they generate, we decided to stick to this transformation since the purpose of the exercise is to show the effects of deformation not of filtering.

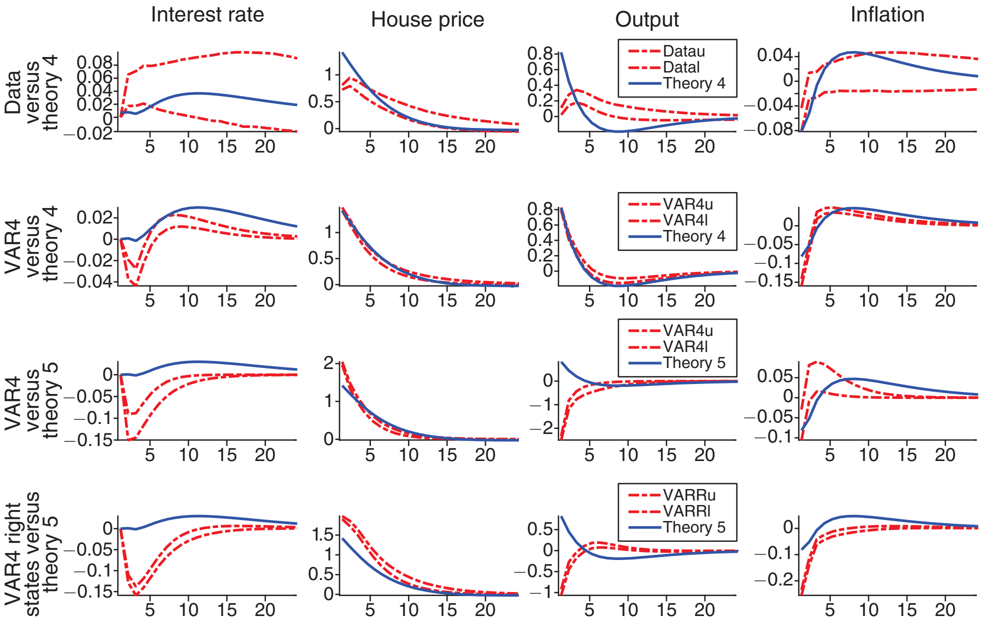

Figure 6. Data and Models, q_t Innovations

Notes: The first row reports the responses to preference disturbances in the Iacoviello (2005) model and the 68 percent highest posterior interval in the data; the second row the responses to preference disturbances in the same model and the 68 percent highest posterior interval in a four-variable VAR on simulated data; the third row the responses to preference disturbances in a model with 5 shocks and the 68 percent highest posterior interval in a four-variable VAR on simulated data when only π_t, r_t, o_t are used as states; the fourth row the responses to preference disturbances in a model with 5 shocks and the 68 percent highest posterior interval in a four-variable VAR on simulated data when all the states are used. VAR parameters and Impulse responses are estimated using the Hitchhiker Guide to Empirical Macro Toolbox, see Canova and Ferroni (2020).

with the extended dataset: after a temporary house price increase, output, inflation and the nominal interest rate persistently rise; and a similar pattern is generated by preference disturbances, although in the data the maximum output response is delayed. The second row demonstrates that truncation lags and the use of recursive restrictions (which fail to hold in the theory) do not affect the mapping between preference disturbances and identified house price shocks. In fact, comparing the data responses with the theoretical responses to preference disturbances or with the Cholesky identified house price responses in a VAR(2) on simulated data gives the same qualitative conclusions, see also Chari, Kehoe, and McGrattan (2005). Hence, in the baseline case, it is legitimate to interpret identified house price responses in the data through the lenses of model-based preference disturbances. This is not necessarily the case when the theory features one additional disturbance for two reasons. Because the five disturbances are mapped into four innovations, cross-sectional deformation matter. Furthermore, because only three state variables (lagged output, lagged inflation and lagged nominal interest rate) enter the VAR, time deformation will also be present.

The third row of Figure 6 plots the responses to a preference disturbance in the theory with 5 disturbances and the posterior 68 percent response interval to a Cholesky

Table 3—Loading of Structural Disturbances on Innovations in (R_t, π_t, q_t, O_t)

Innovations	Disturbances				
	Monetary policy	Preference	Markup	Technology	LTV
R_t	1.0	0	0	0	0
q_t	−2.02	0.06	−0.62	0.15	−1.62
O_t	−2.76	0.01	−1.81	−0.10	4.07
π_t	−0.60	−0.00	1.30	−0.12	0.22

identified house price shock in a VAR including output, nominal rate, inflation and house prices simulated from the theory which only keeps output, inflation, house prices and the nominal rate as endogenous variables.[6] Note that the sign and the persistence of the responses to identified price shocks in the VAR now differ from those of the theory: output and the nominal interest rate respond negatively; and inflation is insignificant after a few quarters. Deformation matters: a four-variable VAR is too small to produce identified house price shocks with the same interpretation as preference disturbances or, put it differently, the mapping between preference disturbances and identified house price shocks is altered.

Explanations.—Why are rows 2 and 3 different? Is it time or cross-sectional deformation that changes the pattern of responses? Row 4 of Figure 6 presents a counterfactual where time deformation is absent.[7] Because the responses in rows 3 and 4 have similar sign and quantitative differences are small, it is cross-sectional deformation that alters the signs of output and interest rate responses. Alternatively, because five structural disturbances are compressed into four VAR innovations, the mapping between identified house price shocks and preference disturbances is polluted by other disturbances. Standard information sufficiency measures are incapable of capturing these distortions. For example, the R^2 of a regression of the theoretical preference disturbances on simulated output, inflation, nominal interest rate, and house prices is 0.94.

To understand what identified house price shocks capture, we compute the matrix of contemporaneous loadings of the four innovations on the five structural disturbances (the λ_0 matrix). House price innovations load on the monetary policy disturbances, e_R (-2.02), on the borrowing constraint disturbances, e_{i1} (-1.62), while the weight on the preference disturbances e_j is small (0.06). Because positive borrowing constraint disturbances increase output and the nominal rate, the negative output and interest rate responses observed in row 3 are due to the large negative loading that borrowing constraint disturbances have on identified house price shocks. To support this interpretation, we compute the contemporaneous correlation between identified house price shocks and preference disturbances in the model with four and five disturbances. The point estimate in the former is 0.91 (95 percent confidence range

[6] The new disturbance has persistence equal 0.75 and standard deviation 1.0. Since we normalize the impulse to unity, the magnitude of the standard deviation is irrelevant.

[7] The scenario is generated simulating data for the four endogenous variables of interest with the decision rules of the model with five disturbances and all the states, identifying house price shocks in a VAR as before. Since all the states are retained, only cross-sectional deformation is present.

across simulations [0.90, 0.92]); in the latter it is only 0.67 (95 percent confidence range [0.63, 0.70]). On the other hand, the contemporaneous correlation of identified house price shocks with the borrowing constraint disturbances is -0.68 (95 percent confidence range [-0.70, -0.65]).

One may ask what is the minimal dimension of the VAR that allows a direct mapping between identified house price shocks and preference disturbances when the DGP has five disturbances. Online Appendix Figure D.1 shows that when the VAR includes the nominal interest rate, house prices, output, inflation and the total stock housing, the dynamics induced by preference disturbances and identified house price shocks are again qualitatively similar.

What have we learned? Our exercise shows that a successful matching exercise becomes a failure when the structure is kept unchanged, but more shocks are added. Because it is standard to specify a theory with as many or fewer shocks in a VAR, and because it is likely that more than four disturbances drive fluctuations in the data, the matching exercise informally run in the literature is justified under the assumption that only the shocks included (and not others or additional ones) are present in the data.

Although the enlarged model generates dynamics on the four-variable VAR which are different than those in the data, one can not conclude that the model is rejected. The correct conclusion is that the data can not be interpreted through the lenses of preference disturbances because house price innovations do not carry enough information about preference disturbances when an enlarged number of disturbances is considered.

V. An Extension

The process in (12)–(13) may be restrictive in certain situations. For example, when analyzing risk or uncertainty disturbances, the model is solved using higher order methods. Hence, a nonlinear DGP specification is needed. This section studies how the conclusions of Section III change in this case.

As shown in Andreasan, Fernández-Villaverde, and Rubio-Ramírez (2018), the pruned solution of a nonlinear state space model approximated with higher order perturbations can be written as

$$X_t = \mu_x(\theta) + \nu_1(\theta)X_{t-1} + \nu_2(\theta)E_t \tag{38}$$

$$Y_t = \mu_y(\theta) + \nu_3(\theta)X_t \tag{39}$$

where, for example in the case of a second-order approximation, $X_t = \Big((x_t^f)', (x_t^s)', \big(x_t^f \otimes x_t^f\big)'\Big)'$, and x_t^f are the states of the first-order system, x_t^s are the states of the second-order system; $E_t = \Big(e_t', \big(e_t \otimes e_t - \mathrm{vec}(I_{n_e})\big)', \big(e_t \otimes x_{t-1}^f\big)'\big(x_{t-1}^f \otimes e_t\big)'\Big)'$,

where e_t are the structural disturbances and I_{n_e} the identity matrix of dimension n_e; Y_t are the controls and the matrices $\mu_x(\theta), \mu_y(\theta), \nu_1(\theta), \nu_2(\theta), \nu_3(\theta)$ are given in the Appendix of Andreasan, Fernández-Villaverde, and Rubio-Ramírez (2018).

Thus, a higher order DGP has a linear state space representation but with a larger number of states and of structural disturbances. If a linear VAR is specified and features $\tilde{Z}_t = \tilde{S}[X_t, Y_t]$ as observables, where $\tilde{S} = [\tilde{S}_1, \tilde{S}_2]$, the conclusions derived in Propositions 1–3 still hold. However, cross-section and time deformations will be more severe because the dimension of E_t is larger, and a larger number of states (in particular, those involving higher order and cross terms) is omitted.[8]

To highlight the effects of deformation in this situation, we take the model of Basu and Bundick (2017), which features disturbances to the volatility of the preference shock, to the level of the technology and to the level of preferences. The model is solved with a third-order perturbation so that $E_t = [E_{1t}', E_{2t}']'$ where

$$\begin{aligned}
E_{1t} &= \left(e_t', \left(e_t \otimes e_t - \operatorname{vec}(I_{n_e})\right)', \left(e_t \otimes x_{t-1}^f\right)'\left(x_{t-1}^f \otimes e_t\right)'\left(e_t \otimes x_{t-1}^s\right)'\right)' \\
E_{2t} &= \left(\left(e_t \otimes x_{t-1}^f \otimes x_{t-1}^f\right)'\left(x_{t-1}^f \otimes x_{t-1}^f \otimes e_t\right)'\left(x_{t-1}^f \otimes e_t \otimes x_{t-1}^s\right)'\right. \\
&\quad \times \left(x_{t-1}^f \otimes e_t \otimes e_t\right)'\left(e_t \otimes x_{t-1}^f \otimes e_t\right)'\left(e_t \otimes e_t \otimes x_{t-1}^f\right)' \\
&\quad \times \left.\left(\left(e_t \otimes e_t \otimes e_t\right) - E\left(e_t \otimes e_t \otimes e_t\right)\right)\right)'.
\end{aligned}$$

Since e_t is a 3×1 vector, and x_t^f a 9×1 vector including lagged values of consumption, capital, hours, output, the nominal rate, expected utility and the three disturbances, X_t is a 432×1 vector and E_t is a 1112×1 vector. They use an eight-variables VAR to trace out the effects of uncertainty shocks, which are identified via a Cholesky decomposition with the VXO index ordered first. The VAR includes four endogenous states (output, consumption, hours and nominal rate), a proxy for the capital state (investment), two controls (inflation, and a volatility measure) and a money supply variable, which is absent from the model.

The Evidence.—The first row of Figure 7 presents the point estimates and the 95 percent response intervals of output, consumption, investment, hours and VXO to an uncertainty shock in the VAR of the data. The second row has the responses to uncertainty shocks in Basu and Bundick's (2017) original setup and parameterization: the dashed line reports theoretical responses, and the solid lines the estimated 95 percent SVAR response intervals in simulated data, identifying the uncertainty shock as in the first row. The match between the theory and the VAR of the data appears to be good. Furthermore, theoretical responses and SVAR responses constructed with simulated data are similar.

Two features of the authors' specification are, however, questionable. Although the nominal interest rate enters the VAR, the model has little to say about it because it posits a deterministic Taylor rule with no persistence (see equation (7), Basu and Bundick 2017). Second, it is not obvious why changes in uncertainty are only demand driven; second moment shocks to the technology could generate similar dynamics in real aggregate variables via a precautionary saving channel. Thus, the

[8] When the class of models suggested by Arouba, Bocola, and Schorfheide (2017) is used, some of the additional deformation problems are eased.

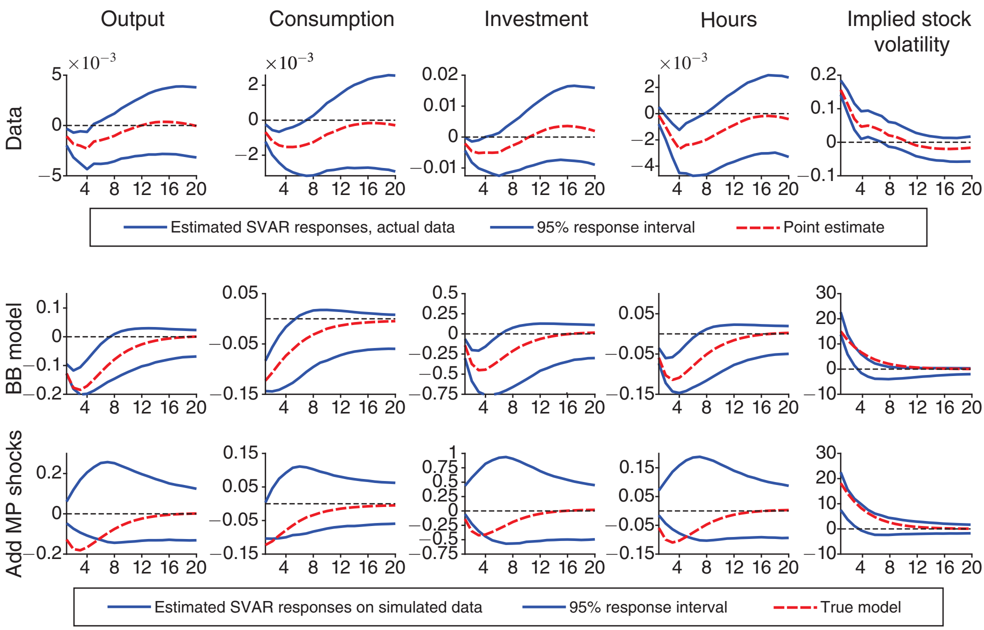

Figure 7. Data and Models, *VXO* Innovations

Notes: The solid lines in the first row report 95 percent response intervals and the dashed line the point estimate using the actual data; the solid lines in the second and third row report the 95 percent response interval in the simulated data and the dashed line the conditional response in the theory.

DGP potentially features more disturbances than those used in the model and the restrictions used to identify uncertainty shocks may be insufficient. For illustration, we add a monetary policy disturbance to the model, keeping the structure and the parameterization unchanged. As row 3 of Figure 7 shows theoretical and the estimated response intervals obtained from simulated data now differ significantly. Moreover, the response intervals in rows 1 and 3 do not line up.

Explanations.—Rows 2 and 3 differ because monetary policy and uncertainty disturbances get mixed up: they both increase the nominal rate and make all other variables fall. While theoretical responses are constructed conditional on the monetary policy disturbances being zero, in the VAR with simulated data, the monetary policy disturbances can be positive and negative. Hence, the sign of the responses of output, consumption, investment and hours to uncertainty shocks depends on the relative importance of uncertainty and monetary disturbances and the sign of the monetary policy disturbances at each t. Given that VAR responses are insignificant, identified uncertainty shocks are likely to pick up positive uncertainty and negative monetary policy disturbances.

To support this conclusion, we compute the contemporaneous correlation of identified volatility shocks with the volatility disturbances in the original and in the extended model with monetary policy disturbances. In the former, the point estimate is 0.88 (95 percent confidence range across simulations [0.86, 0.89]); in the second it is 0.69 (95 percent confidence range [0.66, 0.73]). In the latter system, the contemporaneous correlation between identified volatility shocks and monetary policy disturbances is -0.46 (95 percent confidence range $[-0.50, -0.42]$).

Larger Scale BVAR.—Would the estimation of a larger scale BVAR solve the problems? Because deformation is due both to the fact that an eight-variable VAR is too small and that the volatility and the monetary policy disturbances need both to come first in a Cholesky decomposition to be properly identified, using a larger scale BVAR in the exercise will not necessarily resolve the issue. In addition to the correctly sized VAR, one needs a set of identification restrictions that differentiate the two disturbances, see also Wolf (2020).

VI. Conclusions and Implication for Practice

It is common in macroeconomics to collect stylized facts about the transmission of structural shocks using small-scale VAR models and then build larger scale DSGE models to interpret the dynamics found. This paper argues that important inferential and interpretation distortions may emerge when the process generating the data features more disturbances than the variables entering a VAR.

Cross-sectional deformation makes shock identification hard, because "classes" of structural disturbances need not to be properly compressed into identified shocks, and may make valid theoretical identification restrictions insufficient. Time deformation complicates the matching process because the timing of identified shocks and of structural disturbances differs.

We highlight the practical implications of deformation in two ways. First, we take the DGP as given and show what happens to identified shocks when the empirical model is too small; describe how to reduce time distortions explicitly linking the empirical model to the theory; and highlight the disturbances which are recoverable from different small-scale empirical systems. Second, we take a small-scale VAR as given and ask what would happen to the perceived match between the theory and the data when the DGP includes additional disturbances. In both cases, the gap between the theory and the VAR of the data may be larger than previously thought.

Although it is tempting to associate cross-sectional deformation with the elimination of theoretical controls and time deformation with the elimination of theoretical states, such an association is imperfect. Time distortions emerge also when the empirical system contains all the endogenous states. Conversely, integrating out controls may induce both biases, if the relationship between the remaining controls and the states is altered.

While it is common to sweep deformation under the rug, distortions may be pervasive. For example, Central Banks use structural models with dozens of disturbances to interpret the data and academic researchers often twist standard models in estimation so that structural parameters become exogenous disturbances (e.g., an elasticity of substitution becomes a markup disturbance) to improve their fit. If there are more than two or three disturbances driving macroeconomic variables, it is difficult to take seriously the evidence small-scale VAR models deliver.

Clearly, employing a large-scale VAR can go a long way to ease deformation problems. However, while one can estimate large Bayesian VAR models, even with relatively short datasets, their identification is an issue. Hence, small-scale VARs are still likely to be preferred by macroeconomists. In that case, proceedings as in Sections III and IV, may inform users about potential issues, solidify inference, and avoid interpretation confusions. Changing the VAR information set can also help.

Are there empirical alternatives that could make the gap with the theory smaller? They do exist, but they have to be appropriately rigged to deliver the correct conclusions. For example, one may be able to reduce time deformation if FAVAR models are employed to build dynamic facts, provided factors are constructed using the omitted states. However, FAVARs do not necessarily eliminate cross-sectional distortions. In fact, statistical principal components are unlikely to properly combine classes of structural disturbances and to make the mapping between innovations and structural disturbances better behaved.

It has become common to use IV approaches to identify certain shocks and local projection techniques to compute dynamic responses in the data (see, e.g., Rossi 2018 for a survey). Would such methods reduce the deformation gap? They could, but a number of conditions need to be met. Take, for example, case 2 of Section II, where some states are absent from the empirical model. The DGP for the observables is a VARMA(2,1) which, in a companion from, can be written as $W_t = QW_{t-1} + R v_t$ where $W_t = [y_t', y_{t-1}']' v_t = [e_t', e_{t-1}']'$, $Q = \begin{pmatrix} F_{21} & F_{22} \\ I & 0 \end{pmatrix}$ and $R = \begin{pmatrix} \mathrm{G}_{20} & \mathrm{G}_{21} \\ 0 & 0 \end{pmatrix}$. Projecting $W_{t+h}, h = 1, 2, \ldots$ on $t-1$ information:

$$W_{t+h} = Q^{h+1} W_{t-1} + Q^h R v_{jt} + u_{t+h} \tag{40}$$

where v_{jt} is the disturbance of interest, $u_{t+h} = Q^h R v_{-jt} + Q^{h-1} R v_{t+1} + \ldots + R v_{t+h}$, and v_{-jt} are all the disturbances at t except the $j-th$ one. Because local projections do not rely on VAR innovations, they are less prone to cross-sectional deformation. However, for the projections to be successful in recovering $Q^h R$, the regressors of the projection equation should be W_{t-1} and v_{jt}. When v_{jt} is not observable, we need proxies that capture the effect of both e_{jt} and e_{jt-1}. In general, the conditions stated in Stock and Watson (2018) must be satisfied.

It is well known since Sims and Zha (2006) that if a structural disturbance is invertible in (x_t, y_t), it is unlikely to be invertible in $z_t = S[x_t, y_t]$ only. However, even in that case, impulse responses could be identified, if a proper IV procedure is used, see, e.g., Miranda-Agrippino and Ricco (2019) among others. Propositions 1 and 2 indicate that when deformation is present invertibility in z_t alone is a very low probability event and impulse response hard to identify. In addition, proper instruments may be difficult to find when deformation exists.

Our analysis has implications for two related strands of literature. Rather than using small-scale VARs to validate a theoretical mechanism, it is quite common to employ them to cross off theories inconsistent with the data (see, e.g., Galí 1999; Angeletos, Collard, and Dellas 2020). While the qualitative features of the responses are, at times, unchanged by deformation, magnitudes and persistences are generally affected. Thus, it is dangerous to exclude theories, say, using variance decomposition exercises or the magnitude of multipliers, as it is done in the literature.

It is also popular to estimate the parameters of a theoretical model by matching responses to certain disturbances in the VAR of the data and in the theory, see, e.g., Christiano, Eichenbaum, and Evans (2005). Limited information approaches may avoid certain forms of misspecification of the theoretical model in estimation. However, they are unsuited to reduce the gap that deformation creates unless a large-scale VAR is employed, see, e.g., Christiano, Trabant, and Walentin (2010). In fact, when the DGP features more disturbances than VAR variables, three conditions need to be met to make estimation meaningful. First, to avoid cross-sectional deformation, the theory should be reduced to the same observables used in the empirical model prior to the computation of the decision rules and to estimation. Second, to avoid time deformation, data responses should be computed using a generous lag length and carefully selected variables. Third, one needs to check that the disturbances of interest are identifiable in the small-scale empirical system using theory-based restrictions. When any of these conditions fail, parameter estimates become difficult to interpret.

Appendix A

Derivations of the Equations of the Model of Section II

Combining the optimality conditions yields the Euler equation

$$\alpha\beta E_t\left[\frac{C_t/B_t}{C_{t+1}/B_{t+1}} Z_{t+1}K_t^{\alpha-1}V_t\right] = 1.$$

We guess the solution for investment and consumption are constant fractions of output (up to preference shocks for consumption). Thus, for $0 < s < 1$: $I_t = K_t/V_t = sO_t, C_t = (1-s)O_tB_t$. Plugging the policy functions into the Euler equation, we have:

$$\alpha\beta E_t\left[\frac{(1-s)O_t}{(1-s)O_{t+1}} Z_{t+1}K_t^{\alpha-1}V_t\right] = \alpha\beta E_t\left[\frac{O_t}{O_{t+1}}\frac{O_{t+1}}{sO_t}\right] = 1$$

which holds when $s = \alpha\beta$.

Proof of the Propositions of Section III

PROOF OF PROPOSITION 1:
Simply match (26) with (16)–(17). ■

PROOF OF PROPOSITION 2:
To prove part (i), we first match (28) and (18)–(19). Then $u_{2t} = (S_2FS_2^* - \tilde{F}_2) \times (I - S_2FS_2^*L)^{-1}(G_2e_{t-1} + H_2x_{1t-2}) + G_2e_t + H_2x_{1t-1}$. Because x_{1t} has a VARMA(2,1) format: $M(L)x_{1t} = N(L)e_t$, where $M(L)$ is invertible, we have $u_{2t} = \lambda_2(L)e_t$, where $\lambda_2(L) = G_2 + (S_2FS_2^* - \tilde{F}_2)(I - S_2FS_2^*L)^{-1}\times (G_2 + H_2M(L)^{-1} N(L)L + H_2M(L)^{-1}N(L)L^2$. Matching (28) with (22) one similarly obtains that $u_{3t} = \lambda_3(L)e_t$. Part (ii) and (iii) are immediate. ■

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