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# Online Appendix to "Supply and Demand in a Two-Sector Matching Model"

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The Online Appendix provides a number of results, proofs, figures, examples and derivations that were omitted from the main text. Sections OA.1 and OA.2 provide the proofs, results and derivations omitted from Sections III and IV respectively. Section OA.3 provides details on how I approximate the Cobb–Douglas lognormal specification from Section V. Section OA.4 provides the specifications for which all Figures from the main text have been created, as well as the one figure (Figure OA.1) that has been omitted from the main text. Section OA.5 provides and proves all the formal results on which the discussion in Section VI is based. Finally, Section OA.6 shows that the equilibrium separation function is differentiable with respect to $\theta$.

# OA.1 Section III: Proofs and Omitted Results

### Supermodularity and Imperfect Substitution

In this section, I briefly explain why supermodularity (submodularity) of the surplus function implies that workers of similar rank are imperfect substitutes in production: It needs to be read after Section III.A.2. First, write firm's $h_i$ profit from hiring worker of skill $v_i$ as $r_i(v_i, h_i)$, then

$$\frac{\partial}{\partial v_i} r_i(v_i, h_i) = \frac{\partial}{\partial v_i} \pi_i(v_i, h_i) - \frac{\partial}{\partial v_i} w_i(v_i).$$

This and the first-order condition of the profit maximization problem imply that

$$\frac{\partial}{\partial v_i} r_i(v_i, h_i) = \int_{P_i(x)}^{h_i} \frac{\partial^2}{\partial v_i \partial h_i} \pi_i(v_i, z) \mathrm{d}z$$

where $P_i(v_i) = 1 - S_i(v_i)/R_i$ denotes the matching function, that is, the inverse of $v_i^*$. Under strict supermodularity (submodularity) $P_i$ is strictly increasing (decreasing) and hence $\frac{\partial}{\partial v_i} r_i(v_i, h_i) > 0$ for $v_i < v_i^*(h_i)$ and $\frac{\partial}{\partial v_i} r_i(v_i, h_i) < 0$ for $v_i > v_i^*(h_i)$. Thus, indeed, the firm's profit is unimodal in skill $v_i$.[1]

However, if $\frac{\partial^2}{\partial v_i \partial h_i} \pi_i$ is strictly positive for some worker-firm pairs, and strictly negative for others, then workers of very different ranks might be closer substitutes than workers of very close ranks. Suppose that the surplus function in services is given by

$$\pi_S(v_S, h_S) = ((v_S - 0.5)^2 + h_S)^2 + (2 + v_S^2)^2 - 4,$$

Assumption 5 is satisfied, and $v_S^c = 0$. Note that this surplus function is not supermodular, but it does satisfies all other of my assumptions. The within-sector assignment can be easily determined despite the failure of supermodularity, by noting that the surplus function is supermodular in $\bar{v}_S = |v_S - 0.5|$. It can be thus easily shown that a worker of skill $v_i$ is matched with firm of productivity $|G_i(v_i) - G_i(1 - v_i)|$, which implies that both the best and the worst worker are matched with the most productive firm. It follows, thus, from profit maximization that

$$w_S(1) - w_S(0) = \pi_S(1, 1) - \pi_S(0, 1) = 5,$$

while $\pi_S(1, 1) - \pi_S(0.5, 1) > 5$. As a corollary, any change to the supply of skill in services

[1] In the case of additively separable surpluses $\frac{\partial}{\partial v_i} r_i(v_i, h_i) = 0$ and all workers are perfect substitutes.

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(that preserves $v_S^c = 0$) affects the highest and lowest wage in the economy in the same way, but might affect the wages of the highest and medium ranked workers differently. This implies immediately that, for example, Proposition 1 (ii) would not hold under this surplus function, and neither would Proposition 2 (v).

### Demand: Formal Definition and Shifts

The definition of sectoral demand for skill (Section III) holds for a given hiring function and under the assumption that profit is strictly increasing. However, if, for example, surplus does not depend on firm productivity, then (in equilibrium) (a) firms will be indifferent which worker to hire and there will exist many different hiring functions and (b) all firms will make the same profits. Here, I amend the definition of sectoral demand to allow for such possibilities. Accordingly, the economy will be in equilibrium if there exists at least one demand function consistent with firms maximization problem for which the market clears.

**Definition OA.1.** A mapping $v_i^* : [0,1] \to [0,1] \cup \{-1\}$ is a *hiring function* in sector $i$ for wage function $w_i$, if (a) for $v^*(h) \in [0,1]$, $v_i^*(h) \in \arg\max_{v_i} \pi_i(v_i, h) - w_i(v_i)$ and $\pi_i(v_i^*(h), h) - w_i(v_i^*) \geq 0$ and (b) for $v_i^*(h) = -1$, $\pi_i(v_i, h) - w_i(v_i) \leq 0$ for all $v_i \in [0,1]$ .

Given a talent level $v_i$ and an input function $v_i^*$, define the set $B(v_M, v_i^*) = \{h \in [0,1], v_i^*(h) \geq v_M\}$.

**Definition OA.2.** A mapping $D_i : [0,1] \to [0,R]$ is a *sector $i$ demand function for skill* given wage function $w_i$, if there exists a hiring function such that $R_M \int_{B(v_i,v_i^*)} 1 \mathrm{d}v_i = D_i(v_i)$, for all $v_i \in [0,1]$.

For any matching problem, I will denote as $DC(\theta)$ the set of all possible cumulative demand functions and as $DC(v_M, \theta)$ the set of their values for talent $v_M$.

**Definition OA.3.** Demand for skill *shifts up* if—given the old equilibrium wage function $w_M(\cdot;\theta_1)$ and for all $v_M \in [0,1]$—for any $h'' \in DC(v_M;\theta_2)$ and $h' \in DC(v_M;\theta_2)$ we have that $\max\{h'', h'\} \in DC(v_M;\theta_2)$ and $\min\{h'', h'\} \in DC(v_M;\theta_1)$.

**Proposition OA.1.** If $\frac{\partial}{\partial v_S}\pi_S(v_S, h_S;\theta_2) \geq \frac{\partial}{\partial v_S}\pi_S(v_S, h_S;\theta_1)$ and $\pi_S(v_S, h_S;\theta_2) \geq \pi_S(v_S, h_S;\theta_1)$ for all $(v_S, h_S) \in [0,1]^2$, then the demand for skill shifts up in services. If $R_M + R_S \leq 1$ then, an increase in workers' vertical differentiation alone suffices for an upward shift of skill demand.

*Proof.* The partial order $([0,1], \geq)$ is clearly a lattice and the function $\pi_i(v,h) - w_i(v)$ is supermodular in $v$ (for any $h$). Thus, as an increase in vertical differentiation implies that $\pi_i(v,h) - w_i(v)$ has increasing differences in $c$ it follows from the results in Topkis (1978) and Milgrom and Shannon (1994) that the set $V^*(c_i) = \{v \in [0,1] : v \in \arg\max \pi_i(v, h, (c) - w_i(v)\}$ increases in the strong set order sense with a change from $\theta_1$ to $\theta_2$. This proves the second statement, as $v_i^*(h) \in [0,1]$ for all firms in that case. As for the first claim, note that the increase in surplus levels means that each firm's profit increases for the old choice of inputs, and hence, by profit maximization, also for the new choice. Thus, no firms leave the market and the result follows. $\square$

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### Proof of Lemma 1

Denote $\{v_M \in [0,1] : w_M(v_M) \geq \max\{w_S(0), 0\}\}$ by $A_M$, and $\min A_M$ by $v'_M$; notice that Equation (8) implies that $w_i(\cdot)$ is increasing and continuous.

I will first show that $A_M$ has a positive measure. Suppose not, then $w_M(v_M) < \max\{w_S(0), 0\}$ for almost all $(v_M)$ and hence (a) $w_M(1) \leq \max\{w_S(0), 0\}$ by continuity and (b) $S_M(0) = 0$ by increasingness of $w_S$ and Equation (1). The latter implies that $v_M^c = 1$ and $w_M(v_M) = \pi_M(v_M, 1)$ (as $S_M(0) < R_M$). Overall, this implies $\pi_M(1,1) \leq \{w_S(0), 0\}$. If $R_S < 1$, then $w_S(0) < 0$ and we have $\pi_M(1,1) \leq 0$. However, $\pi_M(1,1) > 0$ by $\pi_M(0,1) \geq 0$ and Assumption A2.2; contradiction! If $R_S \geq 1$, then $w_S(0) \leq \pi_S(0, 1 - \frac{1}{R_S})$, which implies $\pi_M(1,1) \leq \pi_S(0, 1 - \frac{1}{R_S})$ and thus contradicts Assumption 4.

If $A_M$ has a positive measure, then $v'_M$ must be strictly less than 1. First, suppose that $v_M^c < v'_M$, implying that $v'_M > 0$ which implies further (by continuity) that $v'_M = \max\{w_S(0), 0\}$. As $w_M$ is continuous, there must exist some $\epsilon > 0$ such that $w_M(v_M) < \max\{w_S(0), 0\}$ for all workers with $v_M \in [v_M^c, v_M^c + \epsilon]$. As $w_S$ is increasing, this implies that all workers with $v_M \in [v_M^c, v_M^c + \epsilon]$ prefer to remain unemployed or work in services than to join manufacturing and $S_M(v_M^c) = S_M(v'_M)$, which contradicts the definition of $v_M^c$. Thus, $v_M^c \geq v'_M$.

Second, suppose that $v_M^c > v'_M$, which implies that $w_M(v_M^c) > \max\{w_S(0), 0\}$ and $v_M^c > 0$. By continuity of $w_M, w_S$ there exist some $v'_M$ and $v'_S$ such that $w_M(v_M) > \max\{w_S(v_S), 0\}$ for all $(v_M, v_S) \in (v'_M, v_M^c) \times [0, v'_S]$, so that all workers living in this rectangle prefer to join manufacturing than remain unemployed or join services.. As $C$ has full support, a strictly positive measure of workers lives in this rectangle, which contradicts the definition of $v_M^c$. Thus, $v_M^c = v'_M$, as required. The proof for $v_S^c$ is analogous.

Finally, let me prove the last statement. First, I will consider the case of $v_M^c, v_S^c \in (0,1)$. This implies that (a) some workers are unemployed (because workers with $(v_M, v_S) < (v_M^c, v_S^c)$ cannot join either sector by definition of critical skills) and (b) that $w_M(v_M^c) = \max\{w_S(0), 0\}$ and $w_S(v_S^c) = \max\{w_M(0), 0\}$. Suppose $w_S(0) > 0$; then there exists $v''_M < v_M^c$, such that all workers with $(v_M, v_S) \in (v''_M, v_M^c) \times (0, v_S^c)$ prefer to join manufacturing than to remain unemployed, which contradicts the definition of $v_M^c$; thus $w_M(v_M^c) = 0$. An analogous reasoning holds for $w_S(v_S^c)$.

Now suppose that $v_M^c = 0$. It follows immediately that $w_S(v_S^c) \neq w_M(v_M^c)$ only if $w_S(v_S^c) > w_M(v_M^c)$. There must then exist an $\epsilon_2 > 0$ such that $w_M(v_M) < w_S(v_S)$ for all $(v_M, v_S) \in [0, \epsilon_2] \times [v_S^c, v_S^c + \epsilon_2]$, so that all workers with such skill vectors prefer to work in services over manufacturing, which contradicts the definition of $v_M^c$.

### Proof of Lemma 2

I start with manufacturing. The probability that a worker with skill $v_M \geq v_M^c$ chooses services is $\Pr(\psi(V_S) < v_M | V_M = v_M)$. Note that because $\psi$ is weakly increasing, it follows that if $\psi(v'_S) < v_M$ then $\psi(v''_S) < v_M$ for any $v'_S \geq v''_S \geq v_S^c$. Thus:

$$\Pr(\psi(V_S) < v_M | V_M = v_M) = \frac{\partial}{\partial v_M} C(v_M, \phi(v_M)) \qquad \text{for } v_M \geq v_M^c,$$

where $\phi(v_M) = \sup\{v_S \in [v_S^c, 1] : \psi(v_S) < v_M\}$. Because $S_M(1) = 0$, this gives us the required expression for $S(v_M)$ if $v_M \geq v_M^c$. And, of course, for any $v_M < v_M^c$, $S_M(v_M) = S_M(0)$ by the definition of critical skill.

The proof for $S_S(\cdot)$ is analogous.

### Proof of Theorem 1

Define the *extended separating function* $\psi^e : [v_S^c, 1] \to [v_M^c, 1+B]$ as

$$\psi^e(v_S) = v_M^c + \int_{v_S^c}^{v_S} \frac{\frac{\partial}{\partial v_S}\pi_S^e\Big(t, 1 - \frac{\int_t^1 \frac{\partial}{\partial v_S} C^e(\psi(r), r)\mathrm{d}r}{R_S}\Big)}{\frac{\partial}{\partial v_M}\pi_M^e\Big(\psi(t), 1 - \frac{\int_t^1 \frac{\partial}{\partial v_M} C^e(r, \phi(r))\mathrm{d}r}{R_M}\Big)}\mathrm{d}t, \tag{OA.1}$$

where he extended functions $C^e(\bullet)$, $\pi_M^e(\bullet)$ and $\pi_S^e(\bullet)$ are defined as follows (1) $C^e : [0, 1+B] \times [0, 1] \to [0, 1]$

$$C^e(v_M, v_S) = \begin{cases} C(v_M, v_S) & \text{for } (v_M, v_S) \in [0,1] \times [0,1] \\ v_S & \text{for } (v_M, v_S) \in (1, 1+B] \times [0,1], \end{cases}$$

(2): $\pi_M^e(v_M, h) : [0, 1+B] \times [0, \frac{1+R_M}{R_M}] \to \mathbf{R}^+$

$$\pi_M^e(v_M, h) = \begin{cases} \pi_M(v_M, h) & \text{for } (v_M, h) \in [0,1]^2 \\ \pi_M(1, h) + (v_M - 1)\frac{\partial}{\partial v_M}\pi_M(1, h) & \text{for } (v_M, h) \in (1, B] \times [0, 1], \\ \pi_M(v_M, 1) & \text{for } (v_M, h) \in [0, 1] \times (1, \frac{1+R_M}{R_M}], \\ \pi_M(1, 1) + (v_M - 1)\frac{\partial}{\partial v_M}\pi_M(1, 1) & \text{for } (v_M, h) \in (1, B] \times (1, \frac{1+R_M}{R_M}], \end{cases}$$

(3): $\pi_S^e(v_S, h) : [0, 1] \times [0, 1 + \frac{1}{R_S}] \to \mathbf{R}^+$

$$\pi_S^e(v_S, h) = \begin{cases} \pi_S(v_S, h) & \text{for } (v_M, h) \in [0,1]^2 \\ \pi_S(v_S, 1) & \text{for } (v_M, h) \in [0,1] \times (1, 1 + \frac{1}{R_S}], \end{cases}$$

and $B = \frac{\max \frac{\partial}{\partial v_S}\pi_S}{\min \frac{\partial}{\partial v_M}\pi_M}$. Note that $C^e(\cdot, v_S)$, $\frac{\partial}{\partial v_S}C^e(\cdot, v_S)$, $\frac{\partial}{\partial v_M}\pi_M^e(\cdot, \cdot)$ and $\frac{\partial}{\partial v_S}\pi_S^e(v_S, \cdot)$ are Lipschitz continuous[2]; denote their Lipschitz-constants as $L^1$, $L^2$ ,$L^3$, $L^4$ and $L^5$ respectively.

[2] I will do this in detail for $\frac{\partial}{\partial v_S}C^e(v_M, v_S)$—the reasoning for the other two is analogous. $\frac{\partial}{\partial v_S}C^e(v_M, v_S) : [0, 1+B] \times [0, 1] \to [0, 1]$:

$$\frac{\partial}{\partial v_S}C^e(v_M, v_S) = \begin{cases} \frac{\partial}{\partial v_S}C(v_M, v_S) & \text{for } (v_M, v_S) \in [0,1] \times [0,1] \\ 1 & \text{for } (v_M, v_S) \in (1, 1+B] \times [0,1], \end{cases}$$

is clearly continuous in u. It is equally easy to see that the function $\frac{\partial}{\partial v_S}C^e(\cdot, v_S)$ is differentiable almost everywhere and its derivative is Lebesque integrable. It is also the case that for any $(v_M, v_S) \in (1, 1+B] \times [0, 1]$ we have:

$$\frac{\partial}{\partial v_S}C^e(a, v_S) + \int_a^1 C_{uv}^e(r, v_S)\mathrm{d}r + \int_1^{v_M} 0\mathrm{d}r = 1,$$

which means that $\frac{\partial}{\partial v_S}C^e(\cdot, v_S)$ is absolutely continuous. Moreover, as $C^e(\bullet)$ is twice continuously differentiable and any continuous function defined on a compact set is bounded it follows that $\frac{\partial}{\partial v_S}C^e(\cdot, v_S)$ is essentially

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Clearly, given $v_M^c$ and $v_S^c$ the separating function $\psi$ uniquely determines the extended separation function $\psi^e$. Similarly, it should be clear that

$$\psi(v_S) = \begin{cases} \psi^e(v_S) & \text{if } \psi^e(v_S) \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

The result for $\psi^e(v_S) \leq 1$ follows from noting that $\psi^e$ is strictly increasing and then substituting Equation (8) into Equation (11), differentiating wrt $v_S$, dividing both sides by

$$\frac{\partial}{\partial v_M}\pi_M\Big(\psi(v_S), \frac{1}{R_M}\int_{v_M^c}^{\psi(v_S)} \frac{\partial}{\partial v_M}C(r, \psi^{-1}(r))\mathrm{d}t\Big)$$

and then integrating from $v_S^c$ to $v_S$ (and remembering that $\psi(v_S^c) = v_M^c$).[3] The other part follows from the fact that for $v_S$'s such that $w_S(v_S) \leq w_M(1)$ we have $\psi(v_S) = 1$ and $\psi^e(v_S) > 1$ (because $\psi^e$ is strictly increasing).

Thus, it is sufficient to prove that $\psi^e, v_M^c, v_S^c$ exist and are unique. Let me make a few observations that will prove useful.

**Relation Between Supply Functions** By differentiating $C(\psi(r), r)$ rearranging and integrating from $v_S^c$ to $v_S$, we arrive at

$$S_M(0) - S_M(\psi(v_S)) + S_S(0) - S_S(v_S) = C(\psi(v_S), v_S) - C(v_M^c, v_S^c). \tag{OA.2}$$

**Determining the Critical Skills** As the critical skills $v_M^c, v_S^c$ are also unknown, we need to find conditions that will pin them down. Let me start by denoting the measure of employed workers as $M = S_M(0) + S_S(0)$. Clearly, $M = \min\{R_M + R_S, 1\}$ in equilibrium: otherwise we have $S_i(0) < R_i$ in some sector $i$, implying that a positive measure of workers with skill below $(v_M^c, v_S^c)$ would strictly prefer to join sector $i$ than remain unemployed. By Equation (OA.2) this gives $1 - M = C^e(v_M^c, v_S^c)$, determining one of the critical skills as a function of the other. Furthermore, note that Assumption 4 implies that $v_M^c, v_S^c < 1$ and thus $S_M(0), S_S(0) > 0$.[4] Therefore, from Lemma 1 it follows that if $S_M(0) < R_M$ then:

$$\pi_M(v_M^c, 1 - \frac{S_M(0)}{R_M}) = w_M(v_M^c) = w_S(v_S^c) \leq \pi_S(v_S^c, 1 - \frac{M - S_M(0)}{R_S}),$$

and analogously for services. This determines the other critical skill if $R_M + R_S > 1$. Finally, recall that market clearing implies that $S_i(0) \leq R_i$, implying that if $R_M + R_S \leq 1$ we have $S_M(0) = R_M$ and $S_S(0) = R_S$.

bounded; and a differentiable almost everywhere, absolutely continuous function with an essentially bounded derivative is Lipschitz-continuous.

[3]This gives us Equation (OA.1), but with $\psi$ rather than $\psi^e$ on the right hand side.

[4]If $R_i < 1$ this follows immediately from $1 - M = C^e(v_M^c, v_S^c)$. Otherwise, suppose that $v_M^c = 1$; then $S_M(0) = 0 < R_M$ and $w_M(1) = \pi_M(1,1) > \pi_S(0, 1 - \frac{1}{R_S}) \geq w_S(0)$. But then, by continuity of $\pi_M$ and Proposition **??** follows that there must exist some $\epsilon > 0$ such that all workers with $(v_M, v_S) \in [0, \epsilon] \times [1 - \epsilon, 1]$ would prefer to join manufacturing, contradicting $v_M^c = 1$.

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**The Set of Equations and Inequalities** By substituting $S_i(v_i) = S_i(0) - S_i(v_i)$ and Equation (OA.2) into Equation (OA.1) we arrive at

$$\psi^e(v_S) = v_M^c + \int_{v_S^c}^{v_S} \frac{\frac{\partial}{\partial v_S}\pi_S^e\left(t, \frac{R_S - S_S(0) + \int_{v_S^c}^t \frac{\partial}{\partial v_S} C^e(\psi(r), r)\mathrm{d}r}{R_S}\right)}{\frac{\partial}{\partial v_M}\pi_M^e\left(\psi^e(t), \frac{R_M - 1 + S_S(0) + C^e(\psi^e(t), t) - \int_{v_S^c}^t \frac{\partial}{\partial v_S} C^e(\psi^e(r), r)\mathrm{d}r}{R_M}\right)} \mathrm{d}t. \tag{OA.3}$$

This, together with

$$M = \min\{R_M + R_S, 1\} \tag{OA.4}$$
$$1 - M = C^e(v_M^c, v_S^c), \tag{OA.5}$$
$$S_S(0) = \int_{v_S^c}^1 \frac{\partial}{\partial v_S} C^e(\psi(r), r)\mathrm{d}r, \tag{OA.6}$$
$$S_S(0) \in \Theta(M) = [\max\{0, M - R_M\}, \min\{1, R_S\}] \tag{OA.7}$$
$$S_M(0) < R_M \Rightarrow \pi_M^e(v_M^c, 1 - \frac{S_M(0)}{R_M}) \leq \pi_S^e(v_S^c, 1 - \frac{M - S_M(0)}{R_S}), \tag{OA.8}$$
$$S_S(0) < R_S \Rightarrow \pi_S^e(v_S^c, 1 - \frac{M - S_M(0)}{R_S}) \leq \pi_M^e(v_M^c, 1 - \frac{S_M(0)}{R_M}). \tag{OA.9}$$

constitutes the set of Equations and Inequalities that determines $\psi^e, v_M^c, v_S^c$.

The remainder of the proof shows that there exists a unique solution to Equations (OA.3)-(OA.9). Define the set

$$K = \{d \in C[0, 1] : d(v_S) \in [0, 1 + B]\},$$

where $C[0, 1]$ is the set of all continuous functions that map from $[0, 1]$. The constant function $d(v_S) = 1$ lies in $K$ and hence the set is non-empty. Define a (Bielecki) norm, $||\cdot||_\lambda$ on $C[0, 1]$:

$$||h||_\lambda = \sup_{[0,1]} e^{-\lambda v_S}|h(v_S)|,$$

where $\lambda$ is some weakly positive number. $K$ is a complete metric space for the metric implied by this norm.[5]

Endow the sets $[0, 1]^2$ and $\Theta(M)$ with the Euclidean norm and define a mapping $\mathscr{T}$ : $K \times [0, 1]^2 \times \Theta(M) \to K$

$$(\mathscr{T}d)(v_S, v_S^c, v_M^c, S_S(0)) = v_M^c + \begin{cases} 0 & \text{for } v_S < v_S^c \\ \int_{v_S^c}^{v_S} \frac{\frac{\partial}{\partial v_S}\pi_S^e(t, \frac{R_S - S_S(0) + \int_{v_S^c}^t \frac{\partial}{\partial v_S} C^e(d(r), r)}{R_S}\mathrm{d}r)}{\frac{\partial}{\partial v_M}\pi_M^e(d(t), \frac{R_M - 1 + S_S(0) + C^e(d(t), t) - \int_{v_S^c}^t \frac{\partial}{\partial v_S} C^e(d(r), r)\mathrm{d}r}{R_M}\mathrm{d}r)} \mathrm{d}t & \text{for } v_S \geq v_S^c. \end{cases}$$

[5]If we endowed $K$ with the sup-norm, then $K$ would be a closed subset of $C[0, 1]$; since $C[0, 1]$ is complete in the sup-norm, so is $K$. It was shown by Bielecki (1956) that the $||\cdot||_\lambda$ norm is equivalent to the sup-norm for any $C[a, b]$. As $K$ is a closed subset of $C[a, b]$ under the metric implied by Bielecki norm, it is also complete and thus $K$ endowed with the Bielecki metric is a complete metric space for $||\cdot||_\lambda$.

Note that this map is well-defined, as for any $v_S^c \in [0,1]$ and $d \in K$:

$$\frac{R_S - S_S(0) + \int_{v_S^c}^t \frac{\partial}{\partial v_S} C^e(d(r),r)}{R_S}\mathrm{d}r \leq 1 + \int_{v_S^c}^t \frac{1}{R_S}\mathrm{d}r \leq \frac{1}{R_S} + 1$$
$$\frac{R_M - 1 + S_S(0) + C(d(t),t) - \int_{v_S^c}^t \frac{\partial}{\partial v_S} C^e(d(r),r)\mathrm{d}r}{R_M} \leq \frac{R_M + C(d(t),t)}{R_M} \leq \frac{1}{R_M} + 1;$$

and that it is continuous in v, $v_S^c$, $v_M^c$ and $S_S(0)$. Clearly, $(\mathscr{T}d)(v_S, v_S^c, v_M^c, S_S(0)) \geq v_M^c \geq 0$. Further, for $v_S \geq v_S^c$:

$$(\mathscr{T}d)(v_S, v_S^c, v_M^c, S_S(0)) \leq \int_{v_S^c}^{v_S} B\mathrm{d}t + v_M^c \leq 1 + B,$$

and for $v_S < v_S^c$:

$$(\mathscr{T}d)(v_S, v_S^c, v_M^c, S_S(0)) \leq v_M^c - 1 \leq 1 + B,$$

so indeed $\mathscr{T}(K) \subset K$. Finally, it should be clear that for any $(v_S^c, v_M^c, S_S(0))$ the restriction of any fixed point of $(\mathscr{T}d)(\bullet)$ to $[v_S^c, 1]$ gives us the solution to (OA.3) and that any solution to (OA.3) can be easily extended into a fixed point of $(\mathscr{T}d)(\bullet)$. Therefore, it suffices to show that there exists such a $\lambda$ that for any $(v_S^c, v_M^c, S_S(0)) \in [0,1]^2 \times \Theta(M)$, $\mathscr{T}d(\bullet)$ is a contraction wrt to the norm $||\cdot||_\lambda$ to show that (OA.3) has a unique solution for any feasible $(v_M^c, v_S^c, S_S(0))$.

Let us drop $(v_S^c, v_M^c, S_S(0))$ from the definition of the map (remembering that we are keeping them constant) and enhance our notation by new maps: $S_S : [v_S^c, 1] \times K \to [0,1], P_S : [v_S^c, 1] \times K \to [0, 1 + \frac{1}{R_S}]$ and $P_M : [0, B] \times K \to [0, 1 + \frac{1}{R_M}]$

$$(S_S d)(v_S) = S_S(0) - \int_{v_S^c}^{v_S} \frac{\partial}{\partial v_S} C^e(d(r),r)\mathrm{d}r,$$
$$(P_S d)(v_S) = \frac{R_S - (S_S d)(v_S)}{R_S},$$
$$(P_M d)(d(v_S)) = \frac{R_M - 1 + C^e(d(v_S), v_S) + (S_S d)(v_S)}{R_M}.$$

Take any $t \geq v_S^c$ and any $d_1, d_2 \in S$ and for any map $(fd)(t)$ denote $(fd_1)(t) - (fd_2)(t)$ as $\Delta_d(fd)(t)$. Then we have:

$$\begin{aligned}|\Delta_d(S_S(0)d)(t)| &= |\int_{v^c}^t C_v^e(d_1(r),r) - C_v^e(d_2(r),r)\mathrm{d}r| \qquad \text{(OA.10)}\\ &\leq \int_{v^c}^t |C_v^e(d_1(r),r) - C_v^e(d_2(r),r)|\mathrm{d}r \leq \int_{v^c}^t L_2|d_1(r) - d_2(r)|\mathrm{d}r|\\ &= L_2\int_{v^c}^t e^{\lambda r}e^{-\lambda r}|d_1(r) - d_2(r)|\mathrm{d}r \leq L_2||d_1 - d_2||_\lambda \int_{v^c}^t e^{\lambda r}\mathrm{d}r\\ &= \frac{L_2}{\lambda}||d_1 - d_2||_\lambda(e^{\lambda t} - e^{\lambda v^c}) \leq \frac{L_2}{\lambda}||d_1 - d_2||_\lambda e^{\lambda t},\end{aligned}$$

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which can be used to establish

$$
\begin{aligned}
|\Delta_d(P_S d)(t)| &\leq \frac{L_2}{\lambda R_S}||d_1 - d_2||_\lambda e^{\lambda t} && \text{(OA.11)}\\
|(P_M d_1)(d_1(t)) - (P_M d_2)(d_2(t))| &= |\frac{C^e(d_1(v),v) - C^e(d_2(v),v) - \Delta_d(S_S(0)d)(v)}{R_M}| && \text{(OA.12)}\\
&\leq \frac{1}{R_M}(|C^e(d_1(v),v) - C^e(d_2(v),v)| + |\Delta_d(S_S(0)d)(v)|)\\
&\leq \frac{L_2}{\lambda_M}||d_1 - d_2||_\lambda e^{\lambda t} + \frac{L^1}{R_M}|d_1(t) - d_2(t)|.
\end{aligned}
$$

Denote $L_6 = \sup \frac{\partial}{\partial v_S}\pi_S(v_S, h)$, $L_7 = \inf \frac{\partial}{\partial v_M}\pi_M(v_M, h)$ and note that continuity of $\frac{\partial}{\partial v_M}\pi_M$ and $\frac{\partial}{\partial v_S}\pi_S$ and the fact that $\frac{\partial}{\partial v_M}\pi_M > 0$ imply that both $L^6$ and $L^7$ are finite. Using all this, we can write, for any $v_S \geq v_S^c$ and any $d_1, d_2 \in S$:

$$
\begin{aligned}
|\Delta_d(\mathscr{T}d)(v)| &= |\int_{v^c}^{v} \frac{\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_1)(t))}{\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_1(t)))} - \frac{\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_2)(t))}{\frac{\partial}{\partial v_M}\pi_M^e(d_2(r),(P_M d_2)(d_2(t)))}\mathrm{d}t|\\
&\leq \int_{v^c}^{v} |\frac{\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_1)(t))}{\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_1(t)))} - \frac{\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_2)(v,))}{\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_1(t)))}\\
&\quad + \frac{\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_2)(t))}{\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_1(t)))} - \frac{\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_2)(t))}{\frac{\partial}{\partial v_M}\pi_M^e(d_2(r),(P_M d_2)(d_2(t)))}|\mathrm{d}t\\
&\leq \int_{v^c}^{v} \frac{|\frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_1)(t)) - \frac{\partial}{\partial v_S}\pi_S^e(t,(P_S d_2)(t))|}{L_7}\\
&\quad + L_6(|\frac{\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_1(t))) - \frac{\partial}{\partial v_M}\pi_M^e(d_2(r),(P_M d_2)(d_2(t)))}{\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_2(t)))\frac{\partial}{\partial v_M}\pi_M^e(d_2(r),(P_M d_2)(d_2(t)))}|\mathrm{d}t\\
&\leq \int_{v^c}^{v} \frac{L_5}{L_7}|\Delta_d(P_S d)(t)|\\
&\quad + \frac{L_6}{L_7^2}[|\frac{\partial}{\partial v_M}\pi_M^e(d_1(r),(P_M d_1)(d_1(t))) - \frac{\partial}{\partial v_M}\pi_M^e(d_2(t),(P_M d_1)(d_1(t))|]\\
&\quad + \frac{L_6}{L_7^2}[|\frac{\partial}{\partial v_M}\pi_M^e(d_2(t),(P_M d_1)(d_1(t)) - \frac{\partial}{\partial v_M}\pi_M^e(d_2(r),(P_M d_2)(d_2(t)))|]\mathrm{d}t\\
&\leq \int_{v^c}^{v} \frac{L_5 L_2}{\lambda L_7 R_S}||d_1 - d_2||_\lambda e^{\lambda(t-v^c)} + \frac{L_3 L_6}{L_7^2}|d_1(t) - d_2(t)|\\
&\quad + \frac{L_4 L_6}{L_7^2}|(P_M d_1)(d_1(t)) - P_M d_2)(d_2(t))|\mathrm{d}t\\
&\leq \frac{L_5 L_2}{\lambda^2 L_7 R_S}||d_1 - d_2||_\lambda e^{\lambda v} + \frac{L_3 L_6}{\lambda L_7^2}||d_1 - d_2||_\lambda e^{\lambda v}\\
&\quad + \int_{v^c}^{v} \frac{L_4 L_6}{L_7^2}(\frac{L_2}{\lambda_M}||d_1 - d_2||_\lambda e^{\lambda(t-v^c)} + \frac{L_1}{R_M}|d_1(t) - d_2(t)|)\mathrm{d}t\\
&\leq \frac{1}{\lambda}||d_1 - d_2||_\lambda e^{\lambda v}\Big[\frac{L_5 L_2}{\lambda L_7 R_S} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2}\Big(\frac{L_2}{\lambda_M} + \frac{L_1}{R_M}\Big)\Big]
\end{aligned}
$$

Now, for $v_S < v_S^c$ this has to hold as well, as then $|(\mathscr{T}d_1)(v_S) - \mathscr{T}(d_2)(v_S)| = 0$; therefore, for any $v_S \in [0,1]$ we have that

$$
|\Delta_d(\mathscr{T}d)(v_S)| \leq \frac{1}{\lambda}||d_1 - d_2||_\lambda e^{\lambda v_S}\Big[\frac{L_5 L_2}{\lambda L_7 R_S} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2}\Big(\frac{L_2}{\lambda_M} + \frac{L_1}{R_M}\Big)\Big].
$$

Dividing both sides of that by $e^{\lambda v_S}$ and then taking sup on both sides we get

$$||(\mathscr{T}d_1)(t) - \mathscr{T}(d_2)(t)||_\lambda \leq \frac{1}{\lambda}||d_1 - d_2||_\lambda \Big[\frac{L_5 L_2}{\lambda L_7 R_S} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2}\Big(\frac{L_2}{\lambda_M} + \frac{L_1}{R_M}\Big)\Big]. \quad \text{(OA.13)}$$

Therefore, there has to exist a high enough $\lambda$ for which our map $(\mathscr{T}d)(v_S)$ is a contraction in the metric space $(S, ||\cdot||_\lambda)$—which, by Banach's Fixed-Point Theorem means that $(\mathscr{T}d)(v_S)$ has a unique fixed point, which in turn means that Equation (OA.3) has a single solution for any given $(v_S^c, v_M^c, S_S(0)) \in [0,1]^2 \times \Theta(M)$. Note that Equation (OA.13) does not depend on $(v_S^c, v_M^c, S_S(0))$—and thus, by standard results (see e.g. Hasselblatt and Katok, 2003, p. 68) it follows that as $(\mathscr{T}d)(v_S, v_S^c, v_M^c, S_S(0))$ is continuous in $v_S^c$, $v_M^c$ and $S_S(0)$ the fixed point—and thus the solution of (OA.3)— is continuous in them as well.

Denote the fixed point of $(\mathscr{T}d)(\cdot, v_S^c, v_M^c, S_S(0))$ as $d^*(\cdot, v_S^c, v_M^c, S_S(0))$—then the following result holds

**Lemma OA.1.** The function $d^*(\cdot, v_S^c, v_M^c, S_S(0))$ is weakly decreasing in $v_S^c$ and $S_S(0)$ and weakly increasing in $v_M^c$ for all $v_S$'s. Moreover, for some $v_S$'s, $d^*(\cdot, v_S^c, v_M^c, S_S(0))$ is strictly decreasing in $v_S^c$ and $S_S(0)$ (strictly increasing in $v_M^c$).

*Proof.* I start with the claims regarding $d(v_S, \cdot, v_M^c, S_S(0))$ and suppress $v_M^c$ and $S_S(0)$ from notation for that part of the proof. Take any $v_{S2}^c > v_{S1}^c \in [0,1]$, denote $d^*(v_S, v_{S2}^c) - d^*(v_S, v_{S1}^c)$ as $\Delta_{v_S^c} d^*(v_S, v_S^c)$ and for $v_S \geq v_S^c$ define

$$\begin{aligned} S_S(v_S, v_S^c) &= S_S(0) - \int_{v_S^c}^{v_S} \frac{\partial}{\partial v_S} C(d^*(r, v_S^c), r)\mathrm{d}r, \\ P_S(v_S, v_S^c) &= \frac{R_S - S_S(v_S, v_S^c)}{R_S}, \\ P_M(d^*(v_S, v_S^c), v_S^c) &= \frac{R_M - 1 + C(d^*(v_S, v_S^c), r) + S_S(v_S, v_S^c)}{R_M}. \end{aligned}$$

Then for any $v_S \geq v_{S2}^c$ we have

$$\begin{aligned} \Delta_{v^c} d^*(v, v^c) &= v_{S2}^c - v_{S1}^c \\ &+ \int_{v_S^c}^{v} \frac{\frac{\partial}{\partial v_S}\pi_S^e(t, P_S(t, v_{S2}^c))}{\frac{\partial}{\partial v_M}\pi_M^e(d^*(t, v_{S2}^c), P_M(d^*(t, v_{S2}^c), v_{S2}^c))} - \frac{\frac{\partial}{\partial v_S}\pi_S^e(t, P_S(t, v_{S1}^c))}{\frac{\partial}{\partial v_M}\pi_M^e(d^*(t, v_{S1}^c), P_M(d^*(t, v_{S1}^c), v_{S1}^c))}\mathrm{d}t. \end{aligned}$$

It is trivial that for any $v_S \in [v_{S1}^c, v_{S2}^c)$ we have $\Delta_{v_S^c} d^*(v_S, v_S^c) < 0$, which proves the second (strict) part of this claim. Thus, we only need to show now that $\Delta_{v_S^c} d^*(v_S, v_S^c) \leq 0$ for all $v_S \in [v_{S2}^c, 1]$. Suppose not. Then the set $\Omega^{gen} = \{v_S \in [v_{S2}^c, 1] : \ \Delta_{v_S^c} d^*(v_S, v_S^c) > 0\}$ has to be non-empty. Then we have that for $v_S{}^g = \inf \Omega^{gen}$, $\Delta_{v_S^c} d^*(v_S{}^g, v_S^c) = 0$ and $\Delta_{v_S^c} \frac{\partial}{\partial v_S} d^*(v_S{}^g, v_S^c) \geq 0$. The sign of $\Delta_{v_S^c} \frac{\partial}{\partial v_S} d^*(v_S{}^g, v_S^c)$ depends only on the signs of

$$\frac{\partial}{\partial v_S}\pi_S^e(v_S{}^g, P_S(v_{S2}^c, v_S{}^g)) - \frac{\partial}{\partial v_S}\pi_S^e(v_S{}^g, P_S(v_{S1}^c, v_S{}^g))$$

and

$$\frac{\partial}{\partial v_M}\pi_M^e(d^*(v_S{}^g, v_{S1}^c), P_M(d^*(v_S{}^g, v_{S1}^c), v_{S1}^c)) - \frac{\partial}{\partial v_M}\pi_M^e(d^*(v_S{}^g, v_{S2}^c), P_M(d^*(v_S{}^g, v_{S2}^c), v_{S2}^c)).$$

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This can be easily seen by differentiation the expression that gives $\Delta_{v^c} d^*(v, v^c)$.[6] However, as $\Delta_{v_S^c} d^*({v_S}^g, v_S^c) = 0$ and both surplus functions are weakly supermodular, these in turn depend only on the sign of $S_S(v_{S2}^c, {v_S}^g) - S_S(v_{S1}^c, {v_S}^g)$. As for any $v_S \leq {v_S}^g$ it was the case that $\Delta_{v_S^c} d^*({v_S}^g, v_S^c) \leq 0$ and $v_{S2}^c > v_{S1}^c$, it follows that: $S_S(v_{S2}^c, {v_S}^g) - S_S(v_{S1}^c, {v_S}^g) < 0$ and thus:

$$\Delta_{v_S^c} \frac{\partial}{\partial v_S} d^*(v_S, v_S^c) < 0,$$

which means that $\Omega^{gen}$ has to be empty and proves our first claim.

The proof for $S_S(0)$ is analogous.[7] For $v_M^c$, note that for a change in $v_M^c$, $\Delta_{v_M^c} d^*(v_S^c, v_M^c)$ is positive. The subsequent reasoning is analogous, but with opposite signs (the strict decreasingness follows from $\Delta_{v_M^c} d^*(v_S^c, v_M^c) < 0$ and continuity). $\square$

Everything I derived so far applies both for cases with abundant and scarce jobs. From now on, however, I will consider those cases separately.

**Scarce jobs** If $R_M + R_S < 1$, then $M = R_M + R_S$, which reduces (OA.9) to $S_S(0) = R_S$ and gives $C(v_M^c, v_S^c) = 1 - R_M - R_S > 0$. For $(v_M, v_S) > 0$, $C(\bullet)$ is strictly increasing in both parameters, which allows us to define $v_M^c$ as a strictly decreasing, continuous function of $v_S^c$. Define $\underline{v_S}$ as $v_M^c(\underline{v_S}) = 1$ and note that, as $v_M^c \in [0, 1]$, Equation (OA.5) shrinks the range of feasible $v_S^c$'s to $[\underline{v_S}, 1]$. Hence, $d^*(v_S, v_S^c, v_M^c, S_S(0))$ depends only on $v_S$ and $v_S^c$ and is decreasing and continuous in $v_S^c$—I will denote it as $d^*(v_S, v_S^c)$ from now on. Thus, the modified system of equations reduces to

$$R_S = \int_{v_S^c}^1 \frac{\partial}{\partial v_S} C^e(d^*(r, v_S^c), r) \mathrm{d}r.$$

The RHS is continuous in $v_S^c$, as $d^*(v_S, v_S^c))$ is continuous in $v_S^c$. For $v_S^c = \underline{v_S}$, we have $d^*(v_S, v_S^c) \geq 1$ regardless of $v_S$ and therefore $\int_0^1 \frac{\partial}{\partial v_S} C^e(d^*(r, v_S^c), r) \mathrm{d}r = 1$, whereas for $v_S^c = 1$, $\int_1^1 \frac{\partial}{\partial v_S} C^e(d^*(r, v_S^c), r) \mathrm{d}r = 0$; thus, a solution to (OA.6) (given $R_S \in (0, 1)$ ) exists. It is unique, as $d^*(v_S, \cdot)$ is weakly decreasing for all $v_S$ and strictly decreasing for some $v_S$ and thus the RHS crosses $R_S$ only once from above.

**Abundant jobs** If $R_M + R_S \geq 1$, then $M = 1$ and thus $C(v_M^c, v_S^c) = 0$. Hence, $\min\{v_M^c, v_S^c\} = 0$ and I cannot define $v_M^c$ as a function of $v_S^c$, as there is a continuum of $v_S^c$'s for which $C(0, v_S^c) =$

[6]To see this, note that

$$\begin{aligned}
\Delta_{v^c} \frac{\partial}{\partial v_S} d^*(v_S^g, v_S^c) &= \frac{\frac{\partial}{\partial v_S} \pi_S^e(v_S^g, P_S(v_S^g, v_{S2}^c))}{\frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S2}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c))} - \frac{\frac{\partial}{\partial v_S} \pi_S^e(v_S^g, P_S(v_S^g, v_{S1}^c))}{\frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S1}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c))} \\
&= \frac{\frac{\partial}{\partial v_S} \pi_S^e(v_S^g, P_S(v_S^g, v_{S2}^c)) - \frac{\partial}{\partial v_S} \pi_S^e(v_S^g, P_S(v_S^g, v_{S1}^c))}{\frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S2}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c))} \\
&+ \frac{\partial}{\partial v_S} \pi_S^e(v_S^g, P_S(v_S^g, v_{S1}^c)) \frac{\frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S1}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c)) - \frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S2}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c))}{\frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S1}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c)) \frac{\partial}{\partial v_M} \pi_M^e(d^*(v_S^g, v_{S2}^c), P_M(d^*(v_S^g, v_{S1}^c), v_{S1}^c))}.
\end{aligned}$$

[7]For $v_S = v_S^c$ we have $\Delta_{S_S(0)} d^*(v_S, S_S(0)) = 0$ and $\Delta_{S_S(0)} \frac{\partial}{\partial v_S} d^*(v_S, S_S(0)) < 0$. The sign of $\Delta_{S_S(0)} \frac{\partial}{\partial v_S} d^*({v_S}^g, S_S(0))$ depends on $S_{S1}(0) - S_{S2}(0) < 0$ and the difference in $S_S(v_S, S_S(0))$, which is weakly negative for the same reasons as above. Thus, $\Delta_{S_S(0)} \frac{\partial}{\partial v_S} d^*({v_S}^e, S_S(0)) \leq 0$, which implies that $d^*(v_S, \cdot)$ will never strictly increase.

0. I address this by defining the set $\Gamma^c = \{(v_M^c, v_S^c) : \min\{v_M^c, v_S^c\} = 0\}$, a new variable $a \in [-1, 1]$ and writing $v_M^c$ and $v_S^c$ as

$$v_M^c(a) = \begin{cases} -a & \text{for } a \leq 0, \\ 0 & \text{for } a > 0, \end{cases} \qquad v_S^c(a) = \begin{cases} 0 & \text{for } a \leq 0, \\ a & \text{for } a > 0. \end{cases} \tag{OA.14}$$

For any $a$, $(v_M^c(a), v_S^c(a)) \in \Gamma^c$ and for any $(v_M^c, v_S^c) \in \Gamma^c$ there exists a unique $a$, such that $(v_M^c(a), v_S^c(a)) = (v_M^c, v_S^c)$. Thus, if there exists a unique $a$ that solves Equation (OA.6), there also exists a unique $(v_M^c, v_S^c)$ that solves it. Moreover, $v_S^c(a)$ is continuous and increasing, and $v_M^c(a)$ is continuous and decreasing. Therefore the function $d^*(v_S, a, S_S(0)) = d^*(v_S, v_S^c(a), v_M^c(a), S_S(0))$ is continuous and decreasing (strictly for some $v_S$'s) in $a$. Thus, I can write Equation (OA.6) as

$$S_S(0) = \begin{cases} \int_0^1 \frac{\partial}{\partial v_S} C^e(d^*(r, a, S_S(0)), r)\mathrm{d}r & \text{for } a < 0, \\ \int_a^1 \frac{\partial}{\partial v_S} C^e(d^*(r, a, S_S(0)), r)\mathrm{d}r & \text{for } a \geq 0. \end{cases}$$

The RHS is continuous in $a$, as $d^*(v_S, a, S_S(0))$ is continuous in $a$. For $a = -1$, we have $\int_0^1 \frac{\partial}{\partial v_S} C^e(d^*(r, a, S_S(0)), r)\mathrm{d}r = 1$; for $a = 1$, we have $\int_a^1 \frac{\partial}{\partial v_S} C^e(d^*(r, a, S_S(0)), r)\mathrm{d}r = 0$; thus, a solution to (OA.6) (given $S_S(0) \in \Theta(1)$ ) exists. It is unique, as $d^*(v_S, \cdot, S_S(0))$ is weakly decreasing for all and strictly decreasing for some $v_S$ and thus the RHS crosses $S_S(0)$ only once from above.

As $d^*(v_S, \cdot, \cdot)$ is continuous, $a(S_S(0))$ is continuous as well. It is strictly decreasing in $S_S(0)$, as the LHS is strictly increasing in $S_S(0)$ and the RHS is weakly decreasing in $S_S(0)$ and strictly decreasing in $a$; thus, if $S_S(0)$ increases, Equation (OA.6) is met only if $a$ decreases. As $a(S_S(0))$ is unique and $a$ defines uniquely $(v_M^c, v_S^c)$, there exist unique $v_M^c(S_S(0))$ and $v_S^c(S_S(0))$; the former is non-decreasing and the latter non-increasing; and for any $S_{S2}(0) > S_{S1}(0)$ we have that $v_M^c(S_{S2}(0)) > v_M^c(S_{S1}(0))$ or $v_S^c(S_{S2}(0)) < v_S^c(S_{S1}(0))$.

The modified set reduces to

$$S_S(0) > 1 - R_M \Rightarrow \pi_M\Big(u^c(S_S(0)), \frac{R_M - 1 + S_S(0)}{R_M}\Big) \leq \pi_S\Big(v^c(S_S(0)), \frac{R_S - S_S(0)}{R_S}\Big) \tag{OA.15}$$

$$S_S(0) < R_S \Rightarrow \pi_M\Big(u^c(S_S(0)), \frac{R_M - 1 + S_S(0)}{R_M}\Big) \geq \pi_S\Big(v^c(S_S(0)), \frac{R_S - S_S(0)}{R_S}\Big) \tag{OA.16}$$

$$S_S(0) \in \Theta(1). \tag{OA.17}$$

Note that $v_M^c(0) = 0$, $v_S^c(0) = 1$, $v_M^c(1) = 1$ and $v_S^c(1) = 0$. Condition (OA.15)—(OA.16) will be trivially met if there exists some $S_S(0) \in \Theta(1)$ such that

$$\pi_M\Big(v_M^c(S_S(0)), \frac{R_M - 1 + S_S(0)}{R_M}\Big) = \pi_S\Big(v_S^c(S_S(0)), \frac{R_S - S_S(0)}{R_S}\Big).$$

If there is no such $S_S(0)$, then it has to be the case that either (a) $LHS > RHS$ for all $S_S(0) \in \Theta(1)$ or (b) $RHS > LHS$ for all $S_S(0) \in \Theta(1)$. However, (a) is possible only if $\max\{0, 1 - R_M\} = 1 - R_M$, as $LHS > RHS$ for $S_S(0) = 0$ violates condition (d). And for $S_S(0) = 1 - R_M$, $LHS > RHS$ meets (OA.15)—(OA.16), as the first inequality doesn't have to hold. For similar reasons, (b) is possible only if $\min\{1, R_S\} = R_S$, in which case $RHS > LHS$ meets (OA.15)—(OA.16). Thus, existence of a solution to (OA.15)—(OA.16) follows. Hence, there exists a solution to the modified and original sets.

For uniqueness, remember that $d^*(v_S, a(S_S(0)), S_S(0))$ is unique and, thus, it suffices to show that the solution to (OA.15)—(OA.16) is unique. Denote the set of all $S_S(0) \in \Theta(1)$ that meet (OA.15)—(OA.16) as $\Omega^M$. Consider $\min \Omega^M = S_{S1}(0)$. Note that $S_{S1}(0)$ exists as $\Omega^M$ is non-empty and $\pi_M(\cdot,\cdot)$, $\pi_S(\cdot,\cdot)$, $v^c_S(\cdot)$ and $v^c_M(\cdot)$ are continuous. Suppose $S_{S1}(0) = \min\{1, R_S\}$—then the solution is unique. Now suppose that $S_{S1}(0) < \min\{1, R_S\}$, which implies that for any $S_{S2}(0) \in \Omega^M$ such that $S_{S2}(0) > S_{S1}(0)$ we need to have

$$\pi_M\Big(v^c_M(S_{S2}(0)), \frac{R_M - 1 + S_{S2}(0)}{R_M}\Big) \leq \pi_S\Big(v^c_S(S_{S2}(0)), \frac{R_S - S_{S2}(0)}{R_S}\Big)$$

and for $S_{S1}(0)$ we have:

$$\pi_M\Big(v^c_M(S_{S1}(0)), \frac{R_M - 1 + S_{S1}(0)}{R_M}\Big) \geq \pi_S\Big(v^c_S(S_{S1}(0)), \frac{R_S - S_{S1}(0)}{R_S}\Big).$$

This is a contradiction, as $\frac{\partial}{\partial v_M}\pi_M > 0$, $\frac{\partial}{\partial h}\pi_M \geq 0$, $\frac{\partial}{\partial v_S}\pi_S > 0$, $\frac{\partial}{\partial h}\pi_S \geq 0$, $v^c_M(\cdot)$ is weakly increasing, $v^c_S(\cdot)$ is weakly decreasing and $v^c_M(S_{S2}(0)) > v^c_M(S_{S1}(0)) \vee v^c_S(S_{S2}(0)) < v^c_S(S_{S1}(0))$. Thus $S_{S2}(0)$ does not exist and $S_{S1}(0)$ is the only element in $\Omega^M$, which completes the proof.

### OA.1.1 Sattinger and Roy: Production Functions

Section III.C in the main text has compared the wage functions in my model to the wage functions that hold in the single-sector assignment model and the Roy's model. In this section, for completeness, I compare the production functions in these classes of models.

Denote the vector of skill $(v_M, v_S)$ by $\mathbf{v}$. Then the overall production function in my model can be written as

$$\sum_{i\in\{M,S\}} \int_{[0,1]^3} \pi_i(v_i, h) \min\{L_i(\mathbf{v}, h), K_i(\mathbf{v}, h)\} \mathrm{d}\mathbf{v}\mathrm{d}h \qquad \text{(OA.18)}$$

where $L_i(\mathbf{v}, h)$ is the number of workers of type $\mathbf{v}$ assigned to a firm of productivity $h$ that operates in sector $i$, and must satisfy

$$\sum_{i\in\{M,S\}} \int_{[0,1]} L_i(\mathbf{v}, h)\mathrm{d}h \leq \frac{\partial^2}{\partial v_M \partial v_S} C(v_M, v_S) \text{ and } L_i(\mathbf{v}, h) \geq 0;$$

whereas $K_i(\mathbf{v}, h)$ is the number of sector $i$ firms of productivity $i$ assigned to workers of type $\mathbf{v}$, and must satisfy

$$\int_{[0,1]^2} K_i(\mathbf{v}, h)\mathrm{d}\mathbf{v} \leq R_i \text{ and } K_i(\mathbf{v}, h) \geq 0.$$

First, if $R_S = 0$, then $K_S(\mathbf{v}, h) = 0$ and the production function reduces to

$$\int_{[0,1]^3} \pi_M(v_M, h) \min\{L_M(\mathbf{v}, h), K_M(\mathbf{v}, h)\} \mathrm{d}\mathbf{v}\mathrm{d}h,$$

which is the production function in Sattinger (1979).

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Second, if $\frac{\partial}{\partial h_i}\pi_i = 0$, and $R_i > 1$, then all firms are of *de facto* the same productivity and there are more of them than there are workers, and thus the production function reduces to

$$\sum_{i\in\{M,S\}} \int_{[0,1]^2} \pi_i(v_i)L_i(\mathbf{v}), \mathrm{d}\mathbf{v}$$

with the usual constraint on $L_i$. This is the production function of a Roy's economy.

Third, we can now easily compare my model to the Heckman and Sedlacek (1985) model, in which the production function is

$$\sum_{i\in\{M,S\}} U_i(\int_{[0,1]^2} \pi_i(v_i)L_i(\mathbf{v}),\ \mathrm{d}\mathbf{v}),$$

where $U_i$ is increasing and concave. It is clear that while $U_i$ introduces a non-linearity—and thus imperfect substitution—across sectors, $\pi_i(v_i)$ can be interpreted as an efficiency unit of skill in sector $i$, and thus workers are perfect substitutes within each sector.

## OA.2 Section IV: Further Results

### Fall in Concordance and Total Output

**Proposition OA.2.** Consider two copulas that meet Assumption 3. $T(C(\theta_1)) \geq T(C(\theta_2)$ for all quadruples $(\pi_M, \pi_S, R_M, R_S)$ that meet Assumptions 1, 2, and 4 if and only if $C(\bullet, \theta_2)$ is more concordant than $C(\bullet, \theta_1)$.

*Proof.* The "if" part has been proven in Section IV.A.1: Specifically, it follows from Equations (15) and (16).

The "only if" part is a simple adaptation of the standard reasoning for first-order stochastic dominance. Suppose that there exists some $(v'_M, v'_S)$ such that $C(v'_M, v'_S, \theta_1) > C(v'_M, v'_S, \theta_2)$. Then there exists a quadruple $(\pi_M, \pi_S, R_M, R_S)$ that meets Assumptions 1 and 4 for which $T(C(\rho_2)) > T(C(\rho_1))$. Consider $R_M = R_S = 1$ and following surplus functions: $\pi_M(v_M) = 0$ if $v_M \leq v'_M$ and $\pi_M(v_M) = 1$ otherwise, whereas $\pi_S(v_S) = 0.5$ if $v_S \leq v'_S$ and $\pi_S(v_S) = 1.5$ otherwise. Then the efficient assignment of workers to sectors is such that any worker with $v_M > v'_M$ and $v_S < v'_S$ works in manufacturing and all other workers work in services. The measure of workers in manufacturing is, thus, $v'_S - C(v'_M, v'_S)$ and the maximal total surplus produced in the economy is $1.5(1-v'_S)+0.5C(v'_M, v'_S)+1(v'_S - C(v'_M, v'_S))$ giving $1.5-0.5v'_S-0.5C(v'_M, v'_S)$ which is then lower for $C(v'_M, v'_S, \theta_1)$ than $C(v'_M, v'_S, \theta_2)$, as required.

The proof is not complete yet, as these surplus functions do not meet the differentiability assumption. However, they can be approximated by the following pair of surplus functions that meet Assumptions 2 and 3: $\pi_M(v_M, h_M) = \frac{1}{1+\exp(-2k(v_M - v'_M))}$ and $\pi_S(v_S, h_S) = \frac{1}{1+\exp(-2k(v_S - v'_S))} + 0.5$. As $k \to \infty$ these two functions approach the functions outlined above pointwise. It follows from the proof of Theorem 1 that the equilibrium is continuous in any parameters in which the surplus functions are continuous. Thus, it follows by the definition of a limit and by Assumption 3 that for any difference in copulas $C(v'_M, v'_S, \theta_1) - C(v'_M, v'_S, \theta_2) > 0$ there exists $k$ large enough that $T(C(\theta_2)) \geq T(C(\theta_1)$. □

Note that this result can be trivially generalized to any finite number of sectors.

**Derivation of Equation** (17)

Differentiating $F_W(W(t)) = t$ with respect to $\theta$ and rearranging yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}W(t) = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}F_W(W(t))}{\frac{\partial}{\partial x}F_W(W(t))} \tag{OA.19}$$

First, as $W(t)$ is the inverse of $F_W(x)$ it follows that

$$W'(t) = \frac{1}{\frac{\partial}{\partial x}F_W(W(t))}. \tag{OA.20}$$

Second, as $F_W(x) = C(w_M^{-1}(x), w_S^{-1}(x))$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F_W(W(t)) = \frac{\partial}{\partial\theta}C(v_M(t), v_S(t) + \sum_{i\in\{M,S\}} \frac{\frac{\partial}{\partial v_i}C(v_M(t), v_S(t))}{\frac{\partial}{\partial v_i}w_i(v_i(t))}\frac{\partial}{\partial\theta}w_i(v_i(t)), \tag{OA.21}$$

because $\frac{\partial}{\partial\theta}w_i^{-1}(x) = \frac{\frac{\partial}{\partial\theta}w_i(w^{-1}(x))}{\frac{\partial}{\partial v_i}w_i(w_i^{-1}(x))}$ and $v_i(t) = w^{-1}(W(t))$. Finally, note that as the density of all workers occupying rank $t$ must be 1, $p_i(t)$—that is, the probability that a worker that occupies the $t$th percentile in the wage distribution works in services—is simply equal to the density of workers with rank $t$ who work in sector $i$. Therefore, it follows from Equation (12) that

$$p_S(t) = \lim_{h\to 0}\frac{\int_{w_S^{-1}(W(t))}^{w_S^{-1}(W(t+h))}\frac{\partial}{\partial v_S}C(\psi(r), r)\mathrm{d}r}{h} = \frac{\frac{\partial}{\partial v_S}C(v_M(t), v_S(t))}{\frac{\partial}{\partial v_S}w_S(v_S(t))}W'(t), \tag{OA.22}$$

and analogously for $p_M(t)$. Substituting Equations (OA.20)–(OA.22) into Equation (OA.19) yields Equation (17).

**Example: Wage Effect and Polarization**

Here I will provide a numerical example demonstrating that the wage effect can lead to an increase in wage polarization.

Consider a specification of the model with unit measure of firms in each sector and surplus functions that depend on firm's type but are not strictly supermodular: $\pi_M(v_M, h_M) = v_M + h_M$ and $\pi_S(v_S, h_S) = v_S^2 + h_S$. Manufacturing is more productive and, thus, employs more workers in equilibrium.[8] As a consequence, the worker with zero skill in each sector prefers to work in services, as she is matched with a more productive firm in that sector. Wages in services are $w_S(v_S) = v_S^2 + 1 - S_S(0)$ and in manufacturing $w_M(v_M) = v_M + S_S(0)$, implying that $w_S(1) > w_M(1)$, and hence $1 = p_S(1) = p_S(0) > p_S(t)$ for $t$ close to 0. As there are overall more workers in manufacturing it must be the case that $p_M(t) > p_S(t)$ for some quantiles $t \in (0, 1)$. Suppose further that skills are distributed according to the FGM copula, with

[8] Suppose not. Then any worker with $v_M \geq v_S$ would earn strictly more in manufacturing and, hence, manufacturing would attract more workers.

| Change | Effect | The baseline[1] | Pure self-selection[2] | Single-sector[3] |
|---|---|---|---|---|
| $\frac{\partial}{\partial\theta}C > 0$ | $\frac{\partial}{\partial\theta}(W(t) - W(0)),\ t \approx 0$ | $> 0^4$ | $> 0^5$ | NA |
| | $\frac{\partial}{\partial\theta}(W(1) - W(0))$ | $\geq 0^4$ | $=^5$ | NA |
| $\frac{\partial}{\partial\theta}\frac{\partial}{\partial x_S}\pi_S > 0$, $\frac{\partial}{\partial\theta}\pi_S > 0^6$ | $\frac{\partial}{\partial\theta}S_S$ | $> 0$ | $> 0^7$ | $=$ |
| | $\frac{\partial}{\partial\theta}S_M$ | $< 0$ | $< 0^7$ | NA |
| | $\frac{\partial}{\partial\theta}(W_S(1) - W_S(0))$ | $> 0$ | $> 0^7$ | $> 0^8$ |
| | $\frac{\partial}{\partial\theta}(W_M(1) - W_M(0))$ | $\geq 0$ | $\leq 0^7$ | NA |
| | $\frac{\partial}{\partial\theta}W(t) - W(0)),\ t \approx 0$ | $< 0^9$ | $> 0^{7,9}$ | $> 0^8$ |

Table OA.1: Summary (and Comparison) of Comparative Statics Results

Notes: [1]Refers to results holding under Assumption 5 (Propositions 1 and 2) [2]The Roy specification, with $R_i > 1$ and $\frac{\partial}{\partial h_i}\pi_i = 0$. [3]Refers to the Sattinger specification, with $R_M = 0$. [4]If $\frac{\partial}{\partial\theta}\psi = 0$. [5] Follows immediately from Equation (17). [6]The second condition is needed only for Roy's model. [7]Proposition OA.3. [8]By inspection of Equation 8. [9]If $v_S^c(\theta_1) > 0$.

$C(v_M, v_S) = v_M v_S(1 + \theta(1 - v_S)(1 - v_M))$. It can be easily shown that the separation function is $\psi(v_S) = v_S^2 + v_M^c$ for $v_S \leq \sqrt{1 - v_M^c}$ and $\psi(v_S) = 1$ otherwise. Therefore, most of the workers who are equally skilled in both sectors work in manufacturing. The total number of workers in services is then $S_S(0) = 1 - \sqrt{1 - v_M^c} + \int_0^{\sqrt{1-v_M^c}}(v^2 + v_M^c)(1 + \theta(1 - 2v)(1 - v^2 - v_M^c)\mathrm{d}v$. Suppose that originally the two skills are independently distributed ($\theta = 0$), then, because $w_M(v_M^c) = w_S(v_S^c)$, $S_S(0) \approx 0.44$ and $v_M^c \approx 0.11$. By expanding $w_M(v_M^c) = w_S(v_S^c)$ we get $v_M^c + S_S(0) = 1 - S_S(0)$; differentiating this wrt $\theta$, we get $\frac{\mathrm{d}}{\mathrm{d}\theta}v_M^c = 2\frac{\mathrm{d}}{\mathrm{d}\theta}S_S(0)$, which can be further rearranged into: $\frac{\partial}{\partial\theta}v_M^c = -\frac{2\frac{\partial}{\partial\theta}S_S(0)}{(1+2\frac{\partial}{\partial v_M^c}S_S(0))}$. It is easy to verify that $\frac{\partial}{\partial\theta}S_S(0) < 0$ and $\frac{\partial}{\partial v_M^c}S_S(0) > 0$, implying $\frac{\partial}{\partial\theta}v_M^c > 0$ and, thus, $\frac{\mathrm{d}}{\mathrm{d}\theta}S_S(0) < 0$. Therefore, an increase in skill interdependence lowers the overall supply of workers in services and increases it in manufacturing. Therefore, wages increase for all skill levels in services (by $-\frac{\mathrm{d}}{\mathrm{d}\theta}S_S(0)$) and fall in manufacturing by exactly the same amount. Thus, the wage effect increases the polarization of wages by Equation 17, as $W(0)$ increases, $W(1) - W(0)$ remains constant, but the wage effect on $W(t)$ is smaller than on $W(0)$ for $t$ close to 0.

### Comparison of the Predictions of This Model, Roy's Model and Sattinger's Model

Table OA.1 summarizes the predictions of my model, a single-sector assignment model and Roy's self-selection model for the two comparative statics exercises conducted in Section IV. The results for Roy's model are proved below.

**Proposition OA.3.** Suppose that $R_M, R_S > 1$ and that $\frac{\partial}{\partial h_i}\pi_i = 0$. If $\frac{\partial}{\partial\theta}\frac{\partial}{\partial x_S}\pi_S(v_S, h_S) > 0$ for all $(v_S, h_S)$ and $\frac{\partial}{\partial\theta}\pi_S(0,0) > 0$, then (i) $\frac{\partial}{\partial\theta}S_S(v_S) > 0$ for all $v_S$, (ii) $\frac{\partial}{\partial\theta}S_M(v_M) < 0$ for all $v_M$ (iii) $\frac{\partial}{\partial\theta}(W_S(1) - W_S(0)) > 0$ (iv) $\frac{\partial}{\partial\theta}(W_M(1) - W_M(0)) \leq 0$ and (v) if $v_S^c(\theta_1) > 0$, then there exists some $\bar{t} \in (0,1)$ such that $\frac{\partial}{\partial\theta}W(t) - W(0)) < 0$ for all $t \in (0, \bar{t})$.

*Proof.* (i) As explained in Section III.C if $R_M, R_S > 1$ and $\frac{\partial}{\partial h_i}\pi_i = 0$, then $w_i(v_i) = \pi_i(v_i, 0)$. $\frac{\partial}{\partial\theta}\frac{\partial}{\partial x_S}\pi_S(v_S, h_S) > 0$ for all $(v_S, h_S)$ and $\frac{\partial}{\partial\theta}\pi_S(0,0) > 0$ imply that $\frac{\partial}{\partial\theta}w_S(v_S) > 0$ for all $v_S$. It follows from Equation (11) that $\frac{\partial}{\partial\theta}\psi(v_S) > 0$ for all $v_S \in (v_S^c, \bar{v}_S)$ from which $\frac{\partial}{\partial\theta}S_S(v_S) > 0$ for all $v_S$ follows by Equation (12).

(ii) Follows immediately from (i) and Equation (OA.2).

(iii) If $R_M, R_S > 1$ then all workers are matched and $C(\psi(v_S^c), v_S^c) = 0$. Together with $\frac{\partial}{\partial\theta}\psi(v_S) > 0$ for all $v_S \in (v_S^c, \bar{v}_S)$ (proof of result (i)) this implies that $\frac{\partial}{\partial\theta}v_S^c \leq 0$. As $\frac{\partial}{\partial\theta}(W_S(1) - W_S(0)) = \int_{v_S^c}^1 \frac{\partial}{\partial\theta}\frac{\partial}{\partial v_S}\pi_S(r,0)\mathrm{d}r - \frac{\partial}{\partial\theta}v_S^c\frac{\partial}{\partial v_S}\pi_S(v_S^c, 0)$ the result follows.

(iv) $\frac{\partial}{\partial\theta}\psi(v_S) > 0$ for all $v_S \in (v_S^c, v_S^*)$ implies that $\frac{\partial}{\partial\theta}\phi(v_M) > 0$ for all $v_M \in (v_M^c, \bar{v}_M)$; it follows by the same reasoning as above that $\frac{\partial}{\partial\theta}v_M^c \geq 0$. As $\frac{\partial}{\partial\theta}(W_M(1) - W_M(0)) = -\frac{\partial}{\partial\theta}v_M^c\frac{\partial}{\partial v_M}\pi_M(v_M^c, 0)$ the result follows.

(v) By the same reasoning as in the proof of Proposition 2(vi) we have that $p_S(0) = 0$ and thus also $\frac{\partial}{\partial t}p_M(0) < 0$. As $\frac{\partial}{\partial\theta}\left(w_M(v_M^c) - w_S(v_S^c)\right) = -\frac{\partial}{\partial\theta}w_S(v_S^c) < 0$ it follows from inspection of Equation (18) that $W'(0) > 0$ and the result follows. $\square$

## OA.3 Approximating Cobb–Douglas lognormal

Consider the following specification in the canonical form:

$$\pi_i(v_i, h_i) = e^{\Phi^{-1}((1-2a)v_i+a)\sigma_i+\mu_i+\alpha_{iF}\left(\Phi^{-1}((1-2a)h_i+a)+\mu_F\right)} + A_i,$$
$$C(v_M, v_S) = \mathbf{\Phi}_\rho\Big(\Phi^{-1}(v_M), \Phi^{-1}(v_S)\Big),$$

where $a \in [0, \frac{1}{2})$. Note that (i) for $a \in (0, 0.5)$, $\pi_i$ is well-defined and twice contiguously differentiable on $[0,1]^2$, and thus meets all conditions imposed by Assumption 2; and (ii) this specification reduces to the Gaussian-Exponential specification for $a = 0$.

## OA.4 Specifications Used to Produce Figures

In this section I report the specification used to produce Figures 3 and 5 in the main text, as well as Figure OA.1 in this appendix. Note that all figures are produced using the CDL specification of the model, with the truncation parameter $a = 0.001$ and firm measures $R_M = R_S = 0.5$. All parameters are approximated to three decimal points.

### Figure 3

Initial specification: $\alpha_{MC} = 0.497, \alpha_{MN} = 0.444, \alpha_{SC} = 0.0, \alpha_{SN} = 0.667, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 22.932, \alpha_{MF} = 0.4, \mu_{SF} = 22.932, \alpha_{SF} = 0.4$ which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 0.667, \sigma_S = 0.667, \rho = 0.667$.

Final specification: $\alpha_{MC} = 0.094, \alpha_{MN} = 0.66, \alpha_{SC} = 0.0, \alpha_{SN} = 0.667, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 22.932, \alpha_{MF} = 0.4, \mu_{SF} = 22.932, \alpha_{SF} = 0.4$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 0.667, \sigma_S = 0.667, \rho = 0.99$.

### Figure 5a

Initial specification: $\alpha_{MC} = 0.335, \alpha_{MN} = 0.3, \alpha_{SC} = 0.0, \alpha_{SN} = 0.45, \mu_{xC} = 0.05, \mu_{xN} = 0.112, \mu_{MF} = 6.881, \alpha_{MF} = 1.2, \mu_{SF} = 6.881, \alpha_{SF} = 1.2$, which implies $\mu_M = 0.05, \mu_S =$

$0.05, \sigma_M = 0.45, \sigma_S = 0.45, \rho = 0.667$.

Final specification: $\alpha_{MC} = 0.335, \alpha_{MN} = 0.3, \alpha_{SC} = 0.045, \alpha_{SN} = 0.45, \mu_{xC} = 0.05, \mu_{xN} = 0.112, \mu_{MF} = 6.881, \alpha_{MF} = 1.2, \mu_{SF} = 6.881, \alpha_{SF} = 1.2$, which implies $\mu_M = 0.05, \mu_S = 0.053, \sigma_M = 0.45, \sigma_S = 0.452, \rho = 0.738$.

**Figure 5b**

Initial specification: $\alpha_{MC} = 0.55, \alpha_{MN} = 0.0, \alpha_{SC} = 0.55, \alpha_{SN} = 0.002, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 49.63, \alpha_{MF} = 0.2, \mu_{SF} = 49.63, \alpha_{SF} = 0.2$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 0.55, \sigma_S = 0.55, \rho = 1.0$.

Final specification: $\alpha_{MC} = 0.55, \alpha_{MN} = 0.0, \alpha_{SC} = 0.55, \alpha_{SN} = 0.057, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 49.63, \alpha_{MF} = 0.2, \mu_{SF} = 49.63, \alpha_{SF} = 0.2$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 0.55, \sigma_S = 0.553, \rho = 0.995$.

**Figure 5c**

Initial specification: $\alpha_{MC} = 0.312, \alpha_{MN} = 0.95, \alpha_{SC} = 0.0, \alpha_{SN} = 1.0, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 19.915, \alpha_{MF} = 0.4, \mu_{SF} = 19.915, \alpha_{SF} = 0.4$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 1.0, \sigma_S = 1.0, \rho = 0.95$.

Final specification: $\alpha_{MC} = 0.433, \alpha_{MN} = 0.95, \alpha_{SC} = 0.3, \alpha_{SN} = 1.0, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 19.915, \alpha_{MF} = 0.4, \mu_{SF} = 19.915, \alpha_{SF} = 0.4$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 1.044, \sigma_S = 1.044, \rho = 0.991$

**Figure OA.1**

Initial specification: $\alpha_{MC} = 0.745, \alpha_{MN} = 0.667, \alpha_{SC} = 0.0, \alpha_{SN} = 4.0, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 1.934, \alpha_{MF} = 2.0, \mu_{SF} = -1.174, \alpha_{SF} = 2.0$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 1.0, \sigma_S = 4.0, \rho = 0.667$

Final specification: $\alpha_{MC} = 0.745, \alpha_{MN} = 0.667, \alpha_{SC} = 0.1, \alpha_{SN} = 4.0, \mu_{xC} = 0.0, \mu_{xN} = 0.0, \mu_{MF} = 1.934, \alpha_{MF} = 2.0, \mu_{SF} = -1.174, \alpha_{SF} = 2.0$, which implies $\mu_M = 0.0, \mu_S = 0.0, \sigma_M = 1.0, \sigma_S = 4.001, \rho = 0.685$.

## OA.5 Formal Results for Section VI

### OA.5.1 General Comparative Statics Result

I will start by providing a very general comparative statics result, which will be then used to derive the specific results discussed in Sections OA.5.2 and OA.5.3. This result is a generalization of Proposition 2 (i) and (ii)—however, the proof does not rely on differentiability of $\psi$ and $G_S$ with respect to $\theta$.

To simplify what follows, I first introduce new notation. The difference between the new and old values of any object $O$ is denoted as $\Delta_\theta O$. The greater of the old and new values of $O$ is denoted as $\max O$. Thus, for instance, the change in the measure of manufacturing workers is denoted by $\Delta_\theta S_M(0)$ and the greater critical skill in services is denoted by $\max v_S^c$.

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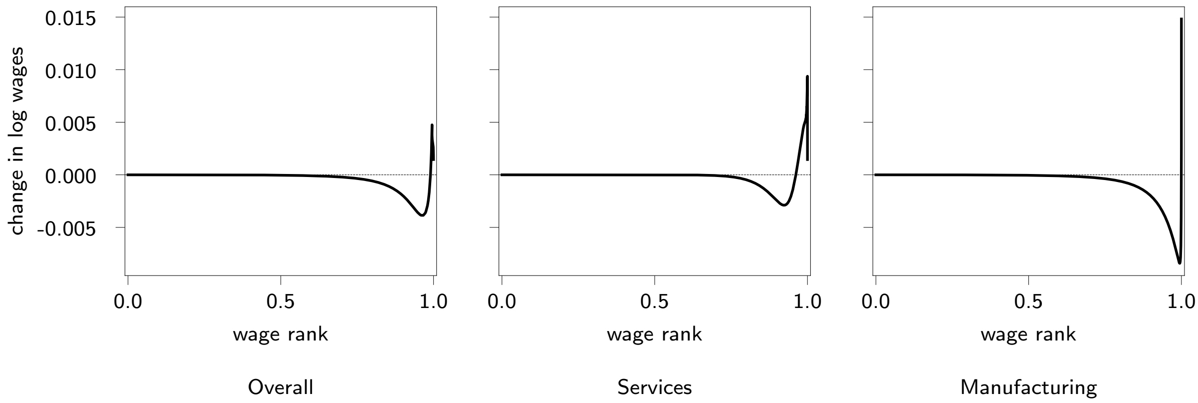


Figure OA.1: Sector-Specific TC: Rise in $\alpha_{SC}$

Notes: Figure OA.1 depicts the effect of an increase in $\alpha_{SC}$ on the inverse distribution of log wages (i.e., $W(t)$).

**Definition OA.4.** *Vertical differentiation* in services increases by (strictly) more than in manufacturing if, for all $(v_S, h)$:

$$\psi_{v_S}(v_S;\theta_2)\Delta_\theta \frac{\partial}{\partial v_M}\pi_M(\psi(v_S;\theta_2), P_M(\psi(v_S);\theta_2)) \leq (<) \Delta_\theta \frac{\partial}{\partial v_S}\pi_S(v_S, P_S(v_S;\theta_2)).$$

Note that a services-specific increase in vertical differentiation (Definition 4) implies trivially that vertical differentiation increased by more in services than in manufacturing.

**Definition OA.5.** The matching problems $(Q(\theta_1), Q(\theta_2))$ have *(strong) impossibility property* if it is impossible that $v_S^c(\theta_2) > (\geq) v_S^c(\theta_1)$ and $\Delta_\theta S_S(0) < (\leq) 0$.

**Theorem OA.1.** Suppose $(Q(\theta_1), Q(\theta_2))$ exhibit the impossibility property, $R_M(\theta_2) \leq R_M(\theta_1)$ and $R_S(\theta_2) = R_S(\theta_1)$. If vertical differentiation increases by more in services than manufacturing then (i) $S_M(v_M;\theta_2) \leq S_M(v_M;\theta_1)$ for all $v_M$ and (ii) $S_S(v_S;\theta_2) \geq S_S(v_S;\theta_1)$ for all $v_S$. If the impossibility property is strong, then (i) holds strictly for a positive measure of $v_M$ and (ii) for a positive measure of $v_S$. If the increase in differentiation is strict, then (iii) $S_M(v_M;\theta_2) < S_M(v_M;\theta_1)$ for all $v_M \in [\max v_M^c, \max \bar{v}_M)$ and (iv) $S_S(v_S;\theta_2) > S_S(v_S;\theta_1)$ for all $v_S \in [\max v_S^c, \max \bar{v}_S)$.

*Proof of Theorem OA.1.* The results for services are proved in a series of lemmas and the result for manufacturing follow easily (details at the end of the proof). But first, I define the following three sets of services talent levels

$$\begin{aligned}
\Xi^0 &= \{v_S \in [\max v_S^c, \min \bar{v}_S] : \, S_S(v_S;\theta_2) \leq S_S(v_S;\theta_1)\} \\
\Xi^1 &= \{v_S \in [\max v_S^c, \min \bar{v}_S] : \, \psi(v_S;\theta_2) > \psi(v_S;\theta_1) \wedge S_S(v_S;\theta_2) < S_S(v_S;\theta_1)\} \\
\Xi^2 &= \{v_S \in [\max v_S^c, \min \bar{v}_S) : \, \psi(v_S;\theta_2) \geq \psi(v_S;\theta_1) \wedge S_S(v_S;\theta_2) \leq S_S(v_S;\theta_1)\}
\end{aligned}$$

as well as the function $\kappa : [\max v_S^c, \min \bar{v}_S] \to \mathbf{R}$:

$$\kappa(v_S) = \Delta_\theta w_S(v_S) - \Delta_\theta w_M(\psi(v_S;\theta_2)).$$

**Lemma OA.2.** For all $v_S \in [\max v_S^c, \min \bar{v}_S]$, if $\Delta_\theta S_S(v_S) \leq (<)0$ then $\Delta_\theta S_M(\psi(v_S;\theta_2)) \geq (>)0$. Similarly, if $\Delta_\theta S_S(v_S) \geq (>)0$ then $\Delta_\theta S_M(\psi(v_S;\theta_1)) \geq (>)0$.

*Proof.* From Equation (OA.2) and Lemma 2 follows that

$$\begin{aligned}\Delta_\theta S_M(\psi(v;\theta_2)) = &-\Delta_\theta S_S(v)\\ &-\Big[\int_{\psi(v;\theta_1)}^{\psi(v;\theta_2)} \frac{\partial}{\partial v_M} C\big(r,v\big)\mathrm{d}r - \int_{\psi(v;\theta_1)}^{\psi(v;\theta_2)} \frac{\partial}{\partial v_M} C\big(r,\phi(r;\theta_1)\big)\mathrm{d}r\Big].\end{aligned}$$

If $\psi(v_S;\theta_2) \geq \psi(v_S;\theta_1)$ then for any $r \in [\psi(v_S;\theta_1),\psi(v_S;\theta_2)]$, $\phi(r;\theta_1) \geq v_S$ and my claim follows. If $\psi(v_S;\theta_2) < \psi(v_S;\theta_1)$ then for any $r \in [\psi(v_S;\theta_2),\psi(v_S;\theta_1)]$, $\phi(r;\theta_1) < v_S$ and my claim follows as well. The second statement follows from an analogous reasoning. □

**Lemma OA.3.** Suppose that vertical differentiation increases by (strictly) more in manufacturing than services. Then $\frac{\partial}{\partial v_S}\kappa(v_S) \geq (>)0$ for all $v_S \in \Xi^0$.

*Proof of Lemma OA.3.* Take any $v_{S0} \in \Xi^0$ and note that by Lemma OA.2 we have $\Delta_\theta P_M(\psi(v_{S0};\theta_2)) \leq 0$. Then we have

$$\begin{aligned}\Delta_\theta \frac{\partial}{\partial v_M} w_M(\psi(v_0;\theta_2)) = &\Delta_\theta \frac{\partial}{\partial v_M}\pi_M(\psi(v_0;\theta_2), P_M(\psi(v_0;\theta_2)))\\ &+\int_{P_M(\psi(v_0;\theta_2);\theta_1)}^{P_M(\psi(v_0;\theta_2);\theta_2)} \frac{\partial^2}{\partial v_M \partial h}\pi_M(\psi(v_0;\theta_2), r;\theta_2)\mathrm{d}r,\end{aligned}$$

and

$$\Delta_\theta \frac{\partial}{\partial v_S} w_S(v_{S0}) = \Delta_\theta \frac{\partial}{\partial v_S}\pi_S(v_S, P_S(v_S;\theta_2)) + \int_{P_S(v_{S0};\theta_1)}^{P_S(v_{S0};\theta_2)} \frac{\partial^2}{\partial v_S \partial h}\pi_S(v_{S0}, r)\mathrm{d}r \geq 0,$$

By differentiating Equation (11) wrt to v for both $\theta_2$ and $\theta_1$, taking differences and rearranging, we arrive at

$$\frac{\partial}{\partial v_S}\kappa(v_S) = \big[\Delta_\theta \frac{\partial}{\partial v_S} w_S(v_{S0}) - \psi_{v_S}(v_{S0};\theta_2)\Delta_\theta \frac{\partial}{\partial v_M} w_M(\psi(v_{S0};\theta_2)\big] \geq (>)0,$$

because vertical differentiation increases by more in services than manufacturing, $\pi_M, \pi_S$ are supermodular, $\Delta_\theta P_M(\psi(v_{S0};\theta_2)) \leq 0$ and $\Delta_\theta P_S(v_{S0}) \geq 0$. □

**Lemma OA.4.** Suppose that $\frac{\partial}{\partial v_S}\kappa(v_S) \geq (>)0$ for all $v_S \in \Xi^0$. Then for any $v_1 \in \Xi^1(\Xi^2)$ it is the case that $(v_1, \min \bar{v}_S] \subset \Xi^1$.

*Proof of Lemma OA.4.* First, note that $\kappa(v_S) = \int_{\psi(v_S;\theta_2)}^{\psi(v_S;\theta_1)} \frac{\partial}{\partial v_S}\pi_M(r, P(r;\theta_1);\theta_1)\mathrm{d}r$. Because $\frac{\partial}{\partial v_S}\pi_M > 0$ it follows that $\mathrm{sgn}(\Delta_\theta\psi(v_S)) = \mathrm{sgn}(\kappa(v_S))$. In particular, this means that $\kappa(v_1) > (\geq)0$. Second, define the set $\Xi^3 = \{v_S \in [v_1, \min \bar{v}_S] : v_S \notin \Xi^1\}$.

I will first show the result for $v_1 \in \Xi^1$. Suppose $\Xi^3$ is non-empty—then continuity of $\psi$ and $S_S$ implies that $\min \Xi^3$ exists; clearly $\min \Xi^3 > v_1$. Further, $[v_1, \min \Xi^3] \subset \Xi^0$. Therefore, $\kappa(v_S) > 0$ for all $v \in [v_1, \min \Xi^3]$, which (together with $\kappa(v_1) > 0$) implies that $\kappa(v_S) > 0$ and thus also $\Delta_\theta \psi(v_S) > 0$ $v \in [v_1, \min \Xi^3]$. However, the last fact implies that

$$\Delta_\theta S_S(v_1) = \Delta_\theta S_S(\min \Xi_3) + \int_{\min v_1}^{\min \Xi_3} \int_{\psi(r;\theta_1)}^{\psi(r;\theta_2)} \frac{\partial^2}{\partial v_M \partial v_S} C(s,r) \mathrm{d}s\mathrm{d}r > 0,$$

and $v_1 \notin \Xi^1$; contradiction!

Now suppose that $v_1 \in \Xi^2$. By continuity of $\kappa$ and the fact that $\frac{\partial}{\partial v_S}\kappa(v_1) > 0$, there must exist some $v_2 > v_1$ such that for all $v_S \in [v_1, v_2]$ we have $\frac{\partial}{\partial v_S}\kappa(v_1) > 0$. It follows that $\kappa(v_S) > 0$ and $\Delta_\theta \psi(v_S) > 0$ for all $v_S \in (v_1, v_2]$, from which follows that $\Delta_{\theta_3} S_S(v_S) < 0$ for all $v_S \in (v_1, v_2]$. Therefore, $(v_1, v_2] \subset \Xi^1$; by the reasoning above follows that $[v_2, \min \bar{v}_S] \subset \Xi^1$; combining these two completes the proof. $\square$

**Lemma OA.5.** Suppose that for any $v_1 \in \Xi^2(\Xi^1)$ it is the case that $[v_1, \min \bar{v}_S] \in \Xi^1$. Then $\Xi^2$ $(\Xi^1)$ is empty.

*Proof of Lemma OA.5.* Take any $v_{S1} \in \Xi^2(\Xi_1)$. This implies that $\Delta_\theta \psi(v_S) > 0$ for all $v_S \in [v_{S1}, \min \bar{v}_S]$, which implies that $\bar{v}_S(\theta_2) < \bar{v}_S(\theta_1)$. $\Delta_\theta S_S(v_{S1})$ can be expanded into:

$$\begin{aligned}\Delta_\theta S_S(v_{S1}) &= \int_{v_{S1}}^{\bar{v}_S(\theta_2)} \frac{\partial}{\partial v_S} C(\psi(v_S;\theta_2), v_S)\mathrm{d}v_S - \int_{v_{S1}}^{\bar{v}_S(\theta_1)} \frac{\partial}{\partial v_S} C(\psi(v_S;\theta_1), v_S)\mathrm{d}v_S - \Delta_\theta \bar{v}_S \\ &= \int_{v_{S1}}^{\bar{v}_S(\theta_1)} \int_{\psi(v_S;\theta_1)}^{\psi(v_S;\theta_2)} \frac{\partial^2}{\partial v_M \partial v_S} C(s, v_S)\mathrm{d}s\mathrm{d}v_S - \int_{\bar{v}_S(\theta_1)}^{\bar{v}_S(\theta_2)} 1 - \frac{\partial}{\partial v_S} C(\psi(v_S;\theta_2), v_S)\mathrm{d}v_S.\end{aligned}$$

The LHS is (strictly) negative, whereas the RHS is strictly positive—contradiction. Thus $\Xi^2(\Xi^1)$ must be empty, as required. $\square$

**Lemma OA.6.** Suppose $\Xi^1$ is empty. Consider some $v_{Se} \in [\max v_S^c, \min \bar{v}_S]$. Then $\Delta_\theta S_S(v_{Se}) \geq 0$ implies $\Delta_\theta S_S(v_{Se}) \geq 0$ for all $v_S \in [v_{Se}, \min \bar{v}_S]$. If $\Xi^2$ is empty, then additionally $\Delta_\theta S_S(v_{Se}) > 0$ implies $\Delta_\theta S_S(v_{Se}) > 0$ for all $v_S \in [v_{Se}, \min \bar{v}_S)$.

*Proof.* I will start with the first claim. Suppose it is false. Then the set $\Upsilon^1 = \{v_S \in [v_{Se}, \min \bar{v}_S] : \Delta_\theta S_S(v_S) < 0\}$ has to be non-empty. Take some ${v_S}^1 \in \Upsilon^1$ and define $\Upsilon^2 = \{v_S \in [v_{Se}, v_{S1}] : \Delta_\theta S_S(v_S) \geq 0\}$. By continuity of $\Delta_\theta S_S(v_S)$ the point ${v_S}^2 = \max \Upsilon^2$ exists and is $< {v_S}^1$. Therefore, for any $v_S \in ({v_S}^2, {v_S}^1]$ we have $\Delta_\theta S_S(v_S) < 0$. However, as:

$$\Delta_\theta S_S({v_S}^1) = \Delta_\theta S_S({v_S}^2) - \int_{{v_S}^2}^{{v_S}^1} \int_{\psi(r;\theta_1)}^{\psi(r;\theta_2)} \frac{\partial^2}{\partial v_M \partial v_S} C(s,r)\mathrm{d}s\mathrm{d}r,$$

this implies that there exists some $v_{S1} \in ({v_S}^2, {v_S}^1]$ such that $\Delta_\theta \psi(v_{S1}) > 0$ and thus $v_{S1} \in \Xi^1$—contradiction.

Let us move to the second claim. Again, suppose it is false. Then the set $\Upsilon^3 = \{v_S \in [v_{Se}, \min \bar{v}_S] : \Delta_\theta S_S(v_S) \leq 0\}$ has to be non-empty; but as $\Delta_\theta S_S(v_S)$ is continuous in v, the non-emptiness implies that ${v_S}^3 = \min \Upsilon^3$ exists. Additionally, ${v_S}^3 > v_{Se}$, as $\Delta_\theta S_S(v_{Se}) > 0$. Define a new set $\Upsilon^4 = \{v_S \in [v_{Se}, {v_S}^3] : \Delta_\theta \psi(v_S) \geq 0\}$ and ${v_S}^4 = \max \Upsilon^4$; by definition of ${v_S}^3$, for any $v_S > {v_S}^3 \wedge \in \Upsilon^4$ we have that $\Delta_\theta S_S(v_S) > 0$. As $[v_{Se}, {v_S}^3]$ is a compact set and $\Delta_\theta \psi(v_S)$ is continuous ${v_S}^4$ won't exist only if $\Upsilon^4$ is empty; but an empty $\Upsilon^4$ implies that

$\Delta_\theta\psi(v_S) < 0$ for any $v_S \in [v_{Se}, {v_S}^3]$, which in turn means that $\Delta_\theta S_S({v_S}^3) > 0$, which contradicts the definition of ${v_S}^3$. Therefore ${v_S}^4$ needs to exist. Now suppose that ${v_S}^4 < {v_S}^3$; then we have $\Delta_\theta S_S({v_S}^4) > 0$ and for any $v_S \in ({v_S}^4, {v_S}^3]$, $\Delta_\theta\psi(v_S) < 0$, which implies that $\Delta_\theta S_S({v_S}^3) > 0$ and also contradicts the definition of ${v_S}^3$. Therefore it has to be the case that ${v_S}^3 = {v_S}^4$; but this implies that $\Delta_\theta(\psi({v_S}^3)) \geq 0$ and $\Delta_\theta S_S({v_S}^3) \leq 0$, which contradicts emptiness of $\Xi^2$. □

**Lemma OA.7.** $\Delta_\theta S_S(\min \bar{v}_S) \geq 0$ implies that (i) for any $v_S > \max \bar{v}_S$ we have $\Delta_\theta S_S(v_S) \geq 0$ and (ii) for all $v_S \in [\min \bar{v}_S, \max \bar{v}_S)$ we have $\Delta_\theta S_S(v_S) > 0$.

*Proof.* Note that $\Delta_\theta S_S(\min \bar{v}_S) > (\geq) 0$ implies that $\bar{v}_S(\theta_2) < (\leq) \bar{v}_S(\theta_1)$[9]. Thus, if $\Delta_\theta S_S(\min \bar{v}_S) = 0$ then $\min \bar{v}_S = \max v_S^c$ and the second claim follows trivially. Whereas if $\Delta_\theta S_S(\min \bar{v}_S) > 0$ then $\bar{v}_S(\theta_2) < \bar{v}_S(\theta_1)$ and by the fact that all agents with $v_S \in (\bar{v}_S, 1]$ join services for sure it follows that for $v_S \in (\bar{v}_S(\theta_1), \bar{v}_S(\theta_2))$ we also have $\Delta_\theta S_S(v_S) > 0$. Claim (i) for $v_S > \max \bar{v}_S$ follows easily from the aforementioned property of $\bar{v}_S$. □

**Lemma OA.8.** The (strong) impossibility property implies that if $v_S^c(\theta_2) > (\geq) v_S^c(\theta_1)$ then $\Delta_\theta S_S(v_S^c(\theta_2)) > 0$.

*Proof.* This follows from the fact that $\Delta_\theta v_S^c > (\geq) 0$ implies that $\Delta_\theta G_S(v_S^c(\theta_1)) < (\leq) 0$, the fact that:

$$\Delta_\theta S_S(v_S) = \big(1 - (G_S(v_S;\theta_2)\big)\Delta_\theta S_S(0) - S_S(0;\theta_1)\Delta_\theta G_S(v_S) \tag{OA.23}$$

and the fact that $v_S^c(\theta_1) < 1$ and thus $1 - G_S(v_S;\theta_2) > 0$. □

**Lemma OA.9.** Empty $\Xi^1$ and the impossibility property jointly imply $\Delta_\theta S_S(\max v_S^c) \geq 0$. If either the increase in vertical differentiation is strict or the property is strong then this inequality holds strictly.

*Proof.* Suppose (strong) impossibility property holds. Define a set $\Xi^5 = \{v_S \in [\max v_S^c, \max \bar{v}_S) : \Delta_\theta\psi(v_{S5}) > 0 \text{ and } \Delta_\theta S_S(v_{S5}) \leq 0\}$. By continuity, there has to exist some arbitrarily small $\epsilon > 0$ such that $v_{S5} + \epsilon \in \Xi^1$; thus, by Lemma OA.5, an increase in vertical differentiation implies that $\Xi^5$ has to be empty.

If $v_S^c(\theta_2) > (\geq) v_S^c(\theta_1)$, then by Lemma OA.8 we have $\Delta_\theta S_S(\max v_S^c) > 0$. If $v_S^c(\theta_2) \leq v_S^c(\theta_1)$ and $\max v_S^c \geq \min \bar{v}_S$, then—as $\bar{v}_S > v_S^c$—it has to be that $\bar{v}_S(\theta_1) > v_S^c(\theta_1) \geq \bar{v}_S(\theta_2)$. But as all agents with $v_S > \bar{v}_S$ join services, this implies $\Delta_\theta S_S(v_S^c(\theta_2)) > 0$.

Thus, we only need to show the result for $\max v_S^c < \min \bar{v}_S$ and $v_S^c(\theta_2) \leq (<) v_S^c(\theta_1)$. As $\Delta_\theta R_M \leq 0$ we have $C(v_M^c(\theta_1), v_S^c(\theta_1)) \leq C(v_M^c(\theta_2), v_S^c(\theta_2))$ and thus $\Delta_\theta v_S^c \leq (<) 0$ implies $\Delta_\theta v_M^c \geq 0$. As $\psi(v_S^c) = v_M^c$ and $\psi(v_S)$ is strictly increasing for any $\theta_i$, we have: $\psi(v_S^c(\theta_1);\theta_2) \geq (>) v_M^c(\theta_2)$, $v_M^c(\theta_2) \leq v_M^c(\theta_1)$ and $v_M^c(\theta_1) = \psi(v_S^c(\theta_1);\theta_1)$, which trivially implies that

$$\Delta_\theta\psi(v_S^c(\theta_1)) \geq (>) 0.$$

[9] To see this, denote the $\theta_j$ for which $\bar{v}_S(c_i) = \max \bar{v}_S$ as $\theta_m$; then we have

$$\Delta_\theta S_S(\min \bar{v}_S;\theta_1) = \int_{\bar{v}_S(\theta_2)}^{\bar{v}_S(\theta_1)} 1 - \frac{\partial}{\partial v_S} C(\psi(v_S, c_m), v_S) dv_S.$$

As $1 - \frac{\partial}{\partial v_S} C(\psi(v_S, c_m), v_S) \geq 0$, the fact that $\Delta_\theta S_S(\min \bar{v}_S) > (\geq) 0$ implies that for this to hold we need $\bar{v}_S(\theta_2) < (\leq) \bar{v}_S(\theta_1)$.

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If this inequality holds weakly, then empty $\Xi^1$ implies $\Delta_\theta S_S(v_S^c(\theta_1)) \geq 0$. If $\Delta_\theta \psi(v_S^c(\theta_2)) > 0$ (i.e. when the impossibility property is strong), then—as $\Xi^5$ is empty—$\Delta_\theta S_S(\max v_S^c) > 0$. If $\Xi^2$ is empty, then $\Delta_\theta \psi(v_S^c(\theta_2)) \geq 0$ implies $\Delta_\theta S_S(\max v_S^c) > 0$, which concludes the proof. □

**Lemma OA.10.** Empty $\Xi^1$ and impossibility properties imply jointly that for any $v_S > \max v_S^c$, $\Delta_\theta S_S(v_S) \geq 0$.

*Proof.* Suppose $\Delta_\theta v_S^c > 0$—then for all $v_S > \max v_S^c$ we have that $\Delta_\theta G_S(v_S^c(\theta_1)) \leq 0$ and by impossibility property that $\Delta_\theta S_S(0) \geq 0$. Thus, the claim follows from Equation (OA.23). Now suppose that $\Delta_\theta v_S^c \leq 0$. This implies that for any $v_S \leq v_S^c(\theta_2)$ it is the case that $\Delta_\theta G_S(v_S^c(\theta_2)) = G_S(v_S;\theta_2) \geq 0$ and this expression is increasing in v. As by Lemma OA.9 $\Delta_\theta S_S(v_S^c(\theta_1)) \geq 0$ it follows from Equation (OA.23) that $\Delta_\theta S_S(v_S) \geq 0$ for all $v_S < \max v_S^c$, as required. Note that this implies also that $\Delta_\theta S_S(0) = \Delta S_S(0) \geq 0$. □

All results for services follow easily from Lemmas OA.3, OA.4, OA.5, OA.6, OA.7, OA.9 and OA.10 as well as continuity of $\Delta_\theta S_S(\cdot)$. As Lemma OA.7 has an exact manufacturing analogue, the manufacturing results for $v_M \geq \max v_M^c$ follow from services results and Lemma OA.2. The results for $v_M < \max v_M^c$ follow from reasoning analogous to that in proof of Lemma OA.10 once we note that $\Delta_\theta S_S(0) \geq 0$ implies $\Delta_\theta S_M(0) \leq 0$. □

### OA.5.2 Scarce and Abundant Jobs

This section provides the formal results on which the discussion in Section VI.A is based.

#### Scarce Jobs Case

**Changes in Concordance** In the scarce jobs case, only the results concerning changes in the difference between the wages earned by highest and lowest earning workers are certain to survive.

**Proposition OA.4.** Suppose that $R_M + R_S \leq 1$, the concordance of the skill distribution increases regularly, and Equation (19) holds. Then (i) $W(1) - W(0)$ increases (strictly if $\frac{\partial^2}{\partial v_i \partial h}\pi_i > 0$ for $i \in \{M, S\}$).

*Proof.* If jobs are scarce, then $w_i(v_i^c) = 0$. Thus, the change in $W(1) - W(0)$ is equal to the change in $\max\{w_M(1), w_S(1)\}$. It follows from Equations (19) and (OA.2) that $0 \leq \frac{\partial}{\partial\theta}\frac{\partial}{\partial v_i} w_i(v_i)$ for all $v_i \in [v_i^c, \bar{v}_i]$. Therefore, the increase in $w_i(1)$ follows by inspection of Equation 8 and the definition of $\bar{v}_i$. □

The impact on lower-tail inequality is ambiguous. The intuition is that while the forces causing a fall in lower-tail inequality are still present, they might operate exclusively on the part of the wage distribution that is occupied by workers that earn reservation wages. To see this, suppose that the two sectors are symmetric; for $t \geq 1 - 2R_S$ Equation (20) becomes then

$$\frac{\mathrm{d}}{\mathrm{d}\theta}W(t) \leq \underbrace{-W'(t)\frac{\partial}{\partial\theta}C(v_M(t), v_S(t))}_{(1)<0} + \underbrace{\int_{v_S^c}^{v_S(t)} \frac{\partial}{\partial\theta}\frac{C(s,s)}{R_S}\frac{\partial^2 \pi_S(s, C(s,s))}{\partial v_S \partial h_S}\,\mathrm{d}s}_{(2)>0} \underbrace{- \frac{\partial}{\partial\theta}v_S^c \frac{\partial \pi_S(v_S^c, 0)}{\partial v_S}}_{(3)>0}$$

We cannot conclude that this expression is negative for two reasons. First, the regularity condition ensures only that $\frac{\partial}{\partial\theta}C(v_M(t), v_S(t))$ is increasing for some $t$ close to 0, and thus it may be decreasing for $t \geq 1 - 2R_S$. Thus, this expression cannot be bound from above in the same way as Equation (20). Second, there is the additional term (3) on the RHS that is part of the wage effect and is finite for all $t \geq 1 - R_M - R_S$. In other words, for the skill levels for which workers start earning 'non-reservation' wages, the wage effect has already accumulated, and there is no guarantee that the composition effect dominates.

**Changes in Surplus** Again, all the results concerning upper-tail wage inequality survive, but the results concerning lower-tail inequality become ambiguous.

**Proposition OA.5.** Suppose $R_M + R_S < 1$ and workers in services become more vertically differentiated, then (i) $G_S(v_S)$ falls for all $v_S$ and (ii) $G_M(v_M)$ increases for all $v_S$. As a consequence, (iii) $\frac{\partial}{\partial v_M}w_M(v_M)$ increases for all $v_M \geq v_M^c(\theta_1)$ and (iv) $W_M(1)-W_M(0)$ increases. In services, (v) $W_S(1) - W_S(0)$ increases by more than $W_M(1) - W_M(0)$.

*Proof.* (i) and (ii) In the scarce jobs case $\Delta_\theta S_S(0) = 0$ and thus the impossibility property (Definition OA.5) is satisfied; the result follows from Theorem OA.1 and the fact that, if $R_M + R_S < 1$, then $G_i(v_i) = 1 - \frac{S_i(v_i)}{R_i}$.

(iii) Follows immediately from (ii) by inspection of Equation (8).

(iv) $\frac{\partial}{\partial\theta}v_M^c \leq 0$ by (ii) and (iv) follows by inspection of Equation (8). (v) I will start by showing that $\Delta\bar{v}_M \geq 0$, $\Delta\bar{v}_S \leq 0$, with at least one of these holding strictly. The first part follows trivially from Theorem OA.1 and Lemma OA.7. Suppose $\Delta\bar{v}_S = 0$; consider the set $\Omega^T = \{v_S \in [\max v_S^c, \min\bar{v}_S] : \Delta\psi(v_S) > 0\}$ and its minimum ${v_S}^5$. Suppose ${v_S}^5 \neq \min\bar{v}_S$. By Theorem OA.1 we have then that $\Delta S_S(v_S^5) > 0$, which implies $\Delta S_S(\min\bar{v}_S) > 0$, and thus $\Delta\bar{v}_S < 0$—contradiction. Therefore, if $\Delta\bar{v}_S = 0$, then $\Delta\psi(\min\bar{v}_S) > 0$, which implies $\Delta\bar{v}_M > 0$.

It follows that

$$w_S(\bar{v}_S(\theta_1);\theta_2) \geq w_S(\bar{v}_S(\theta_2);\theta_2) = w_M(\bar{v}_M(\theta_2);\theta_2) \geq w_M(\bar{v}_M(\theta_1);\theta_2)$$
$$w_M(\bar{v}_M(\theta_2);\theta_1) \geq w_M(\bar{v}_M(\theta_1);\theta_1) = w_S(\bar{v}_S(\theta_1);\theta_1) \geq w_S(\bar{v}_S(\theta_2);\theta_1)$$

with at least one inequality holding strictly, which trivially implies

$$w_S(\bar{v}_S(\theta_1);\theta_2)w_S(\bar{v}_S(\theta_2);\theta_1) > w_M(\bar{v}_M(\theta_1);\theta_2) - w_M(\bar{v}_M(\theta_2);\theta_1). \quad \text{(OA.24)}$$

Thus, $w_M(\bar{v}_M(\theta_1))$ increases strictly. For any $v_i > \bar{v}_i$ we have that

$$w_i(v_i) = \int_{\bar{v}_i}^{v_i} \frac{\partial}{\partial v_i}\pi_i(r, G_i(r))\mathrm{d}r + w_i(\bar{v}_i(\theta_1)). \quad \text{(OA.25)}$$

For $v_i > \bar{v}_i(\theta_1)$, $G_M(v_i)$ does not change; and as surplus' spread implies that $\frac{\partial}{\partial v_S}\pi_S(v_S, h)$ strictly increases, it follows that $w_S(1)$ increases by more than $w_M(1)$; the result follows as $W_i(0) = 0$. □

The change in lower-tail inequality is ambiguous even if $v_S^c(\theta_1) > v_M^c(\theta_1)$ because the change in lower-tail within-sector wage inequality (Term (1) in Equation (18)) is non-zero in the scarce jobs case. In particular,

$$p_S(1-R_M-R_S)=\frac{\frac{\partial}{\partial v_S}C(v_M^c,v_S^c)}{\frac{\partial}{\partial v_S}w_S(v_S^c))}W'(1-R_M-R_S)>0$$

and $\frac{\partial}{\partial\theta}\frac{\partial}{\partial v_M}w_M(v_M(0)) = -\frac{\partial}{\partial\theta}v_M^c\frac{\frac{\partial}{\partial v_M}C(v_M^c,v_S^c)}{R_M}\frac{\partial^2}{\partial v_M\partial h_M}\pi_M(v_M^c,G_M(v_M^c)) \geq 0$. While between-sector lower-tail inequality still falls, we cannot be certain anymore that this fall dominates the increase in within-sector inequality.

### Abundant Jobs Case

In the main text I have claimed (Equation (23)) that the change in the inverse distribution of wages can be decomposed into the baseline effect and the size effect. To see that the effect of a change in selection that leaves $S_i(0)$ unchanged really is the same as in the baseline model, note that $P_i(v_i)\equiv 1-S_i(v_i)/R_i=\frac{S_i(0)}{R_i}G_i(v_i)+1-\frac{S_i(0)}{R_i}$ and thus the impact on wages a change from $\theta_3$ to $\theta_2$ is the same as the impact of a change from $\pi_S(v_i,h_i;\theta_4)=\pi_S(v_i,h_i\frac{R_S(\theta_1)}{S_i(0;\theta_3)}+1-\frac{R_S(\theta_1)}{S_i(0;\theta_3)};\theta_3)$ to $\pi_S(v_i,h_i;\theta_5)=\pi_S(v_i,h_i\frac{R_S(\theta_1)}{S_i(0;\theta_3)}+1-\frac{R_S(\theta_1)}{S_i(0;\theta_3)};\theta_2)$ in a model in which $R_M=S_M(0;\theta_3)$ and $R_S=S_S(0;\theta_3)$.

**Changes in Concordance** In the abundant jobs case the results with respect to changes in skill concordance carry through unchanged because (a) if $\frac{\partial}{\partial h_i}\pi_i(v_i^c,1-\frac{S_i(0;\theta)}{R_i})>0$ in at least one sector, then Equation (19) necessitates that $S_i(0)$ remains constant, and thus that there is no size effect and (b) otherwise, the lowest wage is unchanged. The former is easiest to see in the case of strictly supermodular surplus $\frac{\partial^2}{\partial v_i\partial h_i}\pi_i>0$, as then $\frac{\partial^2}{\partial\theta\partial v_i}w_i(v_i^c)=-\frac{\partial}{\partial\theta}\frac{S_i(0)}{R_i}\frac{\partial^2}{\partial v_i\partial h_i}\pi_i(v_i^c,P_i(v_i^c))$, and thus by differentiating Equation (19) and by $S_M(0)+S_S(0)=1$ follows that $\frac{\partial}{\partial\theta}S_i(0)=0$.

**Proposition OA.6.** Suppose that $R_M+R_S>1$, then the concordance of the skill distribution increases regularly, and Equation (19) holds. Then (i) wage polarization increases in both absolute and relative terms, with both $W(t)-W(0)$ and $\log W(t)-\log W(0)$ falling strictly for some $t\in(0,\bar{t})$. In addition, (ii) if $\frac{\partial^2}{\partial v_i\partial h}\pi_i>0$ for $i\in\{M,S\}$, then $W(1)-W(0)$ and $\log W(1)-\log W(0)$ increase strictly.

*Proof.* (i) The proof for changes in absolute terms is analogous to the proof of Proposition 1, with Equation (OA.2) yielding

$$\frac{\partial}{\partial\theta}P_M(v_M)=\frac{1}{R_M}\frac{\partial}{\partial\theta}\left(C(v_M,\psi(v_M))-R_SP(v_S)\right).$$

which (together with Equation (19)) implies that

$$0\leq\frac{\partial}{\partial\theta}\frac{\partial}{\partial v_S}w_S(v_S)\leq\frac{\partial^2}{\partial v_S\partial h_S}\pi_S(v_S,P_S(v_S)))\frac{\partial}{\partial\theta}\frac{C(\psi(v_S),v_S)}{R_S}$$

and thus—substituting Equation19 into Equation (17)—we have that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(W(t)-W(0))\leq\frac{\partial}{\partial\theta}C(v_S(t),v_S(t))\Big[\int_{v_S^c}^{v_S(t)}\frac{1}{R_S}\frac{\partial^2}{\partial v_S\partial h_S}\pi_S(s,P_S(s))\,\mathrm{d}s-W'(t)\Big]<0.$$

for $t \approx 0$ and thus the result follows.

For changes in relative terms, it suffices to show what happens to the lowest wage in the economy $w_i(v_i^c)$. First, note that if $S_i(0;\theta) < R_i$ for both $i$ then $w_i(v_i^c;\theta) = \pi_i(v_i^c, 1 - \frac{S_i(0;\theta)}{R_i})$. Thus, because $S_M(0) + S_S(0) = 1$ it follows from Equation (19) that either $\frac{\partial}{\partial\theta} S_i(0) = 0$ or $\frac{\partial}{\partial h_i}\pi_i(v_i^c, 1 - \frac{S_i(0;\theta)}{R_i}) = 0$ in both sectors; in either case, the lowest wage remains unchanged. Suppose, instead that $S_M(0;\theta) = R_M$, in which case $w_i(v_i^c;\theta) = \pi_M(v_M^c, 1 - \frac{S_M(0;\theta)}{R_M})$. However, as $\frac{\partial}{\partial\theta} S_M(0;\theta) = S_M(0;\theta_2) - S_M(0;\theta_1)$, if $\frac{\partial}{\partial\theta} S_i(0;\theta) < 0$ then $S_M(0;\theta) < R_M$ for all $\theta \in (0,1]$; contradiction! It follows that $w_i(v_i^c)$ is unchanged and the result follows.

(ii) follows by the same reasoning as Proposition 1 (ii).

□

**Changes in Surplus** If jobs are abundant ($R_M + R_S > 1$), the level of surplus plays a role in determining whether a firm hires any worker at all or exits the market, that is, in determining the extensive margin of a firm's hiring decision. In particular, if there is no change in workers' vertical differentiation in services but the level of surplus falls, then some low-productivity services firms will likely decide to leave the market, which will shift the demand for services skill downward. To address this, in this section I focus on changes in surplus that both increase workers' vertical differentiation and increase the levels of surplus.

**Definition OA.6** (Increase in Levels)**.** The level of surplus produced in services *increases universally* if, for all $(v_S, h_S) \in [0,1]^2$, $\pi_S(v_S, h_S; \theta_2) \geq \pi_S(v_S, h_S; \theta_1)$.

**Proposition OA.7.** If (a) $R_M + R_S > 1$, (b) surplus levels in services increase universally, and (c) services workers become more vertically differentiated, then more skill is supplied to services and less to manufacturing in equilibrium ($S_S(v_S)$ increases and $S_M(v_M)$ falls for all $v_i$).

*Proof.* Suppose the impossibility property does not hold, and thus $\Delta_\theta S_S(0) < 0$, $\Delta_\theta v_S^c > 0$ which implies $\Delta_\theta S_M(0) > 0$ and $\Delta_\theta v_M^c \leq 0$. Denote $P_i(v_i^c)$ by $h_i^c$. It follows trivially that $\Delta_\theta h_M^c < 0$ and $\Delta_\theta h_S^c > 0$. $\Delta_\theta S_M(0;\theta_1) > 0$ implies $S_M(0;\theta_1) < R_M$; $\Delta_\theta S_S(0) < 0$ implies $R_S > S_S(0;\theta_2)$, and thus from (OA.8)—(OA.9) in the proof of Theorem 1 follows that:

$$\pi_S(v_S^c(\theta_2), h_S^c(\theta_2); \theta_2) \leq \pi_M(v_M^c(\theta_2), h_M^c(\theta_2)) \tag{OA.26}$$

$$\pi_M(v_M^c(\theta_1), h_M^c(\theta_1)) \leq \pi_S(v_S^c(\theta_1), h_S^c(\theta_1); \theta_1). \tag{OA.27}$$

Given that $\frac{\partial}{\partial v_M}\pi_M > 0$ and $\frac{\partial}{\partial h}\pi_M \geq 0$, we have that RHS of (OA.26) is less than the LHS of (OA.27) and therefore $\pi_S(v_S^c(\theta_2), h_M^c(\theta_2); \theta_2) \leq \pi_S(v_S^c(\theta_1), h_M^c(\theta_1); \theta_1)$. However, as $\Delta_\theta \pi_S(v_S^c(\theta_1), h_S^c(\theta_1)) \geq 0$, $\frac{\partial}{\partial v_S}\pi_S > 0$ and $\frac{\partial}{\partial h}\pi_S \geq 0$ this is impossible and the impossibility property holds; the result follows from Theorem OA.1. □

Let us again consider the change in services firms' hiring decisions after the surplus function has changed but before wage functions have adjusted. By the logic outlined in Section IV.B, every firm will want to hire a more skilled worker than previously. Additionally, some firms that did not find it profitable to hire anyone previously will now decide to hire a low-skilled worker, because of the increase in surplus levels. Thus again the demand for skill in services shifts upward, which draws in additional workers from manufacturing, so that employment rises in services and falls in manufacturing.[10] Note that this time some of those additional workers could be of low skill, as the increase in surplus levels implies that services generally becomes more productive relative to manufacturing.

As I explained in the main text, the change in overall inverse distribution of wages (or, in fact, the change in any outcome) can be decomposed into the baseline and size effects (Equation (23)). The size effect is defined as the effect o a change from the original specification to an intermediate specification, in which the gradient of the surplus function is the same as initially, but $S_i(0)$ are as in the final equilibrium. As $S_S(0;\theta_2) \geq S_S(0;\theta_1))$ it follows from Proposition OA.7 that the intermediate specification must shift surplus in services upward by a constant compared to the initial specification.

**Proposition OA.8.** Suppose that $R_M + R_S > 1$ and that $\pi_M(\theta_2) = \pi_M(\theta_1) + C_1$ for some $C_1 \geq 0$. Then (i) $\frac{\partial}{\partial v_S} w_S(v_S) \leq 0$ falls for all $v_S$ and $\frac{\partial}{\partial v_M} w_M(v_M) \geq 0$ increases for all $v_M$ (ii) $\Delta_\theta w_S(v_S) \geq \Delta_\theta w_M(\psi(v_S)) \geq 0$ for all $v_S \in [\max v_S^c, \min \bar{v}_S]$ and (iii) if $v_S^c(\theta_2) > 0$ then $W'(0)$ increases.

*Proof.* (i) Follows immediately by Proposition OA.7 and inspection of Equation (7).

(ii) First, note that as $\frac{\partial}{\partial v_i}\pi_i$ is unchanged for both $i$, it is both the case that vertical differentiation increases by more in services than in manufacturing and that it is increases by more in manufacturing than in services. Due to the latter it follows from Lemma OA.5 and Proposition OA.7 that $\Delta_\theta \psi(v_S) \geq 0$ for all $v_S \in [\max v_S^c, \min \bar{v}_S]$.

(iii) First, it follows from (i) that $v_S^c(\theta_2) \leq v_S^c(\theta_1)$. From the fact that $F_W(W(t)) = t$ follows

$$W'(0) = \frac{1}{\frac{\frac{\partial}{\partial v_M} C(v_M^c, v_S^c)}{\frac{\partial}{\partial v_M}\pi_M(v_M^c, P_M(v_M^c))} + \frac{\frac{\partial}{\partial v_S} C(v_M^c, v_S^c)}{\frac{\partial}{\partial v_S}\pi_S(v_S^c, P_S(v_S^c))}} = \frac{\frac{\partial}{\partial v_M}\pi_M(0, P_M(0))}{\frac{\partial}{\partial v_M} C(0, v_S^c)},$$

because $0 = v_M(\theta_2) \geq v_M(\theta_1)$. The numerator increases due to (i) and the denominator falls due to $v_S^c(\theta_2) \leq v_S^c(\theta_1)$, and the result follows.

□

### OA.5.3 Formal Results for Section VI.B

In this Section I provide the formal results and definitions on which the discussion in Section VI.B is based.

#### Equilibrium

**Definition OA.7** (Competitive Equilibrium)**.** An equilibrium consists of sectoral skill supply functions $S_M, S_S$, sectoral skill demand functions $D_M, D_S$, and sectoral wage functions $w_M, w_S$ that satisfy conditions (i)–(iii) from Definition 1, as well as (iv) two *sectoral measures of firms*, $R_M, R_S \in \mathbf{R}_{\geq 0}$, such that $\bar{r}_i = c_i$ if $R_i > 0$ and $\bar{r}_i \leq c_i$ otherwise.

[10] The increase (fall) in services' (manufacturing') employment follows immediately from the increase (fall) in skill supply.

It follows that any equilibrium of the extended model must also be an equilibrium of the baseline model.[11] Formally, denote by $E_B$ the set of all such worker and firm allocation quadruples $E = (S_M, S_S, R_M, R_S)$ such that the supply functions $(S_M, S_S)$ hold in an equilibrium of the baseline model if the sectoral firm measures are $R_M, R_S \in \mathbf{R}_{\geq \mathbf{0}}$.[12] For any $R_M, R_S > 0$, the corresponding $S_M, S_S$ were characterized in Section III. If $R_i = 0$, then $S_i(v) = 0$ and $S_j(v) = \min\{1 - v, R_j\}$ in any equilibrium of the baseline model.[13]

Analogously, denote by $\mathbf{E_E}$ the set of all quadruples $E = (S_M, S_S, R_M, R_S)$ such that $S_M, S_S$ are the supply functions and $R_M, R_S$ the sectoral firm measures that hold in an equilibrium of the extended model. Clearly, $E \in \mathbf{E_E}$ if and only if $E \in \mathbf{E_B}$ and satisfies condition (iv) from Definition OA.7.

**Existence and Uniqueness** In order to show existence and uniqueness of the equilibrium, it will be useful to first show that results analogous to the two Fundamental Welfare Theorems hold in the extended model.

It has been known since at least Gretsky, Ostroy, and Zame (1992) that in an assignment model (such as the baseline model from Section III) any equilibrium is efficient and any efficient assignment is an equilibrium. The extended model presented here, however, is not a special case of the model in Gretsky et al. (1992), and thus efficiency of equilibria needs to be established separately.

Total gross surplus produced in sector $i$ in equilibrium $E \in \mathbf{E_B}$ is equal to the sum of surpluses produced by all workers who joined sector $i$, taking into account that within-sector matching is positive and assortative:

$$T_i(E) = \begin{cases} 0 & \text{if } R_i = 0, \\ \int_1^0 \pi_i\left(v_i, 1 - \frac{S_i(v_i)}{R_i}\right) \mathrm{d}S_i(v_i) & \text{otherwise.} \end{cases} \tag{OA.28}$$

The total net surplus produced in the economy is equal to the sum of the gross surplus produced in the two sectors net of entry costs: $V(E) = T_M(E) + T_S(E) - c_M R_M - c_S R_S$.

**Proposition OA.9.** A worker and firm allocation $E \in \mathbf{E_B}$ can hold in an equilibrium of the extended model if and only if it uniquely maximizes the total net surplus, that is,

$$E^* \in \mathbf{E_E} \Leftrightarrow V(E^*) - V(E') > 0 \text{ for all } E \in \mathbf{E_B} \setminus \{E^*\}.$$

*Proof.* The proof will consist of three steps. First, I will prove that

$$E^* \in \mathbf{E_E} \Rightarrow V(E^*) - V(E') \geq 0 \text{ for all } E \in \mathbf{E_B}, \tag{OA.29}$$

that is, that the "if" part must holds weakly. Then I will show the "only if" part. Finally, I

[11] Note that if $R_i = 0$ for some $i \in \{M, S\}$, then the baseline model reduces to the standard single-sector model from Sattinger (1979).

[12] Formally, the quadruples in $E$ are such that if the sectoral firm measures are $R_M, R_S$, then there exist wage functions $w_M, w_S$ that, together with the supply functions $S_M, S_S$ and demand functions $D_M = S_M, D_S = S_S$, satisfy the conditions from Definition 1.

[13] The first part is trivial. The second part follows from the fact that all workers will be available to join the other sector but market clearing requires that at most $R_j$ can actually be hired by sector $j$ firms.

will show that any equilibrium is the unique maximizer.

**'If'** Assume that $\mathbf{E_E}$ is non-empty and consider some $E^*, E'$ such that $E^* \in \mathbf{E_E}, E' \in \mathbf{E_B}$ and $E^* \neq E'$. Denote by $\mathbf{W}$ the set of pairs of sectoral functions $w = (w_M, w_S)$ that are of the form prescribed by Equation (8) given $E^*$ and consider an arbitrary $w^* \in \mathbf{W}$.

Denote the total wage bill in sector $i$ under wage function $w_i$ and supply function $S_i$ as

$$\bar{w}_i(w_i, S_i) = -\int_0^1 w_i(t)s_i(t)\mathrm{d}t.$$

where $\frac{\partial}{\partial v}S_i(v) = s_i(v)$. Note that for $R_i > 0$ we have $\bar{w}_i(w_i, S_i) = R_i \int_{1-\frac{S_i(0)}{R_i}}^1 w_i(S_i^{-1}((1-h)R_i))\mathrm{d}h$. We can now denote the average wage in the economy $i$ under wage schedule $w = (w_M, w_S)$ and supply functions $S = (S_M, S_S)$ as

$$\bar{w}(w, S) = \bar{w}_M(w_M, S_M) + \bar{w}_S(w_S, S_S).$$

Note that by the definition of a sectoral supply function $\bar{w}(w^*, S^*) \geq \bar{w}(w^*, S')$. Further, if $S^* \neq S'$, then this inequality holds strictly, because the measure of workers who are indifferent between joining manufacturing or services is equal to 0.

Profit maximization implies that, if $R_i' > 0$, then

$$\bar{r}_i^* - c_i = \int_0^1 \max\{\pi_i(v_i^*(h), h) - w_i^*(v_i^*(h)), 0\}\mathrm{d}h - c_i \geq \frac{T_i(S_i', R_i') - \bar{w}_i(w_i^*, S_i')}{R_i'} - c_i, \quad \text{(OA.30)}$$

where $v_i^*$ is the hiring function defined in Section III.A.2.

I will prove the result by first assuming that $R_i^*, R_S^* > 0$ and only later considering the alternative. Note that if $R_M^*, R_S^* > 0$, then $v_i^*(h) = (S_i^*)^{-1}((1-h)R_i^*)$ for $h \in [0, 1 - \frac{S_i^*(0)}{R_i^*}]$, whereas for $h \in [0, 1 - \frac{S_i(0)}{R_i}]$ we have $\pi_i(v, h) - w_i^*(v) \leq 0$ for all $v \in [0, 1]$. This gives

$$\bar{r}_i^* - c_i = \frac{T_i(S_i^*, R_i^*) - \bar{w}_i(w_i^*, S_i^*)}{R_i^*} - c_i. \quad \text{(OA.31)}$$

Note also that $R_M'(\bar{r}_M^* - c_M) + R_S'(\bar{r}_S^* - c_S) \geq V(E') - \bar{w}(w^*, S')$. If $R_M', R_S' > 0$ this follows directly from Equation (OA.30). If $R_i' = 0$, then it follows as $T_i(S_i', R_i') - R_i c_i - \bar{w}_i(w_i^*, S_i') \leq 0 = R_i'(\bar{r}_i^* - c_i)$. Thus we can write

$$\begin{aligned} V(E^*) - \bar{w}(w^*, S^*) &= R_M^*(\bar{r}_M^* - c_M) + R_S^*(\bar{r}_S^* - c_S) = R_M'(\bar{r}_M^* - c_M) + R_S'(\bar{r}_S^* - c_S) \\ &\geq V(E') - \bar{w}(w^*, S'). \end{aligned} \quad \text{(OA.32)}$$

Now suppose that $R_i^* = 0$. By definition of equilibrium follows that $r_i - c_i \leq 0$. If $R_i' > 0$ we have that

$$0 = T_i(S_i^*, R_i^*) - \bar{w}_i(w_i^*, S_i^*) - R_i^* c_i \geq R_i'(\bar{r}_i^* - c_i) \geq T_i(S_i', R_i') - \bar{w}_i(w_i^*, S_i') - R_i' c_i \quad \text{(OA.33)}$$

Also, trivially, if $R_i' = 0$, then

$$0 = T_i(S_i^*, R_i^*) - \bar{w}_i(w_i^*, S_i^*) - R_i^* c_i = T_i(S_i', R_i') - \bar{w}_i(w_i^*, S_i') - R_i' c_i.$$

Thus, it follows that $V(E^*) - \bar{w}(w^*, S^*) \geq V(E') - \bar{w}(w^*, S')$.

$V(E^*) - \bar{w}(w^*, S^*) \geq V(E') - \bar{w}(w^*, S')$ and the fact that $\bar{w}(w^*, S^*) > \bar{w}(w^*, S')$ imply that

$$V(E^*) - V(E') \geq \bar{w}(w^*, S^*) - \bar{w}(w^*, S') \geq 0.$$

**'Only if'** Denote as $S_i(\cdot, R_M, R_S)$ the equilibrium supply of skill in sector $i$ in the baseline model if the measures of firms are $R_M, R_S$.[14] Define

$$T(R_M, R_S) = T_M(S_M(R_M, R_S), R_M) + T_S(S_S(R_M, R_S), R_S), \tag{OA.34}$$
$$V(R_M, R_S) = V(S_M(R_M, R_S), S_S(R_M, R_S), R_M, R_S), \tag{OA.35}$$

so the gross and net total surpluses holding in an equilibrium of the baseline model if the measures of firms are $R_M, R_S$. In a direct analogy, we can also denote the average profits in sector $i$ holding in equilibrium for $R_M, R_S$ as $\bar{r}_i(R_M, R_S)$. Note that the profits are defined uniquely if $R_i > 0$ and $R_M + R_S \neq 0$, otherwise they can take a range of values.

**Lemma OA.11.** Consider $(R_M, R_S), (R_M', R_S') \in \mathbf{R}^2_{\geq 0}$. For any $t \in [0, 1]$ define $R_i(t) = R_i + t(R_i' - R_i)$ and $V(t) = V(R_M(t), R_S(t))$. The following is true: (a) $V(\cdot)$ is absolutely continuous; (b) for any $t \in (0, 1)$ for which $V$ is differentiable, we have

$$V_t(t) = (R_M' - R_M)(\bar{r}_M(R_M(t), R_S(t)) - c_M) + (R_S' - R_S)(\bar{r}_S(R_M(t), R_S(t)) - c_S),$$

giving

$$V(t) = V(0) + \int_0^t (R_M' - R_M)(\bar{r}_M(R_M(s), R_S(s)) - c_M) + (R_S' - R_S)(\bar{r}_S(R_M(s), R_S(s)) - c_S) \mathrm{d}s. \tag{OA.36}$$

*Proof.* As the baseline model is an assignment game, it follows from the results in Gretsky et al. (1992) that the equilibrium of the baseline model is efficient, and thus

$$V(R_S, R_M) = \max_{(S_M, S_S) \in \mathbf{S}_B} V(S_M, S_S, R_M, R_S), \tag{OA.37}$$

where $\mathbf{S}_B = \{(S_M, S_S) : \exists_{(R_M, R_S) \in \mathbf{R}^2_{\geq 0}} (S_M, S_S, R_M, R_S) \in \mathbf{E_B}\}$. For any $(R_M, R_S), (R_M', R_S') \in \mathbf{R}^2_{\geq 0}$ we can define $V(S_M, S_S, t) = V(S_M, S_S, R_M(t), R_S(t))$. Note that $V_t(S_M, S_S, t)$ exists as long as $R_i(t) \neq S_i(0)$, and $R_i(t) \neq 0$ so for all $t \in [0, 1]$ but at most four. Further, whenever $V_t(S_M, S_S, t)$ does exist we have that

$$V_t(S_M, S_S, t) = (R_M' - R_M)(\bar{r}_S(S_M, R_M(t)) - c_M) + (R_S' - R_S)(\bar{r}(S_S, R_S(t)) - c_S),$$

[14]Formally, $S_M(\cdot, R_M, R_S) = S_i(\cdot)$ if and only if there exists some function $S_S(\cdot)$ such that $(S_M, S_S, R_M, R_S) \in \mathbf{E_B}$.

where

$$\bar{r}_i(S_i, R_i) = \begin{cases} \int_0^1 \int_0^h \frac{\partial}{\partial h}\pi_i(S_i^{-1}((1-p)R_i), p)\mathrm{d}p + \pi_i(S_i^{-1}(R_i), 0)\,\mathrm{d}h & \text{for } R_i \in (0, S_i(0)), \\ \int_{1-\frac{S_i(0)}{R_i}}^1 \int_{1-\frac{S_i(0)}{R_i}}^h \frac{\partial}{\partial h}\pi_i(S_i^{-1}((1-p)R_i), p)\mathrm{d}p\,\mathrm{d}h & \text{for } R_i > S_i(0). \end{cases} \tag{OA.38}$$

Thus,

$$V(S_M, S_S, t) = V(S_M, S_S, t_1) + \int_{t_1}^t (R'_M - R_M)\bar{r}_M(S_M, R_S(s)) + (R'_S - R_S)\bar{r}_S(S_S, R_S(s))\mathrm{d}s,$$

proving that $V(S_M, S_S, \cdot)$ is absolutely continuous for any $(S_M, S_S) \in \mathbf{S}_B$ and any choice of $(R_M, R_S), (R'_M, R'_S)$. Clearly, $\bar{r}(S_i, R_i(t)) - c_i \in [-c_i, \pi_i(1,1) - c_i]$, implying

$$|V_t(S_M, S_S, t)| \leq (R'_M - R_M)\max\{c_M, \pi_M(1,1)\} + (R'_S - R_S)\max\{c_S, \pi_S(1,1)\}$$

which proves $V(t)$ is absolutely continuous by Theorem 2 in Milgrom and Segal (2002).

Define $T(t) = T(R_M(t), R_S(t))$ and pick any $t \in (0,1)$ for which $T(t)$ is differentiable. Consider two $c'_M, c'_S \in \mathbf{R}_{\geq 0}$ such that $c'_i = \bar{r}_i(R_M(t), R_S(t))$. For entry costs $c'_M, c'_S$, the quadruple $(S_M(R_M(t), R_S(t)), S_S(R_M(t), R_S(t)), R_M(t), R_S(t))$ is an equilibrium of the extended model, implying that it maximizes the function $V'(t) = T(t) - c'_M R_M(t) - c'_M R_M(t)$. Clearly, both $V(\cdot)$ and $V'(\cdot)$ are differentiable at $t$ as well. It follows from first order conditions that $V'_t(t) = 0$ implying that

$$\begin{aligned} T_t(t) &= (R'_M - R_M)c'_M + (R'_S - R_S)c'_S \\ &= (R'_M - R_M)\bar{r}_M(R_M(t), R_S(t)) + (R'_S - R_S)\bar{r}_S(R_M(t), R_S(t)). \end{aligned}$$

This proves that

$$V_t(t) = (R'_M - R_M)(\bar{r}_M(R_M(t), R_S(t)) - c_M) + (R'_S - R_S)(\bar{r}_S(R_M(t), R_S(t)) - c_S),$$

which, together with the absolute continuity of $V(t)$ proves Equation (OA.36) as well.

□

Consider $R_M^M, R_S^M \geq 0$ for which $V(E)$ is maximized. I will show that $R_M^M, R_S^M$ must satisfy condition (iv) of the equilibrium definition and, together with the corresponding supply functions, constitute an equilibrium.

First, I will show that if $R_M^M > 0$ then $\bar{r}_M - c_M \geq 0$. First, pick some $R'_M < R_M^M$ and define $V(t)$ for $(R_M^M, R_S^M)$ and $(R'_M, R_S^M)$. From Lemma OA.11 and the definition of maximum follows that there exists some $t' \in (0,1)$ such that for any $t < t'$ we have $\bar{r}_M(R_M(t), R_S^M(t)) \geq c_M$. If $R_M^M + R_S^M \neq 1$ then this immediately implies $\bar{r}_M(R_M^M, R_S^M) \geq c_M$ by continuity. If $R_M^M + R_S^M = 1$, there exist wage functions for which $\bar{r}_M(R_M^M, R_S^M) \geq c_M$—and condition (iv) is satisfied as well. It remains to show that if $R_M^M \geq 0$ then $\bar{r}_M - c_M \leq 0$. The proof is analogous: pick some $R''_M > R_M^M$ and it follows from an analogous reasoning as for $R_M^M \geq 0$ that there exists some $t' \in (0,1)$ such that for any $t < t'$ we have $\bar{r}_M(R_M(t), R_S^M(t)) \leq c_M$. Thus the result follows from continuity of $\bar{r}_M(R_M(t), R_S^M(t))$.[15] The proof for services is analogous.

**Uniqueness** Again, assume that $\mathbf{E_E}$ is non-empty and consider some $E^*, E'$ such that $E^* \in \mathbf{E_E}, E' \in \mathbf{E_B}$ and $E^* \neq E'$. Suppose that $V(E^*) = V(E')$. From (OA.29) follows that this is possible only if $E' \in \mathbf{E_E}$. Further, if $S^* \neq S'$, then $\bar{w}(w^*, S^*) - \bar{w}(w^*, S') > 0$, and thus $V(E^*) = V(E')$ is possible only if $S^* = S'$ and $R_i^* \neq R_i'$ for some $i \in \{i, j\}$. Finally, it follows from Equation (OA.38) and Assumption 1 that if $R_i^* \neq R_i'$ then $\bar{r}(S_i^*, R_i^*) \neq \bar{r}(S_i^*, R_i') = c_i$, implying that $E^* \notin \mathbf{E_E}$; contradiction! Therefore, $V(E^*) > V(E')$. □

This result can be interpreted as an analogue of the First and Second Welfare Theorems for this economy. First, it means that any equilibrium is efficient. Secondly, it means that any efficient allocation of workers and firms to sectors holds in some equilibrium.[16] Finally, it implies that any equilibrium allocation of workers and firms $E$ maximizes total net surplus *uniquely*. It follows trivially that if an equilibrium exists it must be (essentially) unique.

**Theorem OA.2.** An equilibrium exists and is essentially unique, in that the equilibrium measure of firms entering each sector, as well as skill supply and demand, are unique. Further, the equilibrium wage functions are uniquely determined for all matched workers (i.e., for $v_i \geq v_i^c$).

*Proof.* **Existence.** Denote as $S_i(\cdot, R_M, R_S)$ the equilibrium supply of skill in sector $i$ in the baseline model, holding for $R_M, R_S$. It follows from the proof of Theorem 1 that $S_S$ is continuous in $R_M, R_S$ for any $R_M, R_S > 0$. Thus because $\int_{1-\frac{S_i(0)}{R_i}}^{1} \pi_i(S_i^{-1}((1-h)R_i), h)\mathrm{d}h \leq \pi_i(1,1)$ for any $R_i > 0$, it follows that

$$V(R_M, R_S) = V(S_M(R_M, R_S), S_S(R_M, R_S), R_M, R_S)$$

is continuous in $R_M, R_S$.[17]

**Lemma OA.12.** If $R_M > \bar{R}_M = \frac{\pi_M(1,1)+\pi_S(1,1)}{c_M}$ then $V(R_M, R_S) < 0 = V(0,0)$.

*Proof.* It follows from Equation (OA.28) that, trivially, $T_i(R_M, R_S) \leq \pi_i(1,1)$, where $T_i(R_M, R_S)$ is defined as in the proof of Proposition OA.9. Thus it follows from the definition of net total surplus that

$$V(R_M, R_S) \leq \pi_M(1,1) - R_M c_M + \pi_S(1,1).$$

Note that $\pi_M(1,1) - R_M c_M + \pi_S(1,1) < 0$ for any $R_M > \frac{\pi_M(1,1)+\pi_S(1,1)}{c_M}$, implying that $V(R_M, R_S) < 0 = V(0,0)$, as required. □

Of course, an analogous result holds for services. Define the set $\bar{R} = \{(R_M, R_S) \in \mathbf{R}^{\mathbf{2}}_{\geq \mathbf{0}} : R_M \leq \bar{R}_M, R_S \leq \bar{R}_S$. Because the net total surplus for $(R_M, R_S) = (0,0)$ is zero, it follows

[15] $\bar{r}_M(R_M(t), R_S^M(t))$ is continuous even at $R_M^M = 0$, in the sense that there exists an average manufacturing profit that holds in an equilibrium at $R_M^M = 0$ that is the limit of the average profit that holds for $R_M > 0$, as $R_M \to 0$. This is because the equilibrium wage function that holds in the non-degenerate sector (services) is trivially continuous in $R_M$, and the services wage function determines the lowest wage function in manufacturing that prevents any worker from joining that sector. A similar reasoning holds even if both sectors are degenerate.

[16] This is because any efficient allocation of workers given $R_M, R_S$ is an equilibrium of the baseline model, which follows from the results in Gretsky et al. (1992).

[17] This is because $\lim_{R_i \to 0} V_i(S_i, R_i) = 0 \cdot \lim_{R_i \to 0} \int_{1-\frac{S_i(0)}{R_i}}^{1} \pi_i(S_i^{-1}((1-h)R_i), h)\mathrm{d}h \leq 0 \cdot \pi_M(0,0) = 0$.

from Lemma OA.12 that

$$\max_{(R_M,R_S)\in\mathbf{R}^2_{\geq 0}} V(R_M,R_S) = \max_{(R_M,R_S)\in\bar{R}} V(R_M,R_S).$$

As $\bar{R}$ is closed and bounded, it follows from Weierstrass' Theorem that $V(R_M, R_S)$ admits a global maximum on $\mathbf{R}^2_{\geq 0}$. As by Proposition OA.9 any global maximum must be en equilibrium, existence follows.

**Uniqueness.** Follows immediately from Proposition OA.9.[18]

**Wages** First, if $R_M + R_S \neq 1$ in equilibrium, then this follows from the Equation (8) and Lemma 1. Otherwise the constant of integration $C_i$ is not uniquely determined in the baseline model; here, however, if $C_i' > C_i$, then $\bar{r}_i' > \bar{r}_i$ contradicting the requirement that both have to be equal to $c_i$. □

Both the existence results and the uniqueness results are new.[19] This is in contrast to the baseline model, where uniqueness of the equilibrium was a new result but its existence could have easily been shown from existing results for assignment models (Gretsky et al., 1992). Further, the uniqueness of equilibrium is stronger here than in the baseline model, as wages are *de facto* uniquely determined even if $R_M + R_S = 1$. This is because constant average profits pinpoint the split of surplus in the least productive match (see below).

### Costrell and Loury (2004)

In the hierarchical job assignment model of Costrell and Loury (2004), firms are homogeneous but consist of a hierarchy of heterogeneous jobs. The surplus produced by a firm is simply the sum of the surpluses produced by all the jobs (and all of them need to be filled to produce anything). Surplus produced in any job is supermodular in the job's rank and the skill of the worker assigned to it. The zero profit condition ensures that, in equilibrium, the measure of all jobs is equal to the measure of workers. Because of positive and assortative matching, a worker with skill $v$ is assigned to a job of rank $h = G(v)$ (where $G(\cdot)$ denotes the cdf of skill). Using my notation, the wage paid to a worker with skill $v_S$ in the Costrell and Loury model is

$$w^{\mathrm{CL}}(v) = \pi(v, G(v)) + \int_0^1 \int_{G(v)}^h \frac{\partial}{\partial h_i}\pi_M\Big(G^{-1}(t), t\Big)\,\mathrm{d}t\,\mathrm{d}h. \tag{OA.39}$$

In equilibrium, the more productive and profitable jobs cross-subsidize the less productive jobs, leading to firm-wide profit of zero.[20]

The wage function in the extended model is a generalization of Equation (OA.39). To see

[18] Consider a pair $E^*, E' \in \mathbf{E_E}$ and $E^* \neq E'$. Then (OA.29) implies that $V(E^*) > V(E)$ and $V(E^*) < V(E)$ which is a contradiction.

[19] The only other paper I am aware of that allows for endogenous entry of firms in an assignment model is Costrell and Loury (2004), which is a single-sector, one-dimensional model.

[20] Technically, Costrell and Loury (2004) allow only for multiplicative surplus functions, in the form $\mu(v)\beta(h)$. There is no problem, however, with generalizing their framework to supermodular surplus functions.

this, note that the profit of firm $h_i$ in sector $i$ can be written as

$$r_i(h_i) = \int_{h_i^c}^{h_i} \frac{\partial}{\partial h_i} \pi_i\Big(P_i^{-1}(h), h\Big)\mathrm{d}h + r_i(h_i^c), \tag{OA.40}$$

where $h_i^c = P(v_i^c)$ denotes the productivity of the least productive matched firm. The profit earned by said firm is pinned down by the zero-expected-profits condition:

$$r_i(h_i^c) + \int_{P_i(v_i^c)}^{1} \int_{P_i(v_i^c)}^{h} \frac{\partial}{\partial h_i} \pi_M\Big(P_M^{-1}(t), t\Big)\mathrm{d}t\mathrm{d}h = c_M. \tag{OA.41}$$

The wage received by a worker of skill $v_i$ is $w_i(v_i) = \pi(v_i, P_i(v_i)) - r_i(P_i(v_i))$. Substituting Equations (OA.40) and (OA.41) into this expression yields

$$w_i(v_i) = \pi(v_i, P_i(v_i)) + \int_{h_i^c}^{1} \int_{P_i(v_i)}^{h} \frac{\partial}{\partial h_i} \pi_i\Big(P_i^{-1}(t), t\Big)\mathrm{d}t\mathrm{d}h - h_i^c r_i(P_i(v_i)) - c_i. \tag{OA.42}$$

This is similar to Equation (OA.39) but the two wage functions differ if the skill and matching functions are not identical, that is, if $R_i \neq S_i(0)$. In addition to this, in the Costrell and Loury (2004) model all workers are employed and all tasks must be filled (by assumption) but this is not necessarily the case here. However, the extended model nests a two-sector version of the hierarchical job assignment model if the total measure of firms is necessarily equal to 1 in equilibrium.

**Assumption OA.1** (Costrell–Loury Specification)**.** $\pi_i(1,1) - \pi_i(1,0) \leq c_i$ for both $i \in \{M, S\}$, and $\pi_i(0,0) > c_i$ for some $i \in \{M, S\}$.

This ensures that the total measure of firms, $R_M + R_S$, is equal to 1 in equilibrium.[21] Any specification of the extended model that meets Assumption OA.1 will be referred to as a *Costrell–Loury (CL) specification*. In Sections OA.5.3 and OA.5.3, I will focus on Costrell–Loury specifications, as they are much more tractable than the general model because of the property that the measures of workers and firms are equal. Further, every firm hires a worker, and all workers are employed. Therefore, the Costrell–Loury specification of my model can be reinterpreted as a model in which firms are homogeneous within each sector but consist of a hierarchy of heterogeneous jobs.

### Skill Interdependence and Wage Polarization

Similarly to the baseline model, the overall effect of an increase in skill interdependence raises wage polarization in absolute terms under fairly general conditions: The only difference is that now I will also assume that the cross-partial of the surplus function is crosses zero at most once,

[21] If $R_M + R_S < 1$, then $\bar{r}_i \geq \pi_i(0,0) > c_i$ in one of the sectors, violating the zero-expected-profit condition. Similarly, if $R_M + R_S > 1$, then $r_i(0) = 0$ in at least one sector, implying that $c_i \geq \int_0^1 \frac{\partial}{\partial h_i}\pi_i\Big(1, h\Big)\mathrm{d}h \geq r_i(1) \geq \bar{r}$, and thus again violating the zero-expected-profit condition. Note, by the way, that Assumption OA.1 can be weakened significantly. For example, $\int_0^1 \int_0^h \frac{\partial}{\partial h_i}\pi_M\Big(1, t\Big)\mathrm{d}t\mathrm{d}h \leq c_M$ is sufficient, as the LHS must be greater than $\bar{r}$.

that is, if $\frac{\partial^2}{\partial v_i \partial h_i}\pi_i(v_i', h_i') = 0$ then $\frac{\partial^2}{\partial v_i \partial h_i}\pi_i(v_i, h_i) = 0$ for all $(v_i, h_i) \leq (v_i', h_i')$. This assumption is needed to ensure that entry can change only if the model is equivalent to Roy's model (in which case only the composition effect is present and the results follow immediately).

However, to ensure that polarization increases also in relative terms, a slightly stronger notion of regularity is needed than the one defined in Section IV.A.2. I will say that a change in interdependence is *strongly regular* if $\frac{\mathrm{d}}{\mathrm{d}v}\big(C_{v_i}(v, v, \theta_2) - C_{v_i}(v, v, \theta_1)\big)|_{v=0} > 0$. This implies that $C_{v_i}(v, v, \theta_2) - C_{v_i}(v, v, \theta_1) > 0$ for $v$ close to 0.

**Proposition OA.10** (Wage Polarization)**.** Suppose that Assumption OA.1 is satisfied, $\frac{\partial^2}{\partial v_i \partial h_i}\pi_i$ crosses zero at most once, the concordance of the skill distribution increases regularly, and Equation (19) is satisfied. Then $W(t) - W(0)$ falls for all $t \leq \bar{t}$ (strictly for some $t \in (0, \bar{t})$) and $W(1) - W(0)$ increases. In addition, if (a) the change in concordance is strongly regular, (b) the model is symmetric and (c) either (i) $w_i(0; \theta_1)$ is sufficiently high or (ii) $\max_{\{M,S\}\times[0,1]^2} \frac{\partial^2}{\partial v_i \partial h_i}\pi_i(v_i, h_i)$ is sufficiently small, then polarization increases in relative terms as well.[22]

*Proof.* I will start by proving that wage polarization must increase in absolute terms. Recall that $\bar{v}_S \equiv \sup\{v_S \in [0,1] : \psi(v_S) < 1\}$ and $\bar{v}_M \equiv \psi(\bar{v}_S)$. First, I will show that if $\frac{\partial^2}{\partial v_S \partial h_S}\pi_S(\bar{v}_S, G_S(\bar{v}_S)) > 0$, then Equation (19) can be satisfied only if $\frac{\partial}{\partial \theta}R_S = 0$, which then implies further that $\frac{\partial}{\partial \theta}R_M = 0$ and the result follows by the same proof as the analogous statement from Proposition 1. If $\bar{v}_S < 1$ then $\psi(\bar{v}_S) = 1$ and $\frac{\partial}{\partial \theta}G_M(\psi(\bar{v}_S)) = 0$, while $G_S(\bar{v}_S) = 1 - \frac{1-\bar{v}_S}{R_S}$ and $\frac{\partial}{\partial \theta}G_S(\bar{v}_S) = \frac{\frac{\partial}{\partial \theta}R_S}{R_S}\frac{1-\bar{v}_S}{R_S}$. As Equation (19) implies that $\frac{\partial}{\partial \theta}\frac{\partial}{\partial v_S}w_S(\bar{v}_S) = \frac{\partial}{\partial v_S}\psi(v_S)\frac{\partial}{\partial \theta}\frac{\partial}{\partial v_M}w_M(\bar{v}_M)$, it follows that $\mathrm{sign}(\frac{\partial}{\partial \theta}G_M(\psi(\bar{v}_S))) = \mathrm{sign}(\frac{\partial}{\partial \theta}G_S(\bar{v}_S))$, which is possible only if $\frac{\partial}{\partial \theta}R_S = 0$ and the result follows. If $\bar{v}_S = 1$ then $\psi(\bar{v}_S) = 1$ or $\psi(\bar{v}_S) < 1$. In the latter case, the result follows by an analogous reasoning as for $\bar{v}_S < 1$. In the former case we have that $g_i(\bar{v}_i) = \frac{\frac{\partial}{\partial v_i}C(1,1)}{R_i} = \frac{1}{R_i}$ so that $\frac{\partial}{\partial \theta}g_S(1) = -\frac{\frac{\partial}{\partial \theta}R_S}{R_S}g_S(1)$ and $\frac{\partial}{\partial \theta}g_M(\psi(1)) = \frac{\frac{\partial}{\partial \theta}R_S}{1-R_S}g_M(1)$. As $G_i(v_i) = G_i(1) - \int_{v_i}^1 g_i(r)\mathrm{d}r$ and $\frac{\partial}{\partial \theta}G_i(1) = 0$ it follows that if $\frac{\partial}{\partial \theta}R_S > (<)0$ then $\frac{\partial}{\partial \theta}G_S(v_S) > (<)0$ and $\frac{\partial}{\partial \theta}G_M(\psi(v_S)) < (>)0$ for all $v_S \approx 1$. This, however, contradicts $\frac{\partial}{\partial \theta}\frac{\partial}{\partial v_S}w_S(v_S) = \frac{\partial}{\partial \theta}\frac{\partial}{\partial v_M}w_M(\psi(v_S))$ and thus also Equation (19). Thus, $\frac{\partial}{\partial \theta}R_S = 0$ and the result follows.

Second, if $\frac{\partial^2}{\partial v_S \partial h_S}\pi_S(\bar{v}_S, G_i(\bar{v}_S)) = 0$, then $\frac{\partial^2}{\partial v_S \partial h_S}\pi_S(v_S, G_i(v_S)) = 0$ for all $v_S \geq v_S^c$ by the fact that $\frac{\partial^2}{\partial v_S \partial h_S}\pi_S$ crosses zero at most once. Thus, $\frac{\partial}{\partial v_S}w_S(v_S)$ is unchanged and $\frac{\mathrm{d}}{\mathrm{d}\theta}(W(t) - W(0)) = -W'(t)\frac{\partial}{\partial \theta}C(v_M(t), v_S(t); \theta)$ from which the result follows trivially.

As far as the second statement is concerned, under symmetry $W(t) = w_S(G_S^{-1}(t))$, implying that

$$W'(t) = \frac{\frac{\partial}{\partial v_S}\pi_S(G_S^{-1}(t), t)}{g_S(G_S^{-1}(t))},$$

where $g_S$ denotes the pdf of $G_S$; thus

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{W'(t)}{W(t)} = \frac{W(t)^{-1}}{g_S(G_S^{-1}(t))}\left(\frac{\partial}{\partial \theta}G_S^{-1}(t)\frac{\partial^2 \pi_S(G_S^{-1}(t))}{\partial v_S \partial v_S} - \frac{\partial \pi_S(G_S^{-1}(t), t)}{\partial v_S}\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}g_S(G_S^{-1}(t))}{g_S(G_S^{-1}(t))} + \frac{\frac{\partial}{\partial \theta}W(t)}{W(t)}\right)\right).$$

[22] The definition of the symmetric case in the extended model differs from the definition for the baseline model only in that the condition $R_M = R_S$ is replaced by $c_M = c_S$, which—provided all other symmetry conditions are met—implies that $R_M = R_S$ in equilibrium.

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Consider $\delta > 0$ such that, for all $v \in [0, \delta]$, (a) $\frac{\mathrm{d}}{\mathrm{d}v}\frac{\partial^2}{\partial\theta\partial v_i}C(v, v) > 0$ and (b) $\frac{\mathrm{d}}{\mathrm{d}v}g_S(v) > 0$—such $\delta$ must exist by the strong regularity condition, the fact that $\frac{\mathrm{d}}{\mathrm{d}v}g_S(0) = \frac{\partial^2}{\partial v_S\partial v_M}C(0, 0) > 0$ and continuity. Note that for $t \leq G_S(\delta)$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}g_S(G_S^{-1}(t)) \geq \frac{\partial}{\partial\theta}g_S(G_S^{-1}(t)) = \int_0^{G_S^{-1}(t)} \frac{\mathrm{d}}{\mathrm{d}v}\frac{\partial^2}{\partial\theta\partial v_i}C(v, v)\mathrm{d}v \geq G_S^{-1}(t)L_8$$

where $L_8 \equiv \min_{v\in[0,\delta]} \frac{\mathrm{d}}{\mathrm{d}v}\frac{\partial^2}{\partial\theta\partial v_i}C(v, v)$, whereas

$$g_S(G_S^{-1}(t)) = \int_0^{G_S^{-1}(t)} \frac{\mathrm{d}}{\mathrm{d}v}g_S(r)\mathrm{d}r \leq G_S^{-1}(t)L_9,$$

with $L_9 \equiv \max_{v\in[0,\delta]} \frac{\mathrm{d}}{\mathrm{d}v}g_S(v)$. Note that $L_8, L_9 > 0$ and both are finite, because $C$ is twice continuously differentiable. It follows then that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{W'(t)}{W(t)} \leq \frac{W(t)^{-1}}{g_S(G_S^{-1}(t))}\left(\frac{\partial}{\partial\theta}G_S^{-1}(t)\frac{\partial^2\pi_S(G_S^{-1}(t))}{\partial v_S\partial v_S} - \frac{\partial\pi_S(G_S^{-1}(t), t)}{\partial v_S}\left(\frac{L_8}{L_9} + \frac{\frac{\partial}{\partial\theta}W(t)}{W(t)}\right)\right).$$

for $t \leq G_S(\delta)$. Because $\frac{\partial}{\partial\theta}G_S^{-1}(0) = 0$ it follows by continuity that if $W(0)$ is large enough or $\frac{\partial}{\partial\theta}W(0)$ is sufficiently small, then there must exist $\bar{t} \leq G_S(\delta)$ such that $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{W'(t)}{W(t)} < 0$ for all $t \in (0, \bar{t})$.[23] The first is ensured by a high enough reservation value, whereas the latter by weak enough supermodularity of the surplus function. This proves the second statement by the fact that $\ln W(t) - \ln W(0) = \int_0^t \frac{W'(t)}{W(t)}$. □

### Changes to Surplus Functions

In this section I investigate the effects of changes to manufacturing's surplus function. I again focus on Costrell–Loury specifications but the main result (Proposition OA.11) holds in general. I find that with endogenous entry, sorting depends not only on the vertical differentiation of workers but also on the vertical differentiation of firms.

As in Section IV, in the comparative statics results I will consider only specifications that result in non-degenerate equilibria, that is, those with $R_i(c_j) > 0$ for $i \in \{M, S\}$ and $j \in \{1, 2\}$.

**Definition OA.8** (Firms' Vertical Differentiation)**.** Firms in services become (strictly) more *vertically differentiated* if, for any $v_S \in [0, 1]$ and any $0 \leq h'_S < h''_S \leq 1$

$$\pi_S(v_S, h''_S; \theta_2) - \pi_S(v_S, h'_S; \theta_2)(>) \geq \pi_S(v_S, h''_S; \theta_1) - \pi_S(v_S, h'_S; \theta_1).$$

This is an exact analogue of an increase in workers' vertical differentiation (see Definition 4 in Section IV.B). To guarantee that the supply of skill increases in services, apart from an increase in workers' differentiation and a universal increase in surplus levels, we also need an increase in differentiation in services firms. To see this, let us consider what could happen otherwise. Keeping sorting constant, a fall in differentiation might reduce firms' profits, thus decreasing entry into services. This decreases the demand for skill in services—and, as a result,

[23] Note that if $\frac{\partial^2}{\partial h_i\partial v_i}\pi_i(v_i, h_i)$ is arbitrarily close to 0 for all $(v_i, h_i)$ then $\frac{\partial}{\partial\theta}W(0)$ is arbitrarily close to 0 as well.

could lower the supply of skill in equilibrium.[24] To rule out this possibility, firms' differentiation needs to increase (weakly).

**Proposition OA.11.** If both workers and firms in services become more vertically differentiated and the surplus produced in services increases universally, then the measure of firms and supply of skill *increase in services* and *fall in manufacturing.*

*Proof.* Endow the space $\mathbf{R}^2_{\geq 0}$ with the following partial order $\succeq$: if $R'_M \leq R_M$ and $R'_S \geq R_S$ then $(R'_M, R'_S) \succeq (R_M, R_S)$. Clearly, $(\mathbf{R}^2_{\geq 0}, \succeq)$ is a lattice.

Recall the function $V : \mathbf{R^2_{\geq 0}} \to \mathbf{R_{\geq 0}}$ defined in Equation (OA.35). I will argue that $V$ is supermodular under order $\succeq$. Consider two points $R''_M, R''_S$ and $R_M, R_S$, such that $R''_i \geq R_i$. $V(\bullet)$ is supermodular if and only if for any such pair of points it is the case that

$$V(R''_M, R_S) + V(R_M, R''_S) \geq V(R''_M, R''_S) + V(R_M, R_S).$$

We can rewrite the above as

$$V(R''_M, R_S) - V(R_M, R_S) \geq V(R''_M, R''_S) - V(R_M, R''_S).$$

By Lemma OA.11 this can be rewritten as

$$\int_{R_M}^{R''_M} \bar{r}_M(s, R_S)\mathrm{d}s \geq \int_{R_M}^{R''_M} \bar{r}_M(s, R''_S)\mathrm{d}s. \tag{OA.43}$$

It follows immediately from Equation (OA.38) and the fact that, by Theorem OA.1, $S^e_M(R_M, R''_S) \leq S^e_M(R_M, R_S)$, that for any $R_M$ we have $\bar{r}_M(R_M, R_S) \geq \bar{r}_M(R_M, R''_S)$, which proves that Equation (OA.43) must hold.

Further, consider some $(R'_M, R'_S) \succeq (R_M, R_S)$, then by Lemma OA.11 follows that

$$\begin{aligned} V(R'_M, R'_S) - V(R_M, R_S) &= V(R'_M, R'_S) - V(R_M, R'_S) + V(R_M, R'_S) - V(R_M, R_S) \\ &= -\int_{R'_M}^{R_M} \bar{r}_M(s, R'_S) - c_M \mathrm{d}s + \int_{R_S}^{R'_S} \bar{r}_S(R_M, s) - c_S \mathrm{d}m \end{aligned}$$

It follows, again, from Equation (OA.38) and Theorem OA.1 that $\bar{r}_M(R_M, R'_S; \theta_2) \leq \bar{r}_M(R_M, R'_S; \theta_1)$ and $\bar{r}_S(R_M, R_S; \theta_2) \geq \bar{r}_S(R_M, R_S; \theta_1)$. Denote the net surplus holding in the equilibrium of the baseline model in specification $\theta_j$, with firm measures $R_M, R_S$ as $V(R_M, R_S, \theta_j)$. Then it follows that

$$V(R'_M, R'_S, \theta_2) - V(R_M, R_S, \theta_1) \geq V(R'_M, R'_S, \theta_2) - V(R_M, R_S, \theta_1).$$

In other words $V(R_M, R_S, \theta)$ has increasing differences in $\theta \in \{\theta_1, \theta_2\}$. Finally, note that the equilibrium sectoral firm measures $R^*_M(\theta_j), R^*_S(\theta_j)$ are given by

$$(R^*_M(\theta_j), R^*_S(\theta_j)) = \underset{(R_M, R_S) \in \mathbf{R}^2_{\geq 0}}{\arg\max} V(R_M, R_S, \theta_j).$$

[24] Of course, the increase in workers' differentiation pushes in the opposite direction, as explained in Section IV.B. Nevertheless, the impact of lower differentiation of firms can easily be the dominant force.

From the facts that $\mathbf{R}^2_{\geq 0}$ endowed with the $\succeq$ is a lattice, $\theta \in \{\theta_1, \theta_2\}$ endowed with the increasing order is a partially ordered set, $V(R_M, R_S, \theta_j)$ is supermodular in $(R_M, R_S)$ and satisfies the increasing differences property in $(R_M, R_S; \theta_j)$ it follows from Theorem 6.1 in Topkis (1978) (or, alternatively, Theorem 4 in Milgrom and Shannon (1994)) that $R_M(\theta_2) \leq R_M(\theta_1)$ and $R_S(\theta_2) \geq R_S(\theta_2)$.

Finally, note that a change from specification $\theta_1$ to specification $\theta_2$ of the extended model constitutes a change from specification $(R_M(\theta_1), R_S(\theta_1), \theta_1)$ to specification $(R_M(\theta_2), R_S(\theta_2), \theta_2)$, i.e. there is a simultaneous fall in $R_M$, increase in $R_S$, universal increase in surplus level in services, and an increase in vertical differentiation of both workers and firms in services. This can be broken down as a change from $(R_M(\theta_1), R_S(\theta_1), \theta_1)$ to $(R_M(\theta_1), R_S(\theta_1), \theta_1)$ and only then $(R_M(\theta_2), R_S(\theta_1), \theta_2)$; applying Theorem OA.1 to both changes proves the result.

□

As explained in Section OA.5.2, for a given measure of firms, an increase in workers' differentiation together with a universal increase in surplus attracts additional high-skilled workers to manufacturing. Combined with an increase in firms' differentiation, this increases profits in manufacturing, induces more firms to enter, and thus further increases demand for manufacturing skill. This results in an increase in equilibrium supply of skill in manufacturing.

**Proposition OA.12.** Suppose Assumption 5 is satisfied in the baseline model, and that $\frac{\partial}{\partial v_S}\pi_S(v_S, h_S; \theta_3) = \frac{\partial}{\partial v_S}\pi_S(v_S, h_S; \theta_1)$. As long as $\frac{R_M(\theta_3)}{R_M(\theta_1)}$ is small enough and $\frac{\partial^2}{\partial v_M \partial h_M}\pi_M > 0$, the wage gradient falls in manufacturing ($\frac{\partial}{\partial v_M} w_i(v_M; \theta_3) - \frac{\partial}{\partial v_M} w_M(v_M; \theta_1) < 0$).

*Proof.* First, note that Equation (11) can be rewritten as $w_M(v_M) = w_S(\phi(v_M))$. Differentiating gives that $\phi_{v_M}(v_M) \geq \frac{\max \frac{\partial}{\partial v_M}\pi_M}{\min \frac{\partial}{\partial v_S}\pi_S}$, for any $v_M \in (v^c_M, \bar{v})$. Note that $\frac{\max \frac{\partial}{\partial v_M}\pi_M}{\min \frac{\partial}{\partial v_S}\pi_S} > 0$ by Assumptions A2.1 and A2.2.

For any $a_R \in (0, 1]$ we can always define a function $\phi(\cdot, a_R) : [v^c_M(\theta_1), 1] \to [0, 1]$ such that $\frac{\frac{\partial}{\partial v_M} C(v_M, \phi(v_M, a_R))}{a_R} = \frac{\partial}{\partial v_M} C(v_M, \phi(v_M; \theta_1))$. I will show that there must exist some $a^*_R > 0$ such that if $a_R < a^*_R$ then $\phi_{v_M}(v_M, a_R) < \frac{\max \frac{\partial}{\partial v_M}\pi_M}{\min \frac{\partial}{\partial v_S}\pi_S}$ for all $v_M \in [v^c_M(\theta_1), \bar{v}(\theta_1)]$. By differentiating the definition of $\phi(\cdot, a_R)$ we get $\phi_{v_M}(v_M, a_R) = \frac{-s'_M(v_M; \theta_1)a_R + C_{v_M v_M}(v_M, \phi(v_M, a_R))}{C_{v_M v_S}(v_M, \phi(v_M, a_R))}$, where $-s'_M(v_M; \theta_1) = \frac{\mathrm{d}}{\mathrm{d}v_M}[\frac{\partial}{\partial v_M} C((v_M, \phi(v_M, \theta_1))]$. Therefore, it is sufficient to show that

$$- s'_M(v_M; \theta_1)a_R + C_{v_M v_M}(v_M, \phi(v_M, a_R)) < \underline{c}\frac{\min \frac{\partial}{\partial v_M}\pi_M}{\max \frac{\partial}{\partial v_M}\pi_M}, \tag{OA.44}$$

where $\underline{c} = \min_{(v_M, v_S) \in [0,1]^2} C_{v_M v_S}(v_M, v_M) > 0$ by Assumption A3.2. Note that $\frac{\partial}{\partial v_M} C(v_M, \cdot)$ is continuously increasing, and that $\frac{\partial}{\partial v_M} C(v_M, 0) = 0$ for all $v_M \in [0, 1]$. From the definition of $\phi(\cdot, a_R)$ this implies that for small enough $a_R$, $\phi(v_M, a_R)$ must be arbitrarily small as well and, hence, so does $C_{v_M v_M}(v_M, \phi(v_M, a_R))$. Altogether, these facts imply that for small enough $a_R$, Equation (OA.44) must be met, as required.

Consider $\frac{R_M(\theta_3)}{R_M(\theta_1)} < a^*_R$ and set $a_R = \frac{R_M(\theta_3)}{R_M(\theta_1)}$. It follows from the definition of $\phi(v_M, a_R)$ and

Assumption OA.1 that

$$P_M(v_M;\theta_1) = \int_{v_M^c(\theta_1)}^{v_M} \frac{\frac{\partial}{\partial v_M} C(v_M, \phi(v_M, a_R))}{R_M(\theta_3)}.$$

Thus, we can define $S_M(v_M; a_R) \equiv R_M(\theta_3) - \int_{v_M^c(\theta_1)}^{v_M} \frac{\partial}{\partial v_M} C(v_M, \phi(v_M, a_R))$; note that $S_M(0; a_R) = R_M(\theta_3)$ and $S_M(1; a_R) = 0$, as required of all supply functions. By the fact that $a_R < a_R^*$ follows that $\phi_{v_M}(v_M, a_R) < \phi_{v_M}(v_M; \theta_3)$ for all $v_M \in E = [\max\{v_M^c(\theta_3), v_M^c(\theta_1)\}, \bar{v}_M(\theta_3)]$. Let me adapt the definitions of the sets $\Xi^1, \Xi^2$ in the proof of Theorem OA.1 as $\Xi^1 = \{v_M \in E : \phi(v_M; a_R) < \phi(v_M; \theta_3) \wedge S_M(v_M; a_R) > S_M(v_M; \theta_3)\}$, $\Xi^2 = \{v_M \in E : \phi(v_M; a_R) \leq \phi(v_M; \theta_3) \wedge S_M(v_M; a_R) \geq S_M(v_M; \theta_3)\}$. It follows trivially that if $v \in \Xi^2$ then $[v, \bar{v}_M(\theta_3)] \subset \Xi_1$, whereas for $v \geq \bar{v}_M(\theta_3)$ we have that

$$\frac{\partial}{\partial v_M} C(v_M, \phi(v_M, a_R)) \leq a_R < \frac{\partial}{\partial v_M} C(v_M, \phi(v_M, \theta_3)).$$

Because $S_M(1; \theta_3) = 1$ it follows that if $v \in \Xi^2$ then $S_M(1; a_R) > 0$—contradiction! Suppose that $v_M^c(\theta_3) \leq v_M^c(\theta_1)$, then $S_M(v_M^c(\theta_3); \theta_3) \leq S_M(v_M^c(\theta_3); a_R)$ and $\phi(v_M; \theta_3) \geq \phi(v_M; a_R)$—contradiction! Thus, $v_M^c(\theta_3) > v_M^c(\theta_1)$, and thus $S_M(v_M^c(\theta_1); \theta_3) > S_M(v_M^c(\theta_1); a_R)$, which—by Lemmas OA.6 and OA.7—implies that $S_M(v_M; \theta_3) > S_M(v_M; a_R)$ and thus $P_M(v_M; \theta_3) < P_M(v_M; \theta_1)$ for all $v_M$. Thus, the result follows by inspection of Equation (7). □

In the case of a large expansion of services, the only workers remaining in manufacturing are those skilled highly in manufacturing but not in services. However, because initially (again, the expansion is large) manufacturing was employing some workers who were not highly skilled in either sector, it follows that *proportionately* more low- than high-skilled workers left manufacturing. Thus the distribution of skill improves in manufacturing, causing a fall in the wage gradient.

### OA.5.4 Dynamics

In this section I develop a dynamic, overlapping generations version of the model with endogenous entry. As I did in Section VI.B, I restrict attention to the canonical formulation of the model, without loss of generality.

**Workers** In each period $t \in \mathbf{N}$ there exist two generations of workers, the old (O) and the young (Y). Period $t$'s young generation is the old generation of period $t+1$, whereas the old generation of period $t$ dies in period $t+1$. Each generation consists of a measure 0.5 of workers, whose skills are given by a copula $C$ that meets Assumption 3.[25] A third random variable $v_P \in \{0, 1\}$ determines workers' sophistication and is distributed independently of both skills. If $v_P = 0$, which happens with probability $sp \in [0, 1]$, the worker correctly foresees future wages; otherwise $v_P = 1$ and the worker uses last period's wage functions to reach her sorting decision. The discount factor is $\delta \in [0, 1]$. When young, workers can costlessly join either occupation.

[25] This implies that each generation has the same distribution of skill. Of course, if there is a shock to that distribution, it will affect only the upcoming generations and, hence, there will exist some period where the two are different.

When old, they can switch from occupation $j \in \{M, S, U\}$ to occupation $i \in \{M, S, U\}$ ($U$ denotes unemployment), but they need to pay a switching cost $sc_{ji}$. I assume that remaining in ones current sector is costless ($sc_{jj} = 0$). The assumptions about reservation payoffs are unchanged from the main model.

**Firms** In each period $t \in \mathbf{N}$ there exists an unlimited supply of identical firms. Following Costrell and Loury, I assume that every firm consists of a continuum of standard uniformly distributed jobs. In each period, there is a fixed cost $c_i > 0$ of operating in sector $i$. If a firm operates in sector $i \in \{M, S\}$ and hires a worker of skill $(v_M, v_S)$ to perform job $h$, then the surplus produced in this job is $\pi_i(v_i, h_i)$, where the function $\pi_i$ meets Assumptions A2.1–A2.3. The overall surplus produced within a firm is equal to the sum of the surpluses produced in all jobs, with unfilled jobs producing surplus equal to 0.

### Supply and Demand

**Supply of Skill** If a worker with skill $(v_M, v_S)$ joins sector $i$ in period $t$, she receives wage $w_i^t(v_i)$, where $w_i^t : [0, 1] \to \mathbf{R}$; an unemployed worker receives 0 in any period. Old workers base their sorting decision only on the payoff they expect to receive in the current period. Therefore, for an old worker of type $\mathbf{v} = (v_M, v_S, v_P)$ who worked in sector $j \in \{M, S, U\}$ in period $t - 1$, the *net present value* of the payoff from joining sector $i$ in period $t$ is

$$NPV_i^t(\mathbf{v}; O, j) = \begin{cases} w_i^{t-v_P}(v_i) - sc_{ji} & \text{for } i \in \{M, S\}, \\ 0 & \text{for } i = U. \end{cases}$$

Note that sophisticated workers anticipate the correct wages, whereas unsophisticated workers use last period's wages to form their expectations.

Due to switching costs, sophisticated young workers take into account the wages they will receive when they are old. For example, if a worker could earn more in services in the current period, but expects manufacturing to pay much better in the future, she might decide to join manufacturing already as a young worker and save on the switching cost in the future. Therefore, for an young worker of type $\mathbf{v}$, the net present value of the payoff from joining sector $i$ in period $t$ is given by:

$$NPV_i^t(\mathbf{v}; Y) = \begin{cases} w_i^{t-v_P}(v_i) + \delta \max_{k \in \{M,S,U\}} NPV_k^{t+1-v_P}(\mathbf{v}; O, i) & \text{for } i \in \{M, S\}, \\ 0 + \delta \max_{k \in \{M,S,U\}} NPV_k^{t+1-v_P}(\mathbf{v}; O, i) & \text{for } i = U. \end{cases}$$

Of course, the worker will join sector $i$ only if her $NPV$ from joining that sector is greater than from joining any other sector.

Given those self-selection rules, for any period $t$ and generation $k \in \{Y, O\}$ we can easily partition the space $[0, 1]^2 \times \{0, 1\}$ into such sets $A_M^t(k), A_S^t(k), A_U^t(k)$ that a worker joins sector

$i \in \{M, S, U\}$ if and only if her $\mathbf{v} \in A_i$. In particular

$$A_i^t(Y) = \Big\{\mathbf{v} : NPV_i^t(\mathbf{v}; Y) \geq \max_{k \neq i} NPV_k^t(\mathbf{v}; Y)\Big\},$$
$$A_i^t(O) = \bigcup_{j \in \{M,S,U\}} \{\mathbf{v} \in A_j^{t-1}(Y) : NPV_i^t(\mathbf{v}; O, j) \geq \max_{k \neq i} NPV_i^t(\mathbf{v}; O, j)\}.$$

The sectoral *supply of skill* of level $v$ in period $t$, $S_i^t(v)$, is defined in the same way as in Section III.A.2, as the measure of workers with sector specific skill of at least $t$ who join sector $i$, for given wage functions $w_M, w_S$:

$$S_i^t(v_i) = 0.5\text{Pr}\Big(V_i \geq v_i, \mathbf{V} \in A_i^t(Y)\Big) + 0.5\text{Pr}\Big(V_i \geq v_i, \mathbf{V} \in A_i^t(O)\Big). \tag{OA.45}$$

**Demand for Skill** The demand for skills in each period is still determined by the firms' hiring decisions. The firms take the period $t$ wage function as given and do not face any frictions in hiring and re-assigning workers who joined the sector in which they produce. Any mapping $v : [0, 1] :\to [0, 1]$ represents some assignment of workers to jobs within a firm. The total profit earned by a firm operating in sector $i$ that assigns workers to jobs according to $v$ is then

$$\bar{r}_i^t(v) = \int_0^1 \max\{\pi_i(v(h), h) - w_i^t(v(h)), 0\}\, \mathrm{d}h - c_i.$$

The contribution of job $h$ to the maximized profit of firm operating in sector $i$ is denoted by $r_i^t(h_i)$ and is earned if worker of skill $v_i^*$ is assigned to the job, where $r_i^t : [0, 1] \to \mathrm{R}$ and $v_i^{*t} : [0, 1] \to [0, 1]$, with

$$r_i^t(h_i) = \max_{v \in [0,1]} \ \pi_i(v, h_i) - w_i^t(v), \tag{OA.46}$$
$$v_i^{t*}(h_i) \in \arg\max_{v \in [0,1]} \ \pi_i(v, h_i) - w_i^t(v). \tag{OA.47}$$

Because profit is additively separable in $\pi_i(v(h), h) - w_i^t(v(h))$, it follows that profit is maximized if and only if a sector $i$ firm (a) hires a worker of skill $v_i^*(h)$ to perform $h$ as long as this produces positive per-job profit (i.e., if $r_i^t(h_i) \geq 0$) and (b) leaves job $h$ vacant otherwise. Therefore:

$$\bar{r}_i^t = \max_v \bar{r}_i^t(v) = \int_0^1 \max\{r_i^t(h), 0\}\, \mathrm{d}h - c_i.$$

In each period firms operate in the sector that maximizes their profit (if any). Denoting the measure of firms that in period $t$ operate in sector $i$ by $R_i^t$, this implies that if entry is positive in sector $i$ ($R_i^t > 0$), then profit must be equal to the cost of entry: $\bar{r}_i^t = c_i$.

Demand for skills can now be defined. The sectoral *demand for skill* of level $v$ in period $t$, $D_i^t(t)$, is equal to the measure of jobs in sector $i$ to which workers with sector-specific skill of at least $t$ are assigned, for a given wage function $w_i^t$:

$$D_i^t(v_i) = R_i^t\text{Pr}\Big(v_i^{t*}(H_i) \geq v_i,\ r_i^t(H_i) \geq 0\Big). \tag{OA.48}$$

**Definition OA.9.** An equilibrium is characterized by:

1. an infinitely countable sequence of pairs of sectoral supply functions: $\{(S_M^1, S_S^1), (S_M^2, S_S^2) \dots (S_M^t, S_S^t) \dots\}$ , where $S_i^t : [0,1] \to [0,1]$, such that $S_i^t$ is consistent with workers' sorting decisions and given by Equation (OA.45);

2. an infinitely countable sequence of pairs of sectoral measures of firms: $\{(R_M^1, R_S^1), (R_M^2, R_S^2) \dots (R_M^t, R_S^t) \dots\}$, where $(R_M^t, R_S^t) \in \mathbf{R}_{\geq \mathbf{0}}$, such that $\bar{r}_i = c_i$ if $R_i^t > 0$ and $\bar{r}_i \leq c_i$ otherwise;

3. an infinitely countable sequence of pairs of sectoral demand functions: $\{(D_M^1, D_S^1), (D_M^2, D_S^2) \dots (D_M^t, D_S^t) \dots\}$, where $D_i^t : [0,1] \to [0,1]$, such that $D_i^t$ is consistent with workers' sorting decisions and given by Equation (OA.48);

4. an infinitely countable sequence of pairs of sectoral wage functions: $\{(w_M^{-1}, w_S^{-1}), (w_M^0, w_S^0) \dots (w_M^t, w_S^t) \dots\}$, where $w_i^t : [0,1] \to \mathbf{R}$, which clear the markets $S_i^t(v_i) = D_i^t(v_i)$ for $i \in \{M, S\}$, $v_i \in [0,1]$ and $t \geq 1$.[26]

An equilibrium is *steady state* if the wage functions do not change over time, i.e. for any $i \in \{M, S\}$ and any pair $t', t'' \in \mathbf{Z}_{\geq -1}$ it is the case that $w_i^{t'}(\cdot) = w_i^{t''}(\cdot)$.

**Proposition OA.13.** The steady-state equilibrium of the dynamic model is identical to the competitive equilibrium of the static model (Definition OA.7), in the sense that the equilibrium supply, demand and wage functions, as well as the sectoral firm measures, are determined by the same set of equations. It follows, therefore, from Theorem OA.2 that the steady state equilibrium exists and is unique.

*Proof.* Denote the steady state wage functions as $w_M, w_S$. It follows by inspection of Equations (3) and (OA.46), as well as (5) and (OA.48), that—given the same wage functions—the sectoral firm measures and the sectoral demand functions are the same in a steady state and competitive equilibria. Therefore it is sufficient to show that the supply functions are identical as well. First, note that in any steady-state equilibrium:

$$\max_{i \in \{M,S,U\}} N_i^t(\mathbf{v}; Y) = (1+\delta) \max\{w_M(v_M), w_S(v_S), 0\}.^{27}$$

Therefore, a young worker joins sector $i$ only if $w_i(v_i) \geq \max\{w_j(v_j), 0\}$, where $j \neq i$. This the very same rule as in the static model. Further, because $sc_{ji} \geq 0$ it follows trivially that for any $i, k \in \{M, S, U\}$ such that $i \neq k$ we have that $A_j^t(Y) \cap A_i^t(O) = \emptyset$, i.e. an old worker will join sector $i$ if and only if they joined that sector as a young worker as well. It follows that $A_i^t(Y) = A_i^t(O)$ and, hence:

$$S_M(v_M) = \Pr\Big(V_M \geq v_M,\ w_M(V_M) \geq w_S(V_S),\ w_M(V_M) \geq 0\Big),$$
$$S_S(v_S) = \Pr\Big(V_S \geq v_S,\ w_M(V_M) < w_S(V_S),\ w_S(V_S) > 0\Big),$$

as required. □

[26] The wage functions for periods $-1$ and $0$ are needed for the supply functions in period 1 to be well defined.

[27] If the worker chose a sector that does not maximize her current wage, in the next period she would either need to bear a switching cost or—because wage functions do not change in steady-state—receive a lower than possible wage in the next period as well.

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Proposition OA.13 shows that the results derived for the static model remain valid even if dynamics are introduced. Note that the equivalence between the static and steady state equilibria holds for any discount factor, switching costs and fraction of sophisticated workers. In particular, the static equilibrium will correctly describe the long-run behavior of the dynamic economy even if the cost of switching sectors is prohibitively high. Nevertheless, the natural replacement of older generations by new ones will lead to the same long run behavior as when there are no switching costs present.

The transitional dynamics, however, do depend on the assumptions about workers' sophistication, switching costs, and discount factors. For example, if all workers were forming rational expectations and switching sectors were costless, the economy would jump to the new equilibrium immediately. If, however, all workers were backward-looking and switching costs were prohibitive, the new equilibrium would be achieved only after many generations. Of course, if different assumptions were made about the within-sector assignment, the dynamics would be different still.[28] What the correct assumptions to make are is an empirical question to which there does not seem to exist a conclusive answer.[29]

Finally, note that this dynamic model could be generalized further and yet the above conclusions would continue to hold. For example, we could have N-generations, or firms that are not fully flexible in their decisions whether to produce or not and Proposition OA.13 would continue to hold. The reason is that the requirement that wage functions are constant across time implies that the choices of all agents in the model do not change across time either.

## OA.6 Differentiability of Solutions

First, note that Equation (19) implies that $\frac{\partial}{\partial \theta}\psi(v_S)$ exists and is equal to 0, and thus all other objects of the model are differentiable wrt $\theta$ as well. Thus, I can take derivatives wrt $\theta$ in the proofs of all results that assume that Equation (19) is satisfied.

**Lemma OA.13.** Suppose that Assumption 5 is satisfied, that one or more of $C, \pi_M, \pi_S$ depend on a parameter $\theta$, and that they are continuously differentiable in $\theta$ for any $(x, y) \in [0,1]^2$.[30] Then $\psi(v;\theta)$ and $G_i(v;\theta)$ are continuously differentiable in $\theta$ as well, and the derivatives are continuous in $v_i$ and $\theta$.

I will show this for the case when only $\pi_S$ depends on $\theta$, to keep notation simple. All the other cases are analogous.

[28] A natural alternative would be to model a decentralized dynamic assignment. Klaus and Newton (2016), for example, show that in the long run the assignment must correspond to the competitive equilibrium assignment with probability one. However, this literature restricts attention to the case of discrete types and finite number of agents, and the results cannot be straightforwardly applied to a model with continuous types.

[29] The huge literature on expectation formation argues convincingly that agents are not fully sophisticated but provides little guidance as to how significant this departure from rational expectations is (Coibion and Gorodnichenko, 2015, page 2645). The literature informs us that occupational mobility is substantial and on the rise (Kambourov and Manovskii, 2008); however, it looks at the number of transitions between occupations, which—in my model—would be determined jointly by the skill copula, the switching costs, and the discount factors.

[30] The result in no way relies on Assumption 5. However, I only need it to hold for the cases that satisfy this assumption, and the proof is simpler in that case.

To prove this result I will make use of the parameter $a$ from the proof of Theorem 1, which in turn determines $v_S^c$ and $v_M^c$ according to Equation (OA.14). Define the vector $\mathbf{p} = [a, \theta]$ and consider the following system of differential equations

$$\frac{\partial}{\partial v_S}\psi^e(v_S;\mathbf{p}) = \frac{\frac{\partial}{\partial v_S}\pi_S^e\Big(v_S, G_S(v_s;\mathbf{p});\theta\Big)}{\frac{\partial}{\partial v_M}\pi_M^e\Big(\psi^e(v_S;\mathbf{p}), \frac{1}{R_M}C(\psi^e(v_S;\mathbf{p}), v_S) - \frac{R_S}{R_M}G_S^e(v_S;\theta)\Big)}, \tag{OA.49}$$

$$\frac{\partial}{\partial v_S}G_S^e(v_S;\mathbf{p}) = \frac{\frac{\partial}{\partial v_S}C^e(\psi^e(v_S;\mathbf{p}), v_S)}{R_S}, \tag{OA.50}$$

with initial conditions $\psi(v_S^c(a)) = v_M^c(a)$ and $G_S(v_S^c(a)) = 0$. Trivially, this system is equivalent to the integral equation $\mathscr{T}$ considered in the proof of Theorem 1. Thus, there exists a unique pair of functions $\psi^e(\cdot;\mathbf{p}), G_S^e(\cdot;\mathbf{p})$ that solve this system. It follows that my system satisfies the conditions from Gronwall (1919) and Theorems 14.3 and 14.4 in Hairer, Norsett, and Wanner (1993), and therefore $\psi^e(\cdot;\mathbf{p}), G_S^e(\cdot;\mathbf{p})$ are differentiable wrt $\alpha \in \{a, \theta\}$ and $\frac{\partial}{\partial \alpha}\psi^e(v_S;\mathbf{p}), \frac{\partial}{\partial \alpha}G_S(v_S;\mathbf{p})$ are differentiable wrt $v_S$ and continuous in $\theta$.

Under Assumption 5, $a$ must solve $G(1; a, \theta) = 1$. From Lemma OA.1 we know that $\frac{\partial}{\partial \alpha}\psi^e(v_S) \geq 0$ for any $v_S$. In addition, the initial value condition together with the continuity of $\frac{\partial}{\partial \theta}\psi^e(v_S)$ implies that $\frac{\partial}{\partial \theta}\psi^e(v_S) > 0$ for some $v_S$. It follows, thus, that $\frac{\partial}{\partial \theta}G(1; a, \theta) > 0$. The implicit function theorem implies, therefore, that the function $a(\theta)$ which solves $G(1; a(\theta), \theta) = 1$ is continuously differentiable in $\theta$. Thus, the (extended) equilibrium separation function is also differentiable in $\theta$ with:

$$\frac{\partial}{\partial v_S}(\frac{\mathrm{d}}{\mathrm{d}\theta}\psi^e(v_S)) = \frac{\partial v_M^c}{\partial \theta}\frac{\partial}{\partial a}(\frac{\partial}{\partial v_S}\psi^e(v_S)) + \frac{\partial}{\partial \theta}(\frac{\partial}{\partial v_S}\psi^e(v_S)),$$

and thus $\frac{\mathrm{d}}{\mathrm{d}\theta}\psi^e(v_S;\theta)$ is continuous in $v_S$ and $\theta$. The result for $\frac{\mathrm{d}}{\mathrm{d}\theta}G_S(v_S)$ follows immediately.

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