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Compartmental market models in the digital economy—extension of the Bass model to complex economic systems

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ABSTRACT

Compartmental models are widely used in epidemiology, engineering, and physics to describe the temporal behavior of complex systems. This paper presents how compartmental models may be applied to the digital economy—more specifically, how the Bass model can be extended to more complex economics systems such as markets with customer churning, competition, multisided platforms, and online games. It is demonstrated that it is straightforward to establish the equations describing the various economic systems under study, however, the equations are often too complex to be solved analytically in the general case. Though the paper presents simple and idealized cases, the solutions may, nevertheless, uncover important strategic aspects that otherwise may be hidden by complexity in the general case, for example, the reasons for slow initial market growth. The paper also discusses how the developed models may be used to evaluate digital economic market evolution and business policy.

1. Introduction

A challenge when marketing new products is to make reasonable forecasts for how the market for the product evolves. In this paper, we promote the intuitively simple and powerful multi-compartmental model as one of the tools for doing such analyses. As for all forecasting methods, compartmental models only provide coarse estimates. This approach to modelling is in the mood of the aphorism: "All models are wrong, but some are useful" (Box, 1976); that is, it must always be borne in mind that models are only approximations, but they may still reveal important strategic truths.

The compartmental models are easy to visualize explicitly showing all compartments (e.g., customers in different states), flows between compartments, and feedback loops controlling the flow. It is equally simple to set up the differential equations for the model. Though compartmental models are simple to construct, they are often hard to solve analytically. Nevertheless, the model allows the product developer to experiment with different marketing models and investigate the effect change of system parameters may have on the market dynamics.

Two important strategic issues are to determine the asymptotic states in which the market may eventually settle, and the initial growth rate of the market (Øverby & Audestad, 2021a, 2021b). The latter is particularly important to avoid a prematurely termination of new services because the initial growth rate is small. If the market growth is subject to strong network effects, the growth may, initially, be slow followed by rapid growth towards market saturation (see Section 4.1). There are several examples in the digital economy where such market behavior is dominant (e.g., social networking services, instant messaging services, and the adaptation of the Internet itself). Most important, compartmental models—though not always solvable in terms of simple expressions—are simple and intuitive and illustrates how the system works and which dependencies exist in the system.

Compartmental models are mathematical models describing the dynamics of how, say, people move from one compartment to

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another (e.g., from potential customer to customer or from being active players to players having left the game). The dynamics can be described by differential equations or difference equations. In general, these equations can only be solved using numerical methods; however, analytical solutions can be found in some simple cases. Such cases are presented in this paper to show that important conclusions can be derived even for such simple models. For more complex analysis, system dynamics may be used (Moorcroft, 2015; Sterman, 2000). System dynamics is also a compartmental model, and the set of differential equations of the analytical model are directly transferable to a system dynamics model. However, mapping a system dynamic model to a system of coupled differential equations may not always be possible.

Other models are computationally more complex and include interacting agents (Fasli, 2011; Tesfatsion., 2002) and Polya urns (Arthur et al., 1986; Øverby et al., 2012). Forecasting estimates can also be based on empiricism and curve fitting; that is, based on observations of how actual businesses evolve and then fitting them to common statistical distributions or other models (Meade & Islam, 2006; Radas, 2005). Since two businesses are not necessarily identical even if they are producing the same good, empiricism may not uncover the underlying mechanisms causing market success or failure (Forrester et al., 2003; Radzicki and Meyers, 2009). However, empiricism may be useful for estimating the range of likely parameter values to be used in compartmental and system dynamic models.

The structure of this paper is as follows. Section 2 discusses some key issues concerning digital products. Section 3 provides a short introduction to compartmental modeling. In Section 4, compartmental models are applied to four central business areas in digital economics:

- Section 4.1 investigates the effect the size of the group influencing other users has on the market growth. One key result is that even small groups of people (about 10 people) will have the same influential effect as the whole population. Another observation is that if the group is smaller than this, it may take a long time before a significant portion of the potential market has been captured. This is based on resent work by Manshadi et al. (Manshadi et al., 2020).
- Section 4.2 contains a multi-compartmental model for competition between several companies offering the same or equivalent services. The simple mathematical model shows that, if all customers are disloyal customers, the final state of the market depends only on the churning rates and not on how the market reached saturation. In some cases, this leads to markets where one of the competitors captures the whole market.
- Section 4.3 investigates the effect of cross-side network effects. It is shown that cross-side network effects may sometimes just enhance the growth of the dependent markets (e.g., the production of video clips on YouTube is stimulated by the number of viewers), while, in other cases, the cross-side network effect may induce considerable latency for the dependent market (e.g., the advertisement market of Facebook will not commence growing unless there are several users of the social media platform).
- Section 4.4 contains a three-compartmental model for massively multiplayer online games (MMOG). This model is similar to the SIR model in epidemics, but the dynamics of MMOGs is, in general, richer than that of the SIR model, in particular, concerning the form of the flow parameters and readoption processes. The model is also applicable to goods and services where the product is replaced by new versions or by equivalent products.

The set of differential equations describing the dynamics of each model is included in the main text to show which mechanisms are at play in each case. For the planner and strategist, this is important information because it reveals otherwise hidden problems (e.g., possible long latency) that should be considered in the early planning stages. The main text also includes solution of the equations for some simple cases. For deeper analysis the reader is referred to other publications by the authors. Mathematical derivations are included in appendices for two reasons: for proof of the validity of the results in the main text and as basis for developing future models. The models presented in Section 4 are based on previous works by the authors (Audestad, 2015; Idland et al., 2015; Øverby and Audestad, 2019, 2021a, 2021b). Finally, Section 5 concludes the paper.

2. Key issues shaping the digital economy

The information and communication technology (ICT) has been the key factor shaping the digital economy. This includes the production of electronic services (e-mail, messaging, and social media) and electronic goods (e-books, apps, web pages, and interactive games), distribution of goods and services over the Internet and other electronic media (e-books and web pages), electronic payment systems (PayPal), electronic interaction between suppliers and consumers (e-commerce), new ways of organizing businesses (commons-based peer production, crowdsourcing, and resource sharing), and evolution of new network capabilities (smartphone technology, real-time streaming, and Internet of Things). The evolution of digital services and applications took off during the early 1990s as a result of the development of the digital mobile phone (GSM) (1991) and the commercialization of the World Wide Web and the Internet (1993). Since then, the Internet technology has replaced the traditional telephone service and, to a large extent, also the broadcast services and, combined with mobile network technology, made telecommunications independent of physical location. Eventually, all telecommunications services will be successfully integrated into a common technology allowing all services—voice, picture, video, messaging, remote monitoring, and remote control—to be carried by the Internet and received on the same user equipment. The timeline for this evolution is found in Chapters 2 and 3 of (Øverby and Audestad, 2021b).

Key issues in the ICT business are the structure of the Internet, the way value is created, network effects, lock-in, and the drivers of the temporal evolution of the market. These subjects form the basis for formulating the compartmental models in each case, as detailed below:

a) The protocol structure of the Internet has split the Internet business into two separate business domains: Internet Service Provider (ISP) and Application Service Provider (ASP). The technical reasons for this are explained in (Audestad, 2007; Øverby & Audestad, 2021a, 2021b). A company may both be an ISP and an ASP; however, national regulations and network neutrality usually require that these business domains are economically separated (Blackman & Srivastava, 2011) to avoid undue competitive advantages. The ISP offers fixed and mobile Internet access, Internet transit, hosting services, processing platforms, and certain cloud services. The ASP offers access to software, web and app services, social media, content, and entertainment. The ASP is a customer of one or more ISPs. Because of the protocol structure of the Internet, message encryption, and over-the-top services, the ISP cannot identify the type of service (voice, picture, film, web search, games, monitoring, remote control, etc.) provided to the users. This puts constraints on the pricing and billing regimes of the ISP. We will not go into this important area in detail but encourage readers to a more detailed treatment in the works by Steinmauer and Gaber in Steinmauer (2014) and Gaber (2005). On the other hand, the ASP can choose any billing method for their customers, for example, monthly charge, pay per view, or freemium (Anderson, 2009). This opens for business opportunities not possible before the Internet era.
b) Several of the businesses in the digital economy are value networks (Stabell & Fjeldstad, 1998). The value network is a business that mediates between users in the same group (e.g., social media users on Facebook), between users in different user groups (e.g., between producers of videos and viewers on YouTube), or both (e.g., simultaneous mediation between social media users and between social media users and advertisers on Facebook). Examples of digital value networks include mobile network providers, electronic newspapers, Facebook, YouTube, Netflix, Uber, eBay, and Airbnb. The value network consists of four components: the organization providing the services, the users of the services and their interaction (the network), services supporting this interaction, and contracts permitting users access to the services. This value generation model is important for determination of the drivers of the temporal evolution of the service: size of network, type of interactions resulting in network effects, type of service, usage conditions, etc.
c) Network effects means that the value of adopting the service depends on the number of existing users of the service. Hence, the network effect is the result of positive feedback from the market (e.g., word of mouth). The feedback may either cause the market to grow until it saturates or enters a stable state (virtuous cycle) or decline until there are no users left (vicious cycle) (Shapiro & Varian, 1998; Øverby & Audestad, 2021a, 2021b). In some cases, the market for particular services does not start to increase unless there are some users there already. Examples include Facebook, Myspace, LinkedIn, and the picturephone—the service is of no value to the first user. For such services, the initial growth of the market is slow, and it may take several years before the revenues turn positive (e.g., see (Øverby & Audestad, 2021a, 2021b) for a discussion of network effects in Facebook).
d) Lock-in incorporates all mechanisms the supplier may apply to keeping the customers by building up barriers that prevent customers switching to competitors (Shapiro & Varian, 1998; Øverby & Audestad, 2021a, 2021b). In some markets, for example, the mobile market, the switching barriers are low, and customer churning may be frequent. The churning rates determines, as is shown in Section 4.2, the long-term market shares of each competitor. In other markets, the barriers are huge and grows larger as the market for the service evolves, leading to de facto monopolies (e.g., Facebook). Lock-in is often the result of strong network effects.
e) Temporal market evolution is about how fast the market evolves and the final state in which the market may settle. Key questions when marketing a new digital product are: is there a market for the product, are there forces that may enhance or hamper the adoption process (e.g., network effects or lock-in barriers), how fast will the product be adopted by the market, and when and under which conditions will it become profitable?

This paper elucidates some of the questions raised in e) for businesses in the digital economy that are subject to network effects. To do so, we use simple and intuitive compartmental models inspired by epidemiology and biology.

3. Compartmental modeling

3.1. General

The basis for all the compartmental models presented below is the "decay equation":

$$\frac{dA}{dt} = -\lambda A,$$

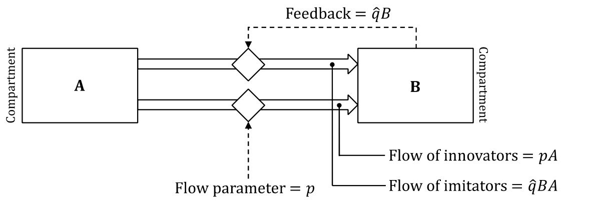
in which dA/dt is the change in the content of a compartment (e.g., the number of potential adopters), A is the current content of the compartment (number of potential adopters), and λ is the intensity by which the content is sent out of the compartment (e.g., λ may represent the adoption rate of a good or service); λ may be constant, depend on time, or depend on other variables (e.g., the number of actual adopters). The equation simply states that the change in the number of potential adopters per unit of time is proportional to the current number of potential adopters. This principle is well founded in physics (e.g., radioactive decay) and biology (e.g., evolution of species) and should be applicable, as a good approximation, to the evolution of digital markets as well: the population of potential adopters is a homogenous group that can be characterized by an average behavior (a uniform adoption rate). The strength of this method, even though it may not provide an accurate reproduction of the evolution, is that it may reveal the basic behavior of the evolution and, therefore, be of explanatory value.

One of the simplest temporal models is the diffusion model of innovations. Because the model can be applied to several different economic problems, in particular, market forecasting, there is a vast literature on diffusion models. The seminal book on diffusion of innovation is the book by Rogers first published in 1962 (Rogers, 2003). This book is mostly a non-mathematical and empirical description of the diffusion process. In 1969, Frank Bass (Bass, 1969) published a mathematical model for the market evolution of consumer durables based on statistical arguments (the Bass diffusion model). An overview of mathematical diffusion models used for forecasting is found in Meade & Islam (2006) and Radas (2005). The models presented there are partly based on theoretical assumptions and partly on empiricism. An overview of statistical distributions that are useful in this type of forecasting is found in Meade & Islam (1998).

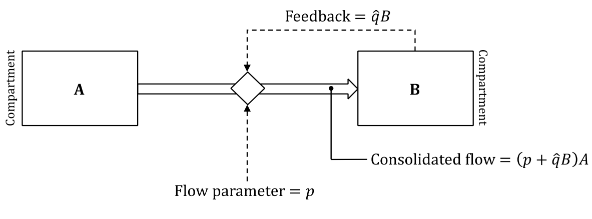
In this paper, we will use compartmental models to describe the dynamics of the market. Compartmental models are common tools for describing the dynamics of complex systems, for example, in epidemiology (Brauer et al., 2008), pharmacokinetics (Jambhekar & Breen, 2012), biology (Murray, 2001), medicine (Hoppensteadt & Peskin, 2002), chemistry (Jourdan et al., 2019), social sciences (Bissell et al., 2014), and system dynamic modelling of complex industrial, economic, and societal systems (Moorcroft, 2015; Radzicki and Meyers, 2009; Sterman, 2000). A complex system consists of compartments containing contents (e.g., people, money, goods, materials, viruses, or energy), flows of contents between the different compartments, and internal and external forces and feedback loops regulating these flows. If the content of the compartments is homogenous and the flows are smooth functions of time, the dynamics of the system can be modelled as a coupled set of nonlinear differential equations. Quite often, even these simple models are too complex to be solved by analytical methods. In such cases, simulation based on system dynamics may be applied. System dynamics is also a compartmental model based on the same principles as the analytical methods (see (Moorcroft, 2015) or (Sterman, 2000) for a comprehensive introduction to system dynamics). System dynamics are also used as a tool to simulate more realistic models and complex systems in which analytical models are out of scope.

Compartmental models are commonly used in epidemiology. The first compartmental models were developed during the 1920s and have since become important tools in several fields of medicine (Hoppensteadt & Peskin, 2002). The objectives of these models are to analyze how diseases spread, to determine epidemiological parameters (e.g., reproduction number, recovery rates, and death rates), and to estimate the effect of countermeasures (e.g., vaccination). There is a vast number of publications in this field. Two comprehensive overviews of the mathematical methods used in epidemiology are found in Brauer et al (2008) and Capasso (2008). During the last two years, several papers have been published in which compartmental models are applied to analysis of the COVID-19 pandemic—two examples are (Leontitsis et al., 2021) and (Abreu et al., 2021). A more general and deeper mathematical approach to compartmental models is found in Jacquez & Simon (1993). The market models studied in this paper are inspired by the epidemiological models since spreading of information, sales of goods, and participation in online games have several characteristics in common with epidemiology.

Compartmental methods are not frequently used in economics. Neither is system dynamics frequently used in mainstream economics despite its potential to handle complex issues. In economics, system dynamics has usually been promoted and applied by researchers not trained as professional economists (Forrester et al., 2003; Radzicki and Meyers, 2009). In the Handbook of Mathematical Economy (Arrow & Intriligator, 1982), compartmental methods and system dynamics are not mentioned at all. In our research, we have found just a few examples where compartmental models have been applied to economic problems (see, e.g. (Castellano et al., 2009; Tramontana & Gallegati, 2010; Yang et al., 2019)) but none of them apply differential equations to study the dynamics of the economic system. A dynamic model based on system theory, a cousin of compartmental modeling and system dynamics, is presented by Granstrand in Granstrand et al. (1991) and Granstrand (1994). This is a quite complex and universal model for linked buyer and seller diffusion describing how diffusion of innovations takes place both among and between sellers and buyers under technological change. The model also includes the effects of price, performance, cost, and other drivers of the diffusion process. The dynamics of multi-phased projects is discussed in Ford & Sterman (1998) using a system dynamic approach. The model consists of five



(a) Imitators and innovators as separate flows



(b) Consolidated flows

Fig. 1. Example of a compartment model with two compartments (A and B), in which the flows of imitators and innovators shown as separate flows (a) and as a consolidated flow (b).

compartments (or stocks) describing the production process from tasks not completed to tasks completed and released. The model is shown both as system dynamic diagrams and the corresponding set of differential equations. An electronic chip development project is used to calibrate the model. In previous research, we have used compartmental models to study temporal evolution of various types of markets in (Audestad, 2015; Idland et al., 2015; Øverby and Audestad, 2019, 2021a, 2021b). After extensive search, we have not identified other papers in the economics literature addressing market evolution in the same way.

The current paper is a comprehensive and pedagogical presentation of the results of our research discussed in the context of business and economic policy. Since studies on temporal market evolution based on compartmental methods similar to those of epidemiology and biology seems to be new in economics, the paper is written in a tutorial manner only considering simple examples. More complex cases are contained in the papers referred to above.

3.2. Simple two-compartmental model

Fig. 1 shows a simple model consisting of two compartments—A and B—and the flow of contents—e.g., people, goods, or money—from compartment A to compartment B. Applied to a market, compartment A consists of potential customers who have not bought the good yet, and compartment B consists of customers having bought the good. This model is a closed model since the total number of customers is constant over time; that is, $A + B = \text{constant} = N$. In an open model, $A + B$ is no longer constant, and the model contains birth and death processes (or creation and annihilation processes).

All models in this paper are closed models since open models usually lead to differential equations that cannot be solved by quadrature. In a closed model, the decay equation is $dB/dt = p(N - B) + \widehat{q}B(N - B)$, in which we have set $A = N - B$ (see Fig. 1(a)). The two flows can be consolidated as a single flow with flow intensity parameter $\lambda = p + \widehat{q}B$ as shown in Fig. 1(b). The term p is the intensity of spontaneous adopters (innovators in the terminology of Bass), and $\widehat{q}B$ is the intensity of stimulated adopters (imitators) (the network effect). Normalizing this equation setting $B = Nu$ and $q = N\widehat{q}$, the decay equation becomes:

$$\frac{du}{dt} = (p + qu)(1 - u).$$

Since the relative number of sales before time t is equal to the cumulative probability of sales before time t, or $u(t) = F(t)$, this equation is identical to the Bass diffusion equation (Bass, 1969).

Most of the literature on the diffusion of new products (or innovations) based on the work of Bass is concerned with parameter estimation (p and q), forecasting, qualitative and quantitative models, and analysis of real cases. Examples include Peers et al. (Peers et al., 2010) extending the Bass model to include seasonality; Bauckhage et al. (Bauckhage et al., 2013, 2014; Bauckhage & Kerstin, 2014) using the model to investigate adoption patterns of several social media services and compare them to the Bass distribution and other statistical distributions (e.g., the Weibull distribution and the shifted Gompertz distribution); and Manshadi et al. (Manshadi et al., 2020) deriving a modified Bass equation for the adoption rate when interactions take place in small groups of acquaintances and not between all potential adopters as in the original Bass model. In this model there are no innovators. The model suggests the following modified differential equation for the cumulative probability:

$$\frac{du}{dt} = q\left[1 - (1 - u)^{1-\frac{2}{k}}\right](1 - u),$$

in which k is the size of each group; that is, each member of the group interacts with exactly k other participants of the group. In graph theory, this type of network is called a k-regular graph—see, for example (Chartrand & Zhang, 2012) for k-regular predictive graphs or (Bollobás, 2001) for k-regular random graphs. For large k, the equation is identical to the Bass equation with only imitators since $\lim_{k\to\infty}\left[1 - (1 - u)^{1-\frac{2}{k}}\right] = u$.

The analysis of two-compartment models in Section 4.1 is based on this model. This model is important because it shows, in contrast to the Bass equation, explicitly the effect of small groups on the diffusion process.

4. Applications

This section considers four different models: the two-compartment model for diffusion of innovations and sales of goods (Section 4.1), competition between two or more providers (Section 4.2), the behavior of multisided platforms (Section 4.3), and distributed online games (Section 4.4). We present numerical examples of cases for which analytically tractable solutions exist. The numerical choice of flow parameters does not reflect particular business cases—the flow parameters are based on estimates given in (Øverby & Audestad, 2021a, 2021b) and (Øverby & Audestad, 2019). The main text contains the description of the model, the main results of the calculations, and some strategic observations. Detailed calculations are contained in the appendices.

4.1. Two-compartment model

In this section, we investigate how the size of the group of people stimulating new customers to buy the product may have on the sales of the product, in particular, the time it may take before the sales-volume becomes big enough to sustain the production.

As shown in Section 3, the Bass model is equivalent to a two-compartment market model. This section analyzes the modified Bass equation proposed by Manshadi et al. (Manshadi et al., 2020) to study the effect of the strength of the market feedback. The model includes the original Bass model with only imitators as a special case. Explicit solution of the market equation is not found elsewhere in the marketing literature. Replacing the cumulative probability, F, by the relative number of people having purchased the good, u, the two-compartment model takes the following form:

$$\frac{du}{dt} = q\left[1 - (1-u)^{1-\frac{2}{k}}\right](1-u),$$

in which u is the relative number of goods sold at time t. The parameter k is a measure for the strength of the market feedback since each individual stimulates exactly k other individuals to buy the good—the smaller k is, the weaker is the market feedback or network effect. In technical terms, the diffusion takes place over a k-regular graph (see (Chartrand & Zhang, 2012) or (Bollobás, 2001) for definition and properties – note, for example, that in a k-regular graph $k \geq 3$). We also observe that, as for the Bass equation with only imitators, the supplier must establish an initial customer base, u_0, before the market starts increasing. Explicit solution of this equation is not found elsewhere in the literature and is, for completeness, derived in Appendix A. The solution is:

$$u = 1 - \frac{1-u_0}{\left[(1-u_0)^{\alpha} + (1-(1-u_0)^{\alpha})e^{\alpha q t}\right]^{1/\alpha}}.$$

For simplicity of notation, we have set $\alpha = 1 - \frac{2}{k}$. Observe that α is a discrete variable with values $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \cdots, 1$ for $k = 3, 4, 5, 6, \cdots, \infty$. Note that $k \geq 3$for a k-regular connected graph. For $k = \infty$, the equation is identical to the Bass equation with only imitators—i.e., $p = 0$—in which case all users are connected to each other.

Fig. 2 shows u as a function of t for $k = \{3, 5, 8, 10, \infty\}$, $u_0 = 0.01$, and $q = 0.46$. The solution reveals several important strategic issues (see Appendix A for the mathematical derivation of latency and inflection point):

1. As already mentioned, an initial base of customers must exist for the market to start increasing.
2. The user adoption u follows an S-curve. This is expected since users adopts the service only because other users have adopted the service—they are all imitators.
3. The strength of the network effect—i.e., the parameter k—determines how fast the market grows and reaches saturation. Observe that the rate of adoption for moderate value of k (e.g., $k = 10$) approaches the rate of adoption for $k \to \infty$. This indicates that each individual only needs to connect to a few other individuals for the maximum network effect to take place.
4. The time taken before the number of customers is large enough to sustain the service (the latency) is usually long as shown in Table 1. Latency decreases as k increases.
5. The demand (i.e., du/dt) increases until it reaches the inflection point at which the demand reaches its maximum; thereafter, the demand diminishes gradually towards zero. Diminishing demand may then just indicate that the market size has increased beyond the inflection point and not that the popularity of the product has diminished. The inflection point is shown in Table 1 for various values of k.

Observe that the strength of the network effect significantly affects the time to reach threshold and, in our example, varies from 5 years to 15 years depending on the parameter k. In a business environment, this may have considerable impacts on earnings and

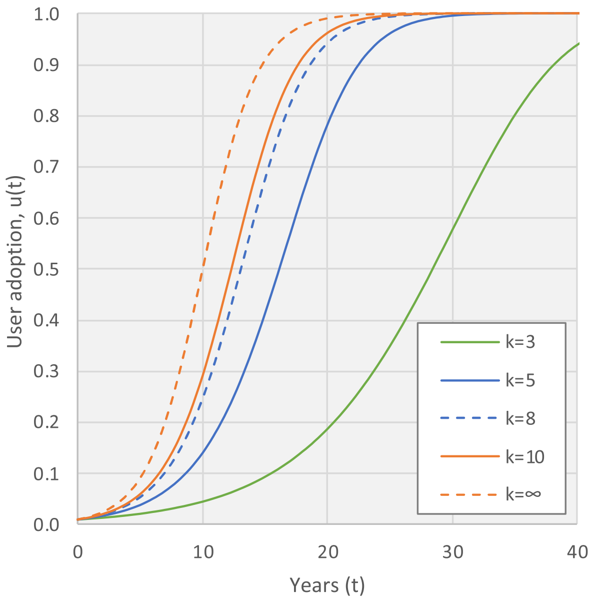


Fig. 2. User adoption $u(t)$ for parameters $u_0 = 0.01$, $q = 0.46$, and $k = \{3, 5, 8, 10, \infty\}$.

whether the service manage to stay in the market or is terminated. Note that, in this case, the time it takes to reach the inflection point is twice as long as the latency.

4.2. Competition and churning

In this section, we show that in a market with churning, the final equilibrium state into which the market may eventually settle, depends on the churning rates only (Audestad, 2015). This model assumes, however, that all users are likely to switch to another supplier from time to time. The model may easily be extended to the case where there are loyal users. In this case the market share of each supplier will consist of loyal customers plus a share of disloyal customers determined by the churning rates. It is also shown that, in some cases, one of the suppliers may end up as a monopoly.

In the main text, we will discuss the case where two providers compete for the market. In Appendix B, the general case of any number of providers is discussed. A two-provider model refers to a duopoly in which the user can only choose between two providers. A user will only be affiliated with (e.g., subscribing to) one of the providers. However, users may freely churn between providers at any time.

Fig. 3 shows the compartmental model (without birth and death processes) for two competing providers sharing a common market (e.g., the mobile smartphone market). There are three compartments of users:

- Potential users not having bought the good ($1-u-v$).
- Users having bought the good from Provider 1 (u).
- Users having bought the good from Provider 2 (v).

There are four flows:

- Flow of users buying the good from Provider 1 $(p_1+q_1u)(1-u-v)$.
- Flow of users buying the good from Provider 2 $(p_2+q_2v)(1-u-v)$.
- Users churning from Provider 1 to Provider 2 $(r_{12}+s_{12}v)u$.
- Users churning from Provider 2 to Provider 1 $(r_{21}+s_{21}u)v$.

The feedback loops controlling each flow are also shown in the figure. It is a reasonable assumption that the churning flows are in accordance with the decay equation given in Section 3.1, that is, containing both a spontaneous component (r_{ij}) and a stimulated component (network effect) (s_{ij}). It is, moreover, assumed that all customers are equally likely to churn, that is, there is no loyal customer base. If there are both loyal and disloyal customers, then the model can be used to study how disloyal customers distribute themselves among the competitors. The stable asymptotic market share for each competitor is then the sum of loyal users and a share of disloyal users.

This leads to a set of two coupled first order differential equations:

$$\frac{du}{dt}=(p_1+q_1u)(1-u-v)+(r_{21}+s_{21}u)v-(r_{12}+s_{12}v)u$$
$$\frac{dv}{dt}=(p_2+q_2v)(1-u-v)+(r_{12}+s_{12}v)u-(r_{21}+s_{21}u)v.$$

This set of differential equations cannot be solved by quadrature in the general case. In Appendix B, a simple solution is derived for the case where all customers are innovators ($q_1=q_2=0$), and the churning is spontaneous ($s_{12}=s_{21}=0$). The general case of any number of providers is discussed in Appendix B. For two providers the solution is:

$$u=\frac{r_{21}}{r_{12}+r_{21}}-\frac{p_2r_{21}-p_1r_{12}}{(r_{12}+r_{21})(p_1+p_2-r_{12}-r_{21})}e^{-(r_{12}+r_{21})t}-\frac{p_1-r_{21}}{p_1+p_2-r_{12}-r_{21}}e^{-(p_1+p_2)t},$$

$$v=\frac{r_{12}}{r_{12}+r_{21}}+\frac{p_2r_{21}-p_1r_{12}}{(r_{12}+r_{21})(p_1+p_2-r_{12}-r_{21})}e^{-(r_{12}+r_{21})t}-\frac{p_2-r_{12}}{p_1+p_2-r_{12}-r_{21}}e^{-(p_1+p_2)t}.$$

The market evolution for this case is shown in Fig. 4 for $p_1=0.23, p_2=2p_1, r_{12}=0.02$, and $r_{21}=4r_{12}$. Observe that the market share for provider 2 increases at about twice the rate compared to provider 1 since $p_2=2p_1$. However, since $r_{21}=4r_{12}$, users affiliated with

Table 1
Latency (t_T) and inflection point (t_{infl}, u_{infl}) for $q=0.46$, $u_0=0.01$, and $u_T=0.1$.

k	t_T	(t_{infl}, u_{infl})
3	15	(30, 0.58)
4	10	(20, 0.56)
5	8.3	(17, 0.54)
10	6.3	(13, 0.52)
∞	5.0	(10, 0.50)

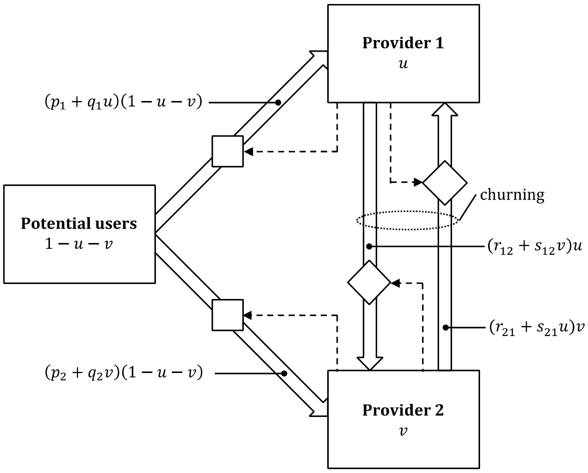


Fig. 3. Competition and churning between two providers. Both spontaneous and stimulated churning are shown. In this model, eventually, all users will be associated with either provider 1 or provider 2.

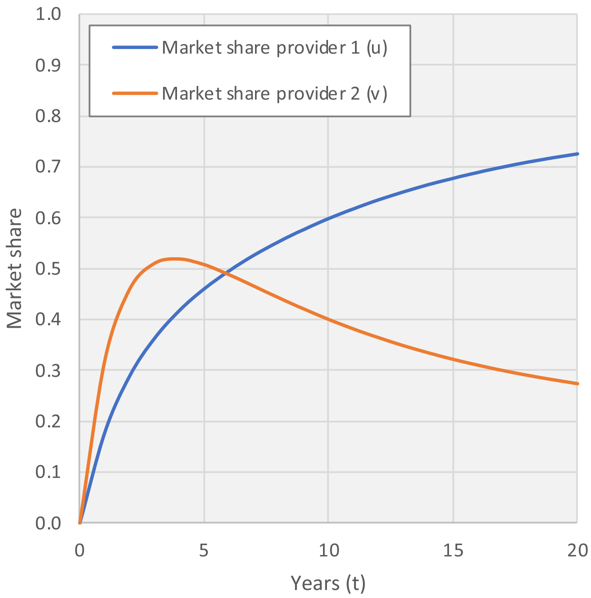


Fig. 4. Market evolution for the case with competition between two providers and churning. The parameters used are $p_1 = 2p_2$, $p_2 = 0.46$, $r_{12} = 0.02$, and $r_{21} = 4r_{12}$.

provider 2 will churn at a higher rate to provider 1 compared to the churn from provider 1 to provider 2. Over time, all users will be affiliated with either provider 1 or 2, and the churning effect results in that approximately $r_{21}/(r_{12} + r_{21}) = 80\%$ of the users will eventually be affiliated with provider 1 and 20% with provider 2.

One commercially important case is $r_{12} = 0$; that is, there is no churning from user category 1 to category 2. As shown in Chapter 11 of (Øverby & Audestad, 2021b), this model can be used to describe markets that develop toward natural monopolies because the market does not accept two or more equivalent services or technologies. This was the case for the standards war between VHS and Betamax in the 1980s. Here, the market never settled at a stable state where the two incompatible technologies offering identical services shared the market but ended up in a monopoly with VHS capturing the whole market.

The solution is in this case:

$$u = 1 - \frac{p_2}{p_1 + p_2 - r_{21}} e^{-r_{21}t} - \frac{p_1 - r_{21}}{p_1 + p_2 - r_{21}} e^{-(p_1+p_2)t},$$

$$v = \frac{p_2}{p_1 + p_2 - r_{21}} e^{-r_{21}t} - \frac{p_2}{p_1 + p_2 - r_{21}} e^{-(p_1+p_2)t}.$$

The market evolution is shown in Fig. 5 for the case where the initial growth is identical for the two goods ($p_1 = p_2$). Observe that both providers increase their market share initially, however, at $t \approx 2$, the churning effect from provider 2 to provider 1 dominates. Eventually, provider 1 becomes a monopolist.

Two important strategic observation based on Appendix B, are:

1. In the case in which the market settles in a stable final state, the market share of each supplier is determined only by the churning parameters and is independent of how the market grew to reach that state.
2. There are cases in which the market ends up as a monopoly. In particular, this is the case when all churning is stimulated, resulting in a winner-takes-all market.

By observing and estimating the model parameters, the model can be used to evaluate strengths and weaknesses when launching new products in a competitive market.

4.3. Multisided platforms

A comprehensive analysis of multisided platforms is given in (Øverby & Audestad, 2021a). In addition to a general discussion of multisided platforms (MSPs), the paper contains coupled sets of differential equations for generic classes of two-sided platforms. The current paper considers only the simplest of these models to illustrate the impact cross-side network effects may have on the overall business opportunities of the platform, for example, the impact the number of customers of one product has on the sales of another product offered by the platform. Applied to Facebook, this model matched well with the observed growth of media users and the revenues from advertisements (, Øverby & Audestad, 2021b).

The model and the results of some simple calculations are presented next in Figs. 6, 7, and 8. The sales of product 1 (A_1) is developing in the natural way, i.e., without network effects, while the sales of product 2 (A_2) depend on the amount of product 1 already sold (U_1)—the cross-side network effect. The set of decay equations can then be written

$$\frac{dA_1}{dt} = -p_1 A_1,$$

$$\frac{dA_2}{dt} = -p_2 f_{12} A_2,$$

in which f_{12} is the cross-side network effect. Detailed calculations are contained in Appendix C.

There are two special cases for the cross-side network effect. In the first case, the cross-side network effect only enhances the sales of product 2. The network effect is then $f_{12} = 1 + r_{12} u_1$. Fig. 7 shows how this type of market evolves for different values of the parameter r_{12}. In the example, we have set $p_1 = 2p_2 = 0.46$ to illustrate how the two markets grow independently of each other also in the case where there is no network effect ($r_{12} = 0$). The curves take similar forms for other values of p_1 and p_2. The essential result is that for any value of t, the number of users in user group 2 increases as r_{12} increases because of the cross-side network effect generated by user

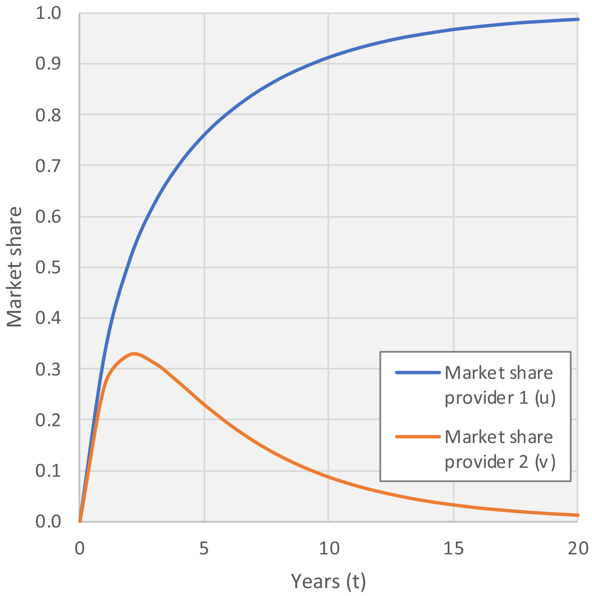


Fig. 5. Market share for provider 1 and 2 in the case of churning only from provider 2 to provider 1. The parameters used are $p_1 = p_2 = 0.46$ and $r_{21} = 0.2$.

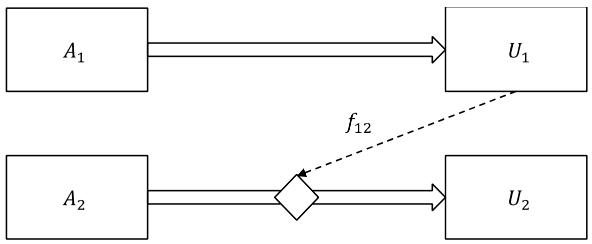


Fig. 6. Two-sided platform with cross-side network effect. A_1 and A_2 are potential users, U_1 and U_2 are actual users, and f_{12} is the cross-side network effect.

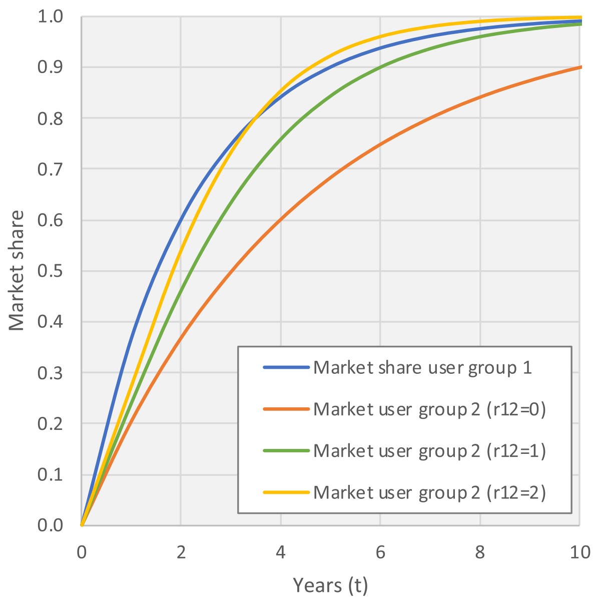


Fig. 7. Market evolution for the case $f_{12} = 1 + r_{12}u_1$ using $p_1 = 2p_2 = 0.46$ and $r_{12} = \{0, 1, 2\}$.

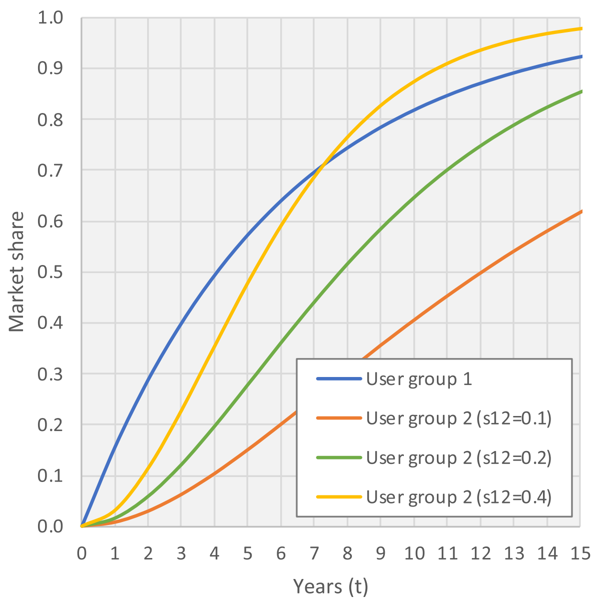


Fig. 8. Market evolution for the case $f_{12} = s_{12}u_1$ using $p_1 = 0.17$ and $s_{12} = \{0.1, 0.2, 0.4\}$.

group 1. In the second case, the network effect takes the form $f_{12} = s_{12}u_1$. This implies that there will be no users of category 2 if there are no users of category 1. Moreover, the intensity of capturing users of category 2 is proportional to the number of users of category 1.

Fig. 8 shows some examples of the market evolution for $p_1 = 0.17$ and $s_{12} = \{0.1, 0.2, 0.4\}$. Observe that the market of users of category 2 grows as an S-curve. This is expected since users of category 2 do not join the market unless there are users of category 1. Also observe that the number of users of category 2 may increase faster that the number of users of category 1 if the feedback is strong enough—for $s_{12} = 0.4$, user segment 2 reaches 90% market share about two years ahead of user segment 1.

The figure also indicates one of the fundamental strategic problems for MSPs where only one of the user groups generates revenues for the company (in this case, user group 2). If the feedback from user group 1 to user group 2 is weak, then it may take a long time before user group 2 contributes to the profit of the MSP. For $p_1 = 0.17$ and $s_{12} = 0.1$, it takes a little more than six month for user group 1 to capture 10% of the market, while it takes about four years until user group 2 reaches the same market share. This may lead to premature shutting down of the MSP because it may be wrongly concluded that it will never become profitable.

4.4. Distributed online games

A comprehensive analysis of massively multiplayer online games is found in (Øverby & Audestad, 2019), which includes: a comparison of the compartmental models of MMOGs with models derived from the epidemiological SIR (Susceptible-Infectious-Recovered) model, models for simple games, competitive games, and complementary games using both coupled sets of differential equations and system dynamics. We will not repeat these arguments here but complete our description of the usefulness of compartmental models in economics by including a general model for simple MMOGs and the solution of the differential equations for two simple cases. The model is shown in Fig. 9. Note that the structure of the general model for MMOGs is richer than the SIR model since the adoption rate of the flow from potential players also may depend on spontaneous adoption and not only on stimulated adoption and that the quitting rate also include players that are stimulated to quit the game. Moreover, readoption is a common event in actual games: introduction of new features may stimulate earlier players to rejoin the game.

The model consists of three compartments:

- *Buyers* are people who have not become an active player yet but are likely to join the game later.
- *Players* are the active players of the game.
- *Quitters* are previous players that have left the game for some reason.

The Bass equation is applied to each of the three flows:

- Flow of new players that join the game either of their own choice (*pu*) or because friends or other people have recommended the game (*qvu*).
- Flow of players leaving the game either because they no longer find the game stimulating (*rv*) or are doing so because other people, e.g., friends, are leaving the game (*swv*).
- Flow of previous players rejoining the game, e.g., because the game has been modified with appealing new features (*gw*) or friends that have rejoined the game (*hvw*).

The readoption flow needs further explanation. If the game stays unchanged throughout its lifetime, we may assume that the flow of re-adopters is negligible since there are no incentives stimulating users to readopt the game. Readoption is likely to take place only if the game is noticeably revised either by adding new features or modifying existing ones. These are discrete events, and the parameters g and h become time-dependent parameters. Realistic readoption flows can then only be studied using simulation techniques such as system dynamics. In (Øverby & Audestad, 2019), it is shown that the actual evolution of the game *World of Warcraft* can be reproduced, accurately, using a system-dynamics model based on Fig. 10. In this model, the effects of each update of the game are estimated (e. g., estimating g and h) to match the evolution of the game. All the flow parameters—and not only g and h—may be changed after each update since an update may also encourage current players to stay longer in the game. Section 4.4.2 consider several simple examples of the effect readoption may have on the evolution of the game. Section 4.4.1 shows the case of no readoption.

The model in Fig. 9 leads to the following set of differential equations:

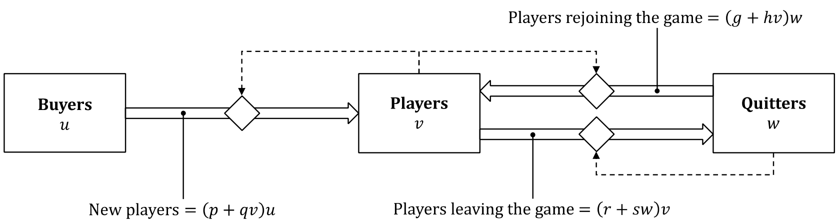


Fig. 9. Compartmental model for a Massively Multiplayer Online Game (MMOG).

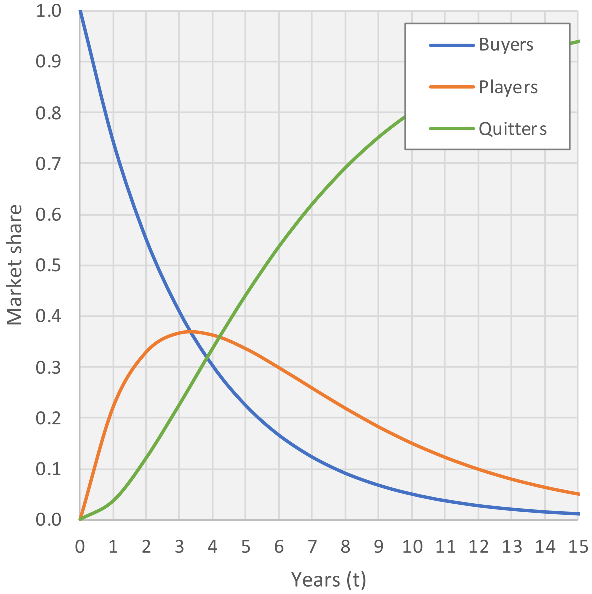


Fig. 10. The relative share of buyers, players, and quitters for the BPQ model without readoption. We have used the parameters $p = r = 0.3$.

$$\frac{du}{dt} = -(p + qv)u,$$

$$\frac{dv}{dt} = (p + qv)u - (r + sw)v + (g + hv)w,$$

$$\frac{dw}{dt} = (r + sw)v - (g + hv)w,$$

in which p, q, r, s, g, h are the flow parameters. Adding the equations, we get

$$\frac{d}{dt}(u + v + w) = 0$$

leading to the conservation equation $u + v + w = 1$.

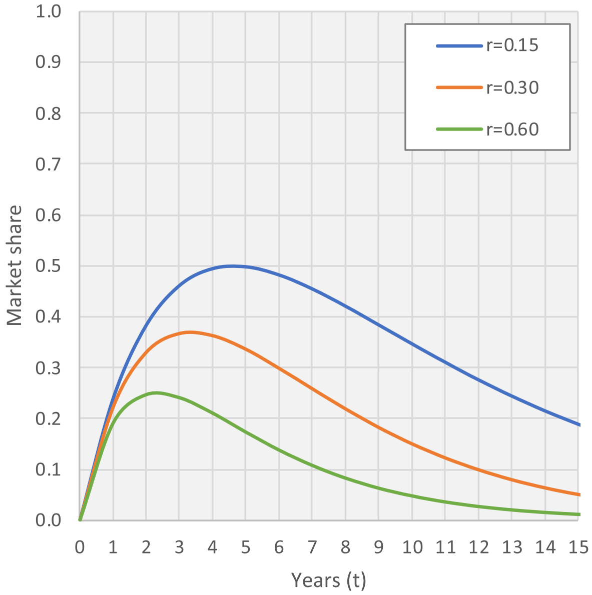


Fig. 11. The relative share of players for $p = 0.30$ and $r = \{0.15, 0.30, 0.60\}$.

4.4.1. Case with No readoption

In the case with no readoption flow and only innovators joining and leaving the game (i.e., $q = s = g = h = 0$) the solution is (see Appendix D for derivation):

$$v = \frac{p}{p-r}\left(e^{-rt} - e^{-pt}\right).$$

The number of players reaches its maximum at time:

$$t_m = \frac{\ln p - \ln r}{p - r}.$$

In this case,

$$v_m = \frac{p}{p-r}\left[e^{-rt_m} - e^{-pt_m}\right].$$

In the special case with $p = r$, we find that $v = pte^{-pt}$. The maximum $v_m = e^{-1}$ occurs at time $t_m = p^{-1}$. Even this simple model may be used to analyze a potential game to establish first estimates for the profitability of the game based on various market assumptions.

Fig. 10 shows the number of buyers, players, and quitters as a function of time for $p = r = 0.3$. The form of the curves is typical for all simple games with constant flow parameters and no re-adopters. Observe that the number of players increases until $t \approx 3.5$, at which time more players quit the game than new players (buyers) join the game. Eventually, all players will have quit the game. Fig. 11 compares the number of players as a function of time for three games with $r = 0.15, r = 0.30$, and $r = 0.60$. Observe that the maximum market share increases as r increases.

4.4.2. Case with readoption

In the case with readoption, the spontaneous and stimulated readoption rates are non-zero, i.e., $g, h \neq 0$. The resulting differential equations do not have a general analytical tractable solution. However, we find that the relative number of active players approaches the limits at $t \to \infty$ as follows:

$$v_\infty = \frac{g + r + s - h \pm \sqrt{(g + r + s - h)^2 - 4g(s - h)}}{2(s - h)}, \text{ for } s \neq h$$

$$v_\infty = \frac{g}{(g + r)} \quad \text{for } s = h.$$

Further in this section we evaluate the equations from Section 4.4 numerically. Fig. 12 shows the evolution of the relative market share for $g = 0.3$, and $h = 0.5$. Observe that relative share of active players reaches an equilibrium of $v_\infty \approx 0.22$. This is because players that have quitted the game may readopt the game later in time.

Fig. 13 shows a scenario in which there is no readoption until $t = 4$, i.e., $g(t), h(t) = 0$ for $t < 4$. However, for $t \geq 4$, we set $g = 0.3$, and $h = 0.5$. This means that readoption is introduced at $t = 4$ due to e.g., a new expansion for game, major updates, or new content making the game more appealing. Observe that this dynamic event changes the behavior of the evolution of the market share. Also observe that the asymptotic value of the number of active players remain the same as in Fig. 14, i.e., $v_\infty \approx 0.22$.

Finally, we consider a scenario of higher complexity—to demonstrate the flexibility of the compartmental models. Assume that the game features periodic updates which momentarily increase the interest of the game. During these periodic updates, there is a surge of quitters that rejoin the game as players. We keep $h = 0.3$ throughout the simulation, but model $g(t)$ as a time-dependent function $g(t) = 9e^{-100 \cdot \sin^2(7.5t)}$. Hence, the periodic updates impact the spontaneous flow of users only. Fig. 14 illustrates this scenario and compares

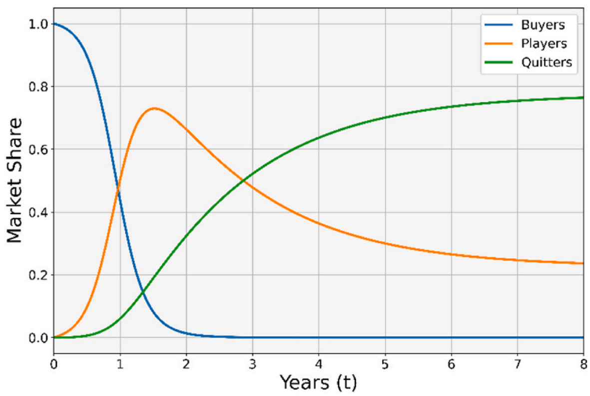


Fig. 12. Numerical solution to the BPQ model with constant rate of readoption, using $p = 0.1, q = 10, r = s = 0.8, g = 0.3$, and $h = 0.5$.

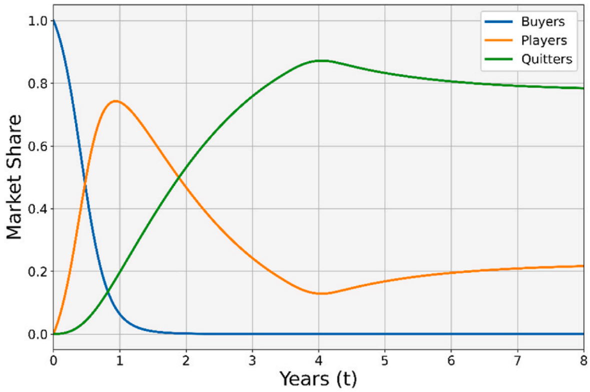


Fig. 13. Numerical solution to the BPQ model with no initial rate of readoption before when $t < 4$, and a constant rate of readoption $g = 0.3$, and $h = 0.5$ for $t \geq 4$. We have used the parameters $p = 1$, $q = 10$, and $r = s = 0.8$.

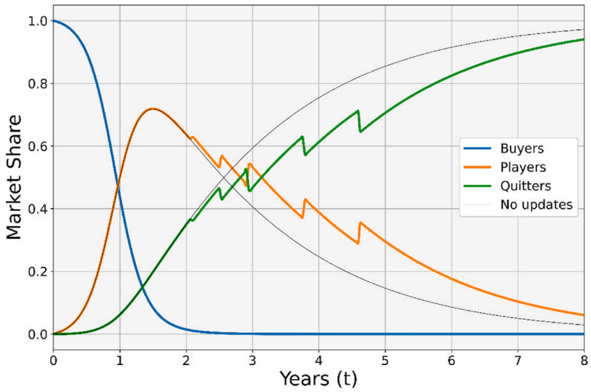


Fig. 14. Numerical solution to the BPQ model with decaying interest in the game and periodical updates that increases the interest. The parameters are $h = 0.3$ and $g(t) = 9e^{-100 \cdot \sin^2(7.5t)}$.

with a situation without periodic updates, i.e., $g(t) = 0$. We observe that the number of players decay slower compared to a situation without periodic updates, which may extend the lifespan of the game.

5. Conclusions

One of the most important tasks when planning a new product is to estimate the market penetration of the product and to uncover particular problems that may hamper the evolution such as latency associated with early market penetration, obstacles caused by existing lock-in barriers, and market saturation. This paper has demonstrated how the simple and intuitive multi-compartmental model can be applied to this type of analysis. The most important observation is that compartmental models are visual models showing compartments (e.g., customers in different states), flows between compartments, and feedback loops controlling the flow. This makes bookkeeping simple; that is, to ensure that all customer states, flows, and feedback loops have been encountered for. It is also easy to add new elements to the model. It is equally simple to set up the differential equations for the models. To study more complex behavior of the model, the model is easily converted to a more complex system dynamic model using one of the several graphical computer tools available for such model-building.

Simple examples of four different ICT service categories are discussed: the simple two-compartment diffusion model, multi-compartment model for competition, four-compartment model for two-sided platforms, and three-compartment model for MMOGs. In all cases, we have been able to draw important strategic conclusions based on simple analysis of the mathematical model. The simple compartmental model is easily extendable to system dynamics models to study more complex system behavior; for example, competition between multisided-platforms, competition and cooperation between several MMOGs, and interaction between ICT and other infrastructure, for example, production, logistics, commerce, and environment.

Future research should examine more complex models and, in particular, to identify whether there are other aspects of economics that may benefit from applying compartmental modelling. Other aspects include combining compartmental modeling and case studies, for comparing theory and empirical results, and for forecasting.

Data availability

Data will be made available on request.

Appendix A. Solution of the Modified Bass Equation

A.1 Solution of Differential Equation

The differential equation for the temporal evolution of the market is (Manshadi et al., 2020)

$$\frac{du}{dt} = q[1 - (1-u)^{\alpha}](1-u),$$

in which $\alpha = 1 - \frac{2}{k}$.

The differential equation is separable:

$$\frac{du}{[1-(1-u)^{\alpha}](1-u)} = qdt.$$

Changing variable

$$1 - u = v^{1/\alpha}, du = -\frac{1}{\alpha}v^{(1/\alpha)-1}dv,$$

gives

$$\frac{dv}{(1-v)v} = -\alpha q dt$$

With initial value $v_0 = (1-u_0)^{\alpha}$, the solution is

$$v = (1-u)^{\alpha} = \frac{(1-u_0)^{\alpha}}{(1-u_0)^{\alpha} + (1-(1-u_0)^{\alpha})e^{\alpha qt}}.$$

Solved for u, this gives:

$$u = 1 - \frac{1-u_0}{[(1-u_0)^{\alpha} + (1-(1-u_0)^{\alpha})e^{\alpha qt}]^{1/\alpha}}.$$

For $u_0 \ll 1$, this is simplified to

$$u = 1 - \frac{1}{(1+\alpha u_0 e^{\alpha qt})^{1/\alpha}}$$

A.2 Latency

Latency can be defined as the time it takes until u reaches a threshold u_T. Solving the approximate equation for t gives:

$$t_T = \frac{1}{q\alpha}\ln\frac{1-(1-u_T)^{\alpha}}{\alpha u_0(1-u_T)^{\alpha}},$$

in which t_T is "time to threshold" or latency, and u_T is the threshold defined, for example, as the point where the sales become profitable or reaches a certain target. In the approximation $u_T \ll 1$, the formula for qt_T becomes

$$t_T = \frac{\ln(u_T/u_0)}{q\alpha}.$$

A.3 Inflection Point

At the inflection point, du/dt changes from an increasing (decreasing) to a decreasing (increasing) function of time; that is, du/dt (i.e., the number of sales per unit of time) has a maximum (minimum) at the inflection point. In our case, the inflection point corresponds to a maximum of the first derivative or, in other words, the instant when the sales reach its maximum.

At the inflection point, $du^2/dt^2 = 0$; that is,

$$\frac{du^2}{dt^2} = \frac{d}{dt}[q[1-(1-u)^\alpha](1-u)] = q\alpha(1-u)^\alpha \frac{du}{dt} - q(1-(1-u)^\alpha)\frac{du}{dt} = 0.$$

Since q and du/dt are nonzero, this equation reduces to:

$$\alpha\left(1-u_{infl}\right)^\alpha - \left(1-\left(1-u_{infl}\right)^\alpha\right) = 0,$$

with solution

$$u_{infl} = 1 - \frac{1}{(1+\alpha)^{1/\alpha}}.$$

The time it takes to reach the inflection point is:

$$t_{infl} = \frac{1}{q\alpha}\ln\frac{1-\left(1-u_{infl}\right)^\alpha}{\alpha u_0\left(1-u_{infl}\right)^\alpha} = -\frac{1}{q\alpha}\ln u_0.$$

Appendix B. Competition Between Any Number of Providers

B.1 General Market Equation

A simple generalization of the two-provider model of Section 4.2.1 leads to the following coupled set of differential equations for $n$providers:

$$\frac{du_i}{dt} = (p_i + q_i u_i)\left(1 - \sum_{j=1}^{n} u_j\right) + \sum_{\substack{j=1 \\ j\neq i}}^{n}\left[\left(r_{ji} + s_{ji}u_i\right)u_j - \left(r_{ij} + s_{ij}u_j\right)u_i\right].$$

The equations are nonlinear and cannot be solved by quadrature in the general case. Next, we will consider a few special cases that will uncover some strategically important aspects of competition. First, we investigate the nature of the asymptotic solutions of the equations. Thereafter, we derive exact solutions in the case where all customers are innovators and there is no stimulated churning.

B.2 Asymptotic Solutions

For $t \to \infty$, everyone has bought the good; that is, $\sum_{j=1}^{n} u_i = 1$. The set of general equations for the market evolution is then:

$$\frac{du_i}{dt} = \sum_{\substack{j=1 \\ j\neq i}}^{n}\left[\left(r_{ji} + s_{ji}u_i\right)u_j - \left(r_{ij} + s_{ij}u_j\right)u_i\right], \sum_{j=1}^{n} u_i = 1.$$

Since the asymptotic state corresponds to a stationary market equilibrium, there is no net churning of customers between the suppliers of the good; that is, the set of equations becomes:

$$\frac{du_i}{dt} = \sum_{\substack{j=1 \\ j\neq i}}^{n}\left[\left(r_{ji} + s_{ji}u_i\right)u_j - \left(r_{ij} + s_{ij}u_j\right)u_i\right] = 0, \sum_{j=1}^{n} u_j = 1.$$

One important observation is that this set of equations is independent of the parameters of innovation (p_i) and imitation (q_i). The final state where everyone has bought the good depends only on the churning parameters and is independent of how the market reached this asymptotic state.

Since

$$\sum_{i=1}^{n}\frac{du_i}{dt} = 0,$$

the n equations derived from the differential equations are not linearly independent. A linearly independent set is obtained by choosing $n-1$ of the equations derived from the differential equations and the conservation equation $\sum_{j=1}^{n} u_j = 1$. However, this is a set of nonlinear equations that cannot be solve analytically except in a few special cases. Two such cases are:

- There is no stimulated churning
- There is only stimulated churning

The following text examines these two cases in detail.

No stimulated churning

If there is no stimulated churning, the set of n linearly independent equations is:

$$\sum_{\substack{j=1 \\ j\neq i \\ i<n}}^{n} \left(r_{ji}u_j - r_{ij}u_i\right) = 0, \sum_{j=1}^{n} u_j = 1.$$

This can be written in matrix form:

$$R\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} -r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & -r_{22} & \cdots & r_{n2} \\ \vdots & \vdots & & \vdots \\ r_{1(n-1)} & r_{2(n-1)} & \cdots & r_{n(n-1)} \\ 1 & 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

in which $r_{ii} = \sum_{\substack{j=1 \\ j\neq i}}^{n} r_{ij}$. The solution of this matrix equation is.

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = R^{-1}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = D^{-1}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1(n-1)} & A_{2(n-1)} & \cdots & A_{n(n-1)} \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

in which D is the determinant and A_{ij} are the cofactors of the matrix R. The solution of this equation is

$$u_i = \frac{A_{ni}}{D}.$$

This is the market share of each competitor if all the customers are innovators, and all customers are likely to churn. It is difficult to derive formulas for the determinant and the cofactors of an $n \times n$ matrix for $n > 3$. However, the formula is easily applicable for numerical evaluation of the market shares of the suppliers for any n. For the case of two competitors, we easily find that:

$$u_1 = \frac{r_{21}}{r_{12} + r_{21}}, u_2 = \frac{r_{12}}{r_{12} + r_{21}} .$$

Only stimulated churning

In the case in which all churning is stimulated, the set of equilibrium equations is:

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(s_{ji} - s_{ij}\right)u_i u_j = 0$$

$$\sum_{j=1}^{n} u_j = 1.$$

Since the stimulated churning parameters are arbitrary, the only solution of these equations is that, say, $u_1 = 1$ and that for $j > 1, s_j = 0$; that is, one of the competitors eventually captures all customers (a winner-takes-all situation).

B. 3 General Solution for a Special Case

This set of market equations is nonlinear and cannot be solved by quadrature. However, if all customers are innovators (i.e., $q_i = 0$ for all i) and churning is spontaneous (i.e., $s_{ij} = s_{ji} = 0$ for all i and j), then the set of equations is linear:

$$\frac{du_i}{dt} = p_i\left(1 - \sum_{j=1}^{n} u_j\right) + \sum_{\substack{j=1 \\ J\neq i}}^{n}\left(r_{ji}u_j - r_{ij}u_i\right) = p_i - \left(p_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} r_{ij}\right)u_i - \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(p_i - r_{ji}\right)u_j.$$

The set of differential equations can be written in matrix form:

$$\frac{d\mathbf{u}}{dt} = \mathbf{p} - M\mathbf{u},$$

in which $\mathbf{u}$ and $\mathbf{p}$ are n-dimensional column vectors with components from top to bottom $\{u_1, u_2, \cdots, u_n\}$ and $\{p_1, p_2, \cdots, p_n\}$, respectively, and M is the $n \times n$ matrix:

$$M = \left(m_{ij}\right) = \begin{pmatrix} p_1 + r_{11} & p_1 - r_{21} & \cdots & p_1 - r_{n1} \\ p_2 - r_{12} & p_2 + r_{22} & & p_2 - r_{n2} \\ \vdots & & \ddots & \vdots \\ p_n - r_{1n} & p_n - r_{2n} & \cdots & p_n + r_{nn} \end{pmatrix},$$

in which $r_{ii} = \sum_{\substack{j=1 \\ j\neq i}}^{n} r_{ij}$. The solution of the matrix differential equation with initial value $\mathbf{u}(0) = \mathbf{0}$ ($\mathbf{0}$ is the null vector) is (Goertzel & Tralli, 1960):

$$\mathbf{u} = \int_0^t e^{-M(t-\tau)}\mathbf{p}d\tau = \int_0^t Se^{-\Lambda(t-\tau)}S^{-1}\mathbf{p}d\tau = S\begin{pmatrix} \lambda_1^{-1}\left(1 - e^{-\lambda_1 t}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^{-1}\left(1 - e^{-\lambda_n t}\right) \end{pmatrix} S^{-1}\mathbf{p},$$

in which S is the matrix of eigenvectors, λ_i are the eigenvalues of M, and Λ is the diagonalization of M:

$$\Lambda = S^{-1}MS = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

The eigenvalues λ_i are the roots of the characteristic polynomial of the matrix M:

$$\text{Det}(M - \lambda I) = \text{Det}\begin{pmatrix} p_1 + r_{11} - \lambda & p_1 - r_{21} & \cdots & p_1 - r_{n1} \\ p_2 - r_{12} & p_2 + r_{22} - \lambda & & p_2 - r_{n2} \\ \vdots & & \ddots & \vdots \\ p_n - r_{1n} & p_n - r_{2n} & \cdots & p_n + r_{nn} - \lambda \end{pmatrix} = 0.$$

The eigenvector corresponding to a given eigenvalue is determined from the equation

$$MS_i = \lambda_i S_i.$$

In this formula, M is a $n \times n$ matrix, S_i is an n-dimensional column vector and λ_i is a number. Note that this equation does not produce a unique value of the eigenvectors; however, any of the allowable values can be used in the matrix S since all of them lead to the same, unique diagonalization of M.

Except for a few special cases, analytical expressions for the eigenvalues λ_i cannot be found by simple methods for matrices with $n > 2$ since the eigenvalues are the roots of the nth order characteristic polynomial defined above. Consequently, analytical expressions for u_i is hard to find for $n > 2$. However, the exact matrix solution given above is still useful since it is well suited for numerical evaluation of the market evolution for any number of competitors since the eigenvalues and the eigenvectors–and hence, u_i –can be determined by simple numerical methods.

Let us illustrate this procedure for the case of two competitors. For $n = 2$,

$$M = \begin{pmatrix} p_1 + r_{12} & p_1 - r_{21} \\ p_2 - r_{12} & p_2 + r_{21} \end{pmatrix}$$

with characteristic polynomial

$$\text{Det}\begin{pmatrix} p_1 + r_{12} - \lambda & p_1 - r_{21} \\ p_2 - r_{12} & p_2 + r_{21} - \lambda \end{pmatrix} = \lambda^2 - (p_1 + p_2 + r_{12} + r_{21})\lambda + (p_1 + p_2)(r_{12} + r_{21}) = 0$$

The eigenvalues are $\lambda_1 = p_1 + p_2$ and $\lambda_2 = r_{12} + r_{21}$ and the matrix of eigenvectors is easily found from the equations $MS_1 = \lambda_1 S_1$ and $MS_2 = \lambda_2 S_2$:

$$S = \begin{pmatrix} (p_2 - r_{12})^{-1} & 1 \\ (p_1 - r_{21})^{-1} & -1 \end{pmatrix}, S^{-1} = \frac{(p_1 - r_{21})(p_2 - r_{12})}{p_1 + p_2 - r_{12} - r_{21}} \begin{pmatrix} 1 & 1 \\ (p_1 - r_{21})^{-1} & -(p_2 - r_{12})^{-1} \end{pmatrix}.$$

The solution for $n = 2$ is then:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = S \begin{pmatrix} \lambda_1^{-1}(1 - e^{-\lambda_1 t}) & 0 \\ 0 & \lambda_2^{-1}(1 - e^{-\lambda_2 t}) \end{pmatrix} S^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Written out in full:

$$u_1 = \frac{r_{21}}{r_{12} + r_{21}} - \frac{p_2 r_{21} - p_1 r_{12}}{(r_{12} + r_{21})(p_1 + p_2 - r_{12} - r_{21})} e^{-(r_{12}+r_{21})t} - \frac{p_1 - r_{21}}{p_1 + p_2 - r_{12} - r_{21}} e^{-(p_1+p_2)t},$$

$$u_2 = \frac{r_{12}}{r_{12} + r_{21}} + \frac{p_2 r_{21} - p_1 r_{12}}{(r_{12} + r_{21})(p_1 + p_2 - r_{12} - r_{21})} e^{-(r_{12}+r_{21})t} - \frac{p_2 - r_{12}}{p_1 + p_2 - r_{12} - r_{21}} e^{-(p_1+p_2)t}.$$

Observe that for $t \to \infty$, the solutions approach the stationary state

$$u_1 = \frac{r_{21}}{r_{12} + r_{21}}, u_2 = \frac{r_{12}}{r_{12} + r_{21}}.$$

This is the same solution we found in Section B.2.

Appendix C. Multisided Platform

The compartmental model for a two-sided platform is shown in Fig. 16. There is a cross-side network effect from user group 1 to user group 2; there are no other network effects. Fig. 15(a) illustrates the general principle using a two-sided platform with mutual cross-side network effects as example, and Fig. 15(b) shows the impact of network effects all other user categories have on the flow of users of category 1 for a general multisided platform. A similar picture as shown in Fig. 15(b) exists for each of the n categories of users. F_{ji} is the cross-side network effect that user category j generates on the flow of users of category i.

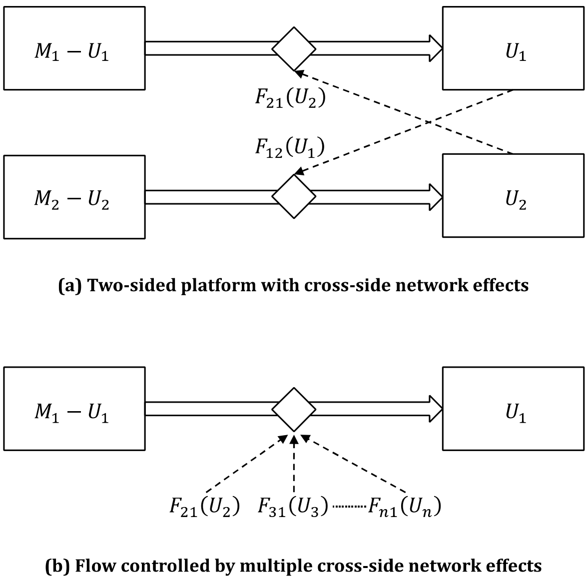


Fig. 15. Multisided platform with cross-side network effects. Two-sided platform (a) and network effects on user category 1 in a n-sided platform (b).

The differential equation for each flow is then:

$$\frac{du_i}{dt} = (p_i + q_i u_i)(1 - u_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} f_{ji},$$

in which $u_i = U_i/M_i$ and f_{ji} is the normalized network effect generated by user category j. M_i is the absolute number of initial customers of category i, and U_i is the number of customers having bought the good at time t. The rationale behind this equation is:

1. The underlying market evolution of each category is assumed to be in accordance with the Bass model; that is, the users of each category may consist of both innovators and imitators. The impact of cross-side network effects is that they modify the flow intensity parameter $(p_i + q_i u_i)$ of the Bass model.
2. The network effects generated by the various user categories are independent of one another, and the combined impact can, therefore, be modelled as product of individual factors.
3. If there are no cross-side network effects on user category i, then $\prod_{j=1, j\neq i}^{n} f_{ji} = 1$ and the equation for that category becomes a Bass equation.

The feedback functions may take two essentially different forms:

1. The market of a certain user category may start to grow even if there are no cross-side network effects. The cross-side network effect may just boost the market growth. The feedback function can then be written $f_{ij} = 1 + r_{ji} u_j$. In (Øverby & Audestad, 2019), we called this "composite growth" since there are two courses for the growth. Examples of such markets are Uber and eBay in which each market segment (e.g., sellers in the case of eBay) may start growing independently of other market segments (e.g., buyers). However, an increase in the number of sellers may be stimulated by an increased number of buyers.
2. The market of one category (i) may not start growing unless there already is a market for another category (j). The cross-side feedback function can then be written $f_{ij} = s_{ij} u_j$. This may be denoted "stimulated growth" since the market segment cannot grow unless there is a cross-side network effect. Examples where such network effects play an essential role are Facebook and Google Search. The market for advertisers depends directly on the market for the number of users of social media services (Facebook) or search engine services (Google Search).

The resulting set of coupled differential equations is, in the general case, nonlinear that cannot be solved analytically except in a few cases. We will, therefore, restrict our analysis to two simple cases of two-sided platforms, one for each of the two forms of the feedback function (see Øverby & Audestad, 2021a) for the discussion of other two-sided platforms). In both cases, there is a cross-side network effect from users of category 1 to users of category 2 but not vice versa as illustrated in Fig. 16. Moreover, all users of category 1 are innovators.

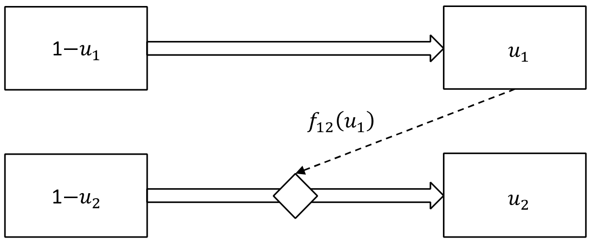


Fig. 16. Simple two-sided platform with cross-side network effects from user category 1 to user category 2.

C.1. Case 1: $f_{12} = 1 + r_{12} u_1$

In this case, the differential equations are

$$\frac{du_1}{dt} = p_1(1 - u_1),$$

$$\frac{du_2}{dt} = p_2(1 - u_2)(1 + r_{12} u_1).$$

The solution of the first equation is $u_1 = 1 - e^{-p_1 t}$.Inserted in the second equation and solved for u_2 gives:

$$u_2 = 1 - e^{-p_2(1+r_{12})t + p_2 r_{12} u_1 / p_1}.$$

C.2. Case 2: $f_{12} = s_{12} u_1$

In this case, the set of differential equations are:

$$\frac{du_1}{dt} = p_1(1 - u_1),$$

$$\frac{du_2}{dt} = s_{12} u_1 (1 - u_2).$$

Inserting the solution $u_1 = 1 - e^{-p_1 t}$ into the second equation and solving for u_2 gives:

$$u_2 = 1 - e^{-s_{12}(t - u_1/p_1)}.$$

We see that $u(t) = 0$ for $s_{12} = 0$ and all t. In this case, users will never join user group 2.

Appendix D. Distributed Online Games

In the case with no readoption flow and only innovators joining and leaving the game (i.e., $q = s = g = h = 0$) the set of equations is:

$$\frac{du}{dt} = -pu,$$

$$\frac{dv}{dt} = pu - rv,$$

$$\frac{dw}{dt} = rv,$$

$$u + v + w = 1.$$

The first equation gives immediately $u = e^{-pt}$. This leads to the linear differential equation for v:

$$\frac{dv}{dt} + rv = pe^{-pt}$$

with solution

$$v = \frac{p}{p - r}(e^{-rt} - e^{-pt}).$$

In the special case with $p = r$, we find that $v = pte^{-pt}$. The maximum $v_m = e^{-1}$ occurs at time $t_m = p^{-1}$.

More complex cases are found in (Øverby & Audestad, 2019).

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