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Applying the Black-Litterman Model Using Analyst Recommendations on the Nordic Stock Market

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Abstract

In this thesis, we are using consensus stock analyst recommendations which are applied to the Black-Litterman optimization model. We create a portfolio consisting of the Nordic Stock market between 2002 and 2017 to examine whether the analyst recommendations could add value to the model. The stocks in the portfolio are separated into portfolios based on the type of recommendation: "buy", "hold", and "sell", and tracked historically to determine the stocks' performance relative to the market. The investment period is divided into three investment periods surrounding the Great Recession. The portfolio created from the Black-Litterman model is then compared against benchmarks to determine the raw excess and risk-adjusted returns based on performance measures. The Black-Litterman performed significantly better than the market portfolio before the Great Recession but underperformed in the period afterwards in terms of raw excess and risk-adjusted returns. This suggests that the consensus analyst recommendations may add value in certain situations for the Black-Litterman model separated by the type of rating.

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1 Introduction

Brokerage firms and analysts spend a large amount of time and resources analyzing stocks to inform and identify attractive stocks for investors. Information gathering is a costly activity that should compensate for their time and efforts with commensurate expected profits for both the investor and analyst through the brokerage services (Grossman and Stiglitz, 1980). A logical reasoning for obtaining brokerage services is excess stock returns following recommendations from equity analysts, i.e. the expected benefit from the advice should exceed the cost of the advice. The aim of this study is to apply consensus recommendations from stock analysts as views in the Black-Litterman optimization model. The Black-Litterman model distinguishes itself from traditional models as it allows the combination of the portfolio manager or investor's tactical views in the portfolio optimization processes with the market equilibrium about the investment opportunities (Litterman, 2003). The Black-Litterman model uses a Bayesian approach to introduce subjective views regarding the equilibrium's expected returns to form a new set of expected returns.

Black and Litterman (1992) argued that Markowitz' (1952) model was too theoretical and difficult to apply in practice. Unless the investor has perfect knowledge about the expected return and covariances, the mean-variance model may not yield the optimal results. It may be unrealistic to assume that a portfolio manager holds this information on all the assets. Also, even the slightest change in any of the input can lead to large differences in the optimal weights, thus the model is extremely sensitive to estimation errors. They started creating the foundation for their own model in 1990, coming up with improvements in 1991 and 1992 before publishing the Black-Litterman model[1] in 1992 in the paper "Global Portfolio Optimization". In the following years after the publication, multiple authors have contributed with extensions to the model.

Previous literature regarding the value of analyst recommendations suggests that analyst recommendations have economic value. Bjerring et al. (1983) found that if investors followed the advices from analysts with favorable ratings they would earn significantly positive abnormal returns due to analysts' possession of market

[1] Black-Litterman will from now on be referred to as BL for simplicity

timing abilities. Elton et al. (1986) found excess returns in the same month and the following two months after a consensus analyst recommendation has changed, and excess returns of 3.43% over three months after the most favorable changes in recommendations. Furthermore, the returns from analyst recommendations are more significant and larger from the consensus compared to the brokerage firm or analyst level. Womack (1996) examined whether analysts can predict or influence stock prices. His findings suggest that the one-month post-recommendation returns are positive and significant for buy recommendations, and large and negative for sell recommendations. The returns are asymmetric with +2.4% for buy recommendations, and -9.1% for sell recommendations over six months. The recommendations align with the direction of the analysts' predictions. Womack's findings support the notion that consensus recommendations have stock selection and market timing capabilities. Stock selection is viewed as the ability to be right on the direction of the stocks, and market timing is the ability to take advantage of favorable market movements.

We will use consensus analyst recommendations rather than top brokerage firms' or star analysts' recommendations. The choice of structuring the analyst recommendations to be applied in BL is imperfect, and consensus recommendations may not be the most optimal way of forming the views. Additionally, empirical evidence suggests that stock prices react slowly to information accommodated in the analyst recommendations (Womack, 1996; Barber et al. 2001), such that the model may fail to capitalize. Factors such as information may already be incorporated, but not all investors have access to analyst recommendations, while capital constraints and transaction costs may limit full incorporation of information from the analyst recommendations to the market prices. Searching for the optimal structure may be interesting, but it is out of scope of the purpose of this paper.

1.1 Research Question

This thesis will focus on the implementation of the BL model using historic consensus analyst recommendations applied on the Nordic stock market. The BL model is motivated by the weaknesses of Markowitz' (1952) mean-variance optimization model due to unrealistic weightings, sensitivity to input parameters and unrealistic implementation. The BL model ought to solve the error-maximization problem with a special case of Bayesian approach with views from the market's equilibrium expected returns. The user introduces an own set of views which acts as a statement for the expected returns of the assets which enables the derivation of the optimal portfolio weights. The portfolio starts from a neutral point determined by the market equilibrium and tilts in the direction of the investor's views. Thus, the optimal portfolio is proportional to the market equilibrium and the investor's portfolio from the views (He & Litterman, 1999). Intuitively, the investor starts with a portfolio determined by the market equilibrium and then invests according to her views, the views' strength, covariance between views and equilibrium returns and the covariance among the views.

The consensus analyst recommendations would be used in a BL setting by structuring the stocks according to the type of recommendation. With the application of the BL model, we can determine the economic and investment value from the consensus analyst recommendations. By dividing the stocks into sub-portfolios based on their consensus analyst recommendation, it allows the BL model to separate between types of stocks based on the recommendation to obtain the optimal portfolio. The type of stocks is separated into "buy", "hold", and "sell"-type recommendations. The performance of the stocks, based on their type, is then tracked, and the value of the analysis is derived from ac calendar-time rather than the event-studies approach on the Nordic stock market. This allows us to test the historic total returns and risk-adjusted performance of the BL model against competing strategies and portfolios, i.e. market portfolio, Markowitz' mean-variance portfolio, minimum variance portfolio, buy-and-hold portfolio, and $1/N$ portfolio on the Nordic stock market.

Overall, the portfolio derived from the BL model can generate superior raw excess return from 2002 to 2017 compared to its peers. Especially before the 2007-08 financial crisis, no other portfolio was able to exceed the performance of the BL model. Furthermore, the BL model generated better risk-adjusted performance measured by Sharpe, Treynor, information and Jensen's alpha before compared to after the Great Recession. The information ratio is consistently high compared to most of its peers in all periods. In contrast, the portfolio underperforms the benchmark portfolios in raw excess and risk-adjusted returns from 2008 to 2017. The rationale behind the performance is its excessive risky positions due to the absent of constraints on the portfolio. Additionally, the consensus analyst recommendation is resistant to making “sell”-type recommendations throughout the investment period, which makes the portfolio tilt towards stocks with poor returns.

Section 2 reviews existing literature surrounding portfolio optimization and various contributions regarding BL. Section 3 explains the theory behind the frameworks of the BL and the Markowitz model. Section 4 describes how the data is structured and managed in this study. Section 5 introduces the methodology of the BL separated into portfolios based on the analyst recommendations. The results and conclusion are presented in section 6 and 7, respectively.

2 Literature

In this section we will provide an overview of the application of the BL model and briefly discuss existing literature regarding portfolio diversification.

Markowitz (1952) argues that an investor should focus on achieving the highest risk-return tradeoff. He claimed that an investor could achieve this through 1) focusing on minimizing the risk while simultaneously keeping returns constant, or 2) keeping the risk constant while maximizing the returns. This can be achieved through diversification, which Markowitz considers to be the only "free lunch" in finance. To maximize diversification benefits, the investor aims to find the assets with the lowest correlation with the existing assets to minimize the overall risk of the portfolio.

The BL model was introduced in the early 1990s as an extended tool to the Markowitz’ mean-variance framework and the Capital Asset Pricing Model (CAPM) by Shape and Lintner. Black and Litterman (1992) argued that the unconstrained Markowitz’ original framework could provide solutions that would result in large short or long positions in a handful of assets. Even with constraints, such as through no shorting of the assets, the Markowitz model would still assign zero weights to multiple assets. They argued that the weights in the mean-variance model are extremely sensitive to even the smallest changes in expected returns. Thus, the results are unstable portfolios due to the high sensitivity to the inputs. Expected returns are hard to estimate, and the authors argued that historical returns are a poor proxy for expected returns. Rather, Black and Litterman (1992) suggested that investors should use their own views about the prices of securities as a proxy for expected returns. By first finding the equilibrium risk premiums, and using them as a neutral point, investors can then add their own views, if they have any, to obtain the expected returns. Should they not have any views, the results of the weightings for the assets will equal the CAPM equilibrium market portfolio.

He and Litterman (1999) gave a more intuitive interpretation of the mathematical workings behind the model than what Black and Litterman (1992) did. The authors explained how an investor should start by investing in the market portfolio and then add views. The views will allow the investor to deviate from the market portfolio according to the views and the confidence of the views. The authors aim to give investors a more general idea of the inner workings and provide a detailed clarification of the model than was provided from the original article.

Idzorek (2005) provided an intuitive and detailed overview of the BL model and its steps by using eight assets from the U.S. stock market. The author introduced an alternative approach to derive the optimal portfolio with the views, called the *New Combined Return Vector*. Using a specified scalar and covariance matrix of the error terms, combined with the other parameters of the BL model, the New Combined Return Vector can be derived. By solving for a defined unconstrained maximization problem, it is possible to obtain the new weights in the optimal portfolio from the model.

He, Grant and Fabre (2013) applied the BL model on the Australian stock market. The authors treated the consensus analyst recommendations as a proxy for the views in the model. Stocks on S&P/ASX 50 index with "buy"-type recommendations on average tend to outperform the market. In contrast, the stocks with unfavorable recommendations tend to underperform the benchmark S&P/ASX 50 index. By separating stocks according to the type of the recommendation and applying the BL model with frequent rebalancing, the investment strategy outperforms the market in terms of raw excess returns and risk-adjusted performance measures.

Walters (2014) provided an overview of the earlier work of the BL model. He introduced three different reference models. The original BL model from Black and Litterman (1992) and He and Litterman (1999) applied a Bayesian approach, while Walters used the term Canonical Reference Model (CRM) for these models. The non-Bayesian were split into two categories, with and without models that included the parameter tau often called "weight on views" (a scalar which will be introduced in the next section).

Bessler, Opfer and Wolff (2017) used a multi-asset portfolio and compared the optimization problem with the BL model, the mean-variance model, and the naïve diversification portfolio. The authors defined the naïve diversification as the equal weighted multi-asset portfolio, while the BL model used the mean-variance model to overcome the problems of estimation. A comparison between the models was used in an out-of-sample context. They concluded that the BL model created an optimized portfolio that outperforms the other models, as the BL model gives the highest Sharpe ratio, even after controlling for different risk-aversion levels and portfolio constraints.

3 Theory

This section will introduce the concepts and parameters of the BL model. The section will start by introduction Markowitz' Modern Portfolio Theory (MPT) before continuing with the BL model, as the latter builds upon the former. Finally, the mathematical and theoretical framework of the BL model will conclude this section.

3.1 Markowitz

The BL model was developed as an extension of Markowitz' mean-variance model and the CAPM. The mean-variance model aims to find an optimal portfolio by minimizing the variance for the assets in the portfolio holding expected return constant. Alternatively, the investor may maximize the expected return for a given level of risk. Thus, the goal is to find a portfolio that maximizes the risk-return trade-off. Markowitz defined risk as the variance of the individual asset returns, but also the covariance in-between. Markowitz argued that by diversifying across assets, an investor can reduce the total risk while achieving the required rate of returns. Diversification was known long before Markowitz introduced his mean-variance model in 1952, but his model illustrated the effect of diversification on efficient and inefficient portfolios taking the risk-return tradeoff into account (Markowitz, 1999).

The assumption regarding the mean-variance model is the maximization of the risk-return trade-off in respect of the risk-aversion of the investor. Once this is established, the model can be applied. Furthermore, a proxy for the expected returns, the variance, and covariance of the returns need to be established to obtain the asset allocation in the portfolio. The following mathematical expressions would need to be satisfied to obtain the weightings of the assets:

$$\min w^T \sum w \tag{1}$$

s.t

$$w^T r = r_p \text{ and } w^T = 1 \tag{2}$$

Alternatively, one can instead maximize the return:

$$\max w^T \mu \tag{3}$$

s.t

$$w^T \sum w = \sigma_p^2 \text{ and } w^T = 1 \tag{4}$$

Where

w^T	is a vector of the portfolio weights, i.e. $w^T = (w_1, w_2, \dots, w_n)^T$
$w^T \sum w$	is the total variance of the portfolio, $\sum$ being a covariance matrix
$w^T r$	is the sum of the expected returns for the portfolio
r_p	is the required return for the portfolio
σ_p^2	is the variance for the portfolio

Solving any of these problems will produce optimal weights for the mean-variance model. For problem (1) there will only be one unique solution that minimizes the variance for any given assets. However, problem (3) can have different suggestions for any given assets, as it depends on the risk tolerance of the investor. An investor with high risk-tolerance would require a higher expected return for the risk taken. In contrast, an investor with low risk-tolerance requires a lower expected return.

3.1.1 Markowitz Limitations

It is necessary to have full knowledge of the expected returns and the covariance matrix of the returns for the Markowitz model to work. However, these inputs are often estimated with errors. The problem with the estimation technique is that even small estimation errors can lead to large adjustments in the portfolio structure. This is especially true for assets with high expected returns and low variance, which are the most prone to estimation errors.

Additionally, the model assumes normal distribution of the returns of the assets. This assumption is often made in the theoretical finance world, but it is not a realistic assumption in practice as the returns tend to have fat tails. The reason being that there tend to be asymmetry in financial returns and volatility (Andersen et al., 2001), and during financial crisis' asset returns have shown to exhibit kurtosis more than three.

The model has shown to give final weights that may be extreme, i.e. it might suggest relatively large long or short positions for the assets. Implementing such a strategy in real life can be quite costly with respect to transaction costs. Let alone, several funds and other practitioners face constraints with respect to short sale, meaning the unconstrained mean-variance model might not be of use. It is possible to implement a no short constraint in the model, but the model will then tend to come up with "corner solutions" where multiple assets will have zero weights in the portfolio. Such solutions are not optimal either, as most of the diversification benefit might vanish, and this combined with the sensitivity of the estimation error make the model undesirable and excessively risky.

3.2 Black-Litterman

When Black and Litterman introduced their model in the early 1990s, they offered solutions to some of the drawbacks of the Markowitz framework. The very basics of the BL model build on the same framework as Markowitz'; maximize the risk-return tradeoff. However, the BL model differs when it comes to the expected returns. Black and Litterman apply the market capitalization weights for the assets as a starting point in the portfolio. Rather than specifying the expected returns from historical returns, the expected returns are a weighted-average of the views and degree of confidences, and the market portfolio. The portfolio manager can then add her own views about the assets, and the new portfolio weights will be tilted towards the assets according to her views. If the portfolio manager does not have any views, the weights from the benchmark[2] will be used, thus the portfolio manager will end up holding the market equilibrium portfolio. Thus, the BL optimal portfolio is a set of deviations from the market capitalization weights tilted towards the views and balances the contributions from the expected returns from the views. The model adjusts the expected returns away from starting values in the direction of the views. In other words, the optimal portfolio is the market equilibrium portfolio plus a weighted sum of portfolios adjusted to the views.

Because the BL model allows the user to specify her views, the model assumes the efficient market hypothesis does not hold. The efficient market hypothesis assumes that all available information is reflected in the stock prices. If the semi-strong form does hold, then the market will already include all public information in the asset prices, and the investor will hold the market portfolio unless the investor is trading on private information. Therefore, the market should occasionally have mispriced securities such that an investor can generate their own views and deviate from the market portfolio. Using the formulas below for the BL model, the optimal weights from problem (3) of the mean-variance model can be obtained.

The following formula represents BL model, and was presented by He and Litterman (1999):

$$\bar{\mu} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q] \quad (5)$$

Where

$\bar{\mu}$ is the vector of mean expected excess returns

τ is a scalar indicating uncertainty of CAPM, also known as the "weight on views" and scales the covariance matrix of returns

Σ is the covariance matrix of historical excess returns

P is the $k \times n$ matrix expressing the k view portfolios in terms of weights in n assets

Ω is a diagonal covariance matrix representing the uncertainty of the views for the error terms

[2] Henceforth, *the* benchmark (or market) portfolio mentioned throughout this paper will be referred to as the market-weighted stocks in the investment portfolio. Even though the 10 largest stocks in each stock exchange may not constitute the market portfolio in the CAPM world, calling it the market portfolio makes it simpler in context of writing the paper.

$\prod$ is a n-vector of the equilibrium risk premiums

Q is a k-vector expressing the expected excess returns on the k view portfolios

The BL model blends the market equilibrium with the views from the investor (Lee, 2000). The first bracket is a common multiplicator of the formula. According to the second bracket of (5), i.e. $P'\Omega^{-1}Q$, the model is a simple weighted average of the market equilibrium and the views. The first term in the second bracket, $(\tau\sum)^{-1}\prod$, is the market equilibrium from prior returns, while the second term, $P'\Omega^{-1}Q$ is the returns determined by the views. Thus, if the distribution of $\tau\sum$ is narrow, then $(\tau\sum)^{-1}$ will be large and more weight will be allocated to the equilibrium, $\prod$. Likewise, if the confidence on the views are high, Ω, is small, $P'\Omega^{-1}Q$ will be high and more weight will be allocated to the views represented by Q. The formula gives an intuitive explanation of how the relationships between the parameters are connected to derive the posterior returns. Otherwise, if the views are absence for the investor, the second term, $P'\Omega^{-1}Q$, vanishes and the expected returns would be determined by the market equilibrium. For this reason, the BL model is the weighted average of the equilibrium and the views, where the weights on former and latter are established by the degree of uncertainty of the views.

3.2.1 Reference Models

To fully understand how the original BL model works, the reference models, CRM and Alternative Reference Model (ARM), are presented here by Walter (2014). The greatest difference between the two reference models are determined by which parameters are random or used as inputs. Upon explaining the difference, we will use the analogy from Walters (2014).

Black and Litterman assumed the prior normal distribution of the expected returns as the following:

$$r \sim N(\mu, \sum) \quad (6)$$

Where μ is the unknown mean and Σ is the variance. These values are needed as inputs into the portfolio optimization processes. Further, μ, the distribution of the random mean of the returns is defined as:

$$\mu \sim N(\pi, \Sigma_\pi) \tag{7}$$

π is the estimate of the mean, and Σ_π is the variance of the estimate for the unknown mean, μ. The variables can be expressed as a linear relationship:

$$\mu = \pi + \varepsilon \tag{8}$$

We can interpret this as the prior returns being normally distributed around the estimates of π with a disturbance value ε. The disturbance value, ε, is also normally distributed, but it has a mean of zero and variance Σ_π, and by assumption it is uncorrelated with μ. Finally, it is possible to define a new equation for the variance, Σ_r, which is a product of the variance of the returns, r, and our estimates, π:

$$\Sigma_r = \Sigma + \Sigma_\pi \tag{9}$$

The CRM can now be defined for the expected returns:

$$r \sim N(\pi, \Sigma_r) \tag{10}$$

Now with the posterior distribution available, the model allows for solving a mean-variance optimization. The main difference between equation (6) and equation (10) lies in the returns. Equation (10) uses a distribution, while equation (6) uses point estimate of the return, μ. Equation (6) is known as the ARM. There should be a clear distinction between the CRM and ARM when the BL model is applied to differentiate between the parameters, which will have a significant effect on the results depending on the model. For this study, we will only use the CRM, and will not go into more details of the ARM.

3.2.2 Equilibrium Returns

The original BL model uses the estimated excess returns from the CAPM market portfolio. In our thesis, we will work with the quadratic utility function and the standard assumptions required for the CAPM to hold. The CAPM can be defined as follows:

$$E(r) = \alpha + r_f + \beta\, r_m \tag{11}$$

Where

$E(r)$ is the expected return

α is the residual, or asset idiosyncratic excess return

r_f is the risk-free rate

β is a regression coefficient of the form $\beta = \rho \frac{\sigma_p}{\sigma_m}$

r_m is the excess market portfolio return, or market risk premium

The residual risk from an asset should be uncorrelated with other assets, and thus it is possible to diversify away this non-systematic risk. Non-systematic risk is specific to the stock and not universal for the stocks in the portfolio or the market. Therefore, the CAPM states that an investor should only be rewarded for taking on systematic risk, that is β, and not residual risk that stems from α. Moreover, in the CAPM world, all investors should hold the same risky portfolio, i.e. the market portfolio, which is also the portfolio for a fully rational investor. If this holds, then in equilibrium the weights of the assets in the market portfolio will be determined by the market capitalization of the assets. No other portfolio will have a higher Sharpe ratio than the market portfolio on the efficient frontier under this scenario.

3.2.3 Reverse Optimization

It can further be stated that once we are in the equilibrium, all sub-markets must exist in equilibrium. That is, any sub-market the investor chooses to invest in will be a part of the global equilibrium. For our context, this implies that the Nordic stock market is also in equilibrium as a sub-market. The market portfolio assumes positions in the entire investable asset universe, which makes it hard to specify the expected returns due to the limitation of available information. Investors only pick a limited investable universe and optimize their portfolio according to the available assets. In equilibrium, however, reverse optimization can derive the expected excess returns given that the market capitalization and the covariance matrix are estimated. To reverse optimization, we start with the following quadratic utility function, as defined by Walters (2014):

$$U = w^T \prod - (\frac{\delta}{2}) w^T \sum w \quad (12)$$

Where

U is the investors utility, also known as the objective function

w is a vector of weights for the assets

$\prod$ is a vector of equilibrium excess returns for the assets

δ is the risk aversion

$\sum$ is the covariance matrix of excess returns

We can maximize the utility function with respect to the weights to get the solution of the reverse optimization by taking the first derivative of equation (12) with respect to *w* and solve for $\prod$ to obtain:

$$\prod = \delta \sum w \tag{13}$$

One common approach for the estimation of the covariance matrix, $\sum$, is the use of historical returns. The market weights, *w*, can be obtained directly from the market capitalization of the stocks. However, we still need a value for the risk aversion parameter; δ. The risk aversion can be obtained by multiplying equation (13) with w^T on both sides and replacing the vector terms with scalar terms, and thereby solving for the risk aversion, such that we get the following equation:

$$\delta = \frac{(r - r_f)}{\sigma^2} \tag{14}$$

Where

δ is the risk-aversion coefficient

r is the total market portfolio return (r = $w^T \prod + r_f$)

r_f is the risk-free rate

σ^2 is the market portfolio variance (σ^2 = $w^T \sum w$)

An alternative method for calculating the risk aversion parameter is through the formula for Sharpe ratio. Equation (14) can then be rewritten as the following expression, where numerator denotes the Sharpe ratio and the denominator is the variance of the market portfolio:

$$\delta = \frac{SR}{\sigma_m} \tag{15}$$

Both equation (14) and (15) provides a risk aversion parameter that can be utilized in equation (13). Once a value for the risk aversion is found, we can plug the covariance matrix ($\sum$), market weights (w), and the risk aversion (δ) into equation (13) to get the (prior) equilibrium returns for the assets.

We can now continue with the expression of the prior distributions. However, we need the $\sum_{\pi}$ from the reference model. Black and Litterman assumed that the covariance matrix of the estimate is proportional to the covariance of the returns. The parameter τ was made as a constant of the proportionality, such that $\sum_{\pi} = \tau\sum$. The prior distribution can then be defined as:

$$P(A) \sim N(\prod, \tau\sum), r_A \sim N(P(A), \sum) \tag{16}$$

This is the prior distribution for the BL model, and it represents the estimate of the mean with a proportional variance. τ is typically given a low value in the literature which is close to zero. For example, Black and Litterman (1992), He and Litterman (1999), and Idzorek (2005) choose values of τ between 0.025 and 0.05. By using the CRM equation (10), we can rewrite the prior distribution equation (16) to the following expression:

$$r_A \sim N(\prod, (1+\tau)\sum) \tag{17}$$

3.2.4 Views Matrix

The BL model distinguishes itself from other optimal portfolios by allowing the investor's or portfolio manager's subjective views to be a function of the expected returns. The stronger the views (either through higher expected returns or lower uncertainty of the views) the greater would the tilt be towards the portfolios formed by the views (recall that the expected returns are a weighted-average of the portfolios derived from the views and market equilibrium). Adding a view creates a positive tilt towards the security if the view is more bullish than the expected return implied by the BL model without the views. In contrast, a negative tilt can be created if it is more bearish than the BL without the views. Black and Litterman (1992) defined the mathematical expressions of the investor's views as following:

$$P * E(R) = Q + \varepsilon \tag{18}$$

Where

$E(R)$ is the expected returns

P is the $k \times n$ matrix expressing the k view portfolios in terms of weights in n assets

Q is a k-vector expressing the expected excess returns on the k view portfolios

ε is the error term

The views can be expressed either as relative or absolute, where the sums of the rows of the weights are 0 and 1, respectively. There is no universal way of how the weights of P can be applied. In practice, the weights of the views depend on the conditions and processes of estimating the expected returns from the views (Walters, 2007; Litterman, 2003). Satchell and Scowcroft (2000) applied an equal weighted scheme in P where each stock applies the same weight on views. However, both He and Litterman (1999) and Idzorek (2005) use the market capitalization as their weighting scheme. The P matrix in its general form can be expressed as following:

$$P = \begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & \ddots & \vdots \\ P_{k,1} & \cdots & P_{k,n} \end{bmatrix} \tag{19}$$

The P matrix is not required to be invertible nor full rank (Walters, 2014), which makes it sometimes difficult to determine a distribution for the views in P. In the case that the P matrix does not have a full rank; the incomplete or relative views may make the variance non-invertible. Walters (2014) provided the following conditional distribution for the views:

$$P(B|A) \sim N(Q\ ,\Omega) \tag{20}$$

$$P(B|A) \sim N(P^{-1}Q, [P^T\Omega^{-1}P]^{-1}) \tag{21}$$

As Walters (2014) pointed out, formula (21) is of no practical use, and no evaluation of formula (21) is needed for the BL model. Since P can be either invertible or non-invertible, formula (21) is difficult to evaluate. However, it turns out that formula (21) can be of aid when discussing Bayes Theorem, which we will see later in the section.

$$Q + \varepsilon = \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix} \tag{22}$$

Q represents a $k \times 1$ vector of the views, and ε is an error term. ε is an unobservable, normally distributed random variable with mean 0 and diagonal covariance matrix of the views, which is the uncertainty in the views. Recall that Ω is the diagonal covariance matrix representing the uncertainty of the views for the error terms. Ω is diagonal because of the assumption that views are uncorrelated and independent. Q contains information about the weight of the views. It may be expressed in terms of expected returns of the assets either in absolute or relative form. In absolute form, the views express expected returns of an asset. In relative form, the views express the differential expected returns between securities. In absence of active views, the model will suggest holding the benchmark portfolio with no deviation and active trades. ε itself is not observable and does not enter the model. An error term is still needed, otherwise the investor would be 100% confident in all the views. The confidence of the views will be expressed as follows:

$$\Omega = diag(P(\tau\sum)P^T) \tag{23}$$

Thus, in the most general case, Ω will have this form:

$$\Omega = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_k \end{bmatrix} \tag{24}$$

Since the error term is unobservable, Ω can be applied to incorporate the error term. Ω contains the variances of the error terms, ω, that are connected to the views. By inverting Ω, a new matrix provides the confidence of the views. The degree of confidence is the standard deviation around the expectation. The views that tilt the weights towards the assets are a function of the confidence and magnitude about the expected returns from the views. The total adjustment away from equilibrium may deviate from the views expressed in Q due to the associated uncertainty. In the case that the investor is absolute certain about a view, the diagonal is zero for Ω and adjustment is fully reflected from the views and exactly proportional given 100% confidence.

In the original BL model (Black and Litterman, 1992), the authors did not provide instructions for the calculation of the variance in Ω. There is no universal agreement for how Ω can be determined, and the most suitable way to specify omega may be context dependent. The most common method is to assume proportionality between the variance of the returns and the variance of the views, and that the two are independent of each other.

3.2.5 Bayes Theorem

The returns calculated from the BL model are a product of the implied equilibrium returns and the views of the investor. Both the views and the implied equilibrium returns are assumed to be normally distributed. We already defined the prior distribution in formula (16) as well as the conditional distribution (21). By applying Bayes Theorem on the prior and conditional distribution, we can create a new posterior distribution. The posterior distribution is defined as the precision weighted average from the prior and conditional estimates. According to Walters (2014), the posterior distribution, which can also be referred to as the BL master formula, can be defined in the following way:

$$P(A|B) \sim N([(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q][(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}, [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}) \quad (25)$$

Further, under this representation of the BL master formula, Walters (2014) provides an alternative representation of the mean returns, $\hat{\Pi}$, and the covariance, M, of expression (25) as following:

$$\hat{\Pi} = \Pi + \tau\Sigma P^T[(P\tau\Sigma P^T) + \Omega]^{-1}[Q - P\Pi] \quad (26)$$

$$M = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1} \quad (27)$$

It is worthwhile to give a better intuition of how these two equations work such that the reader can better understand the model fully. We will first take a closer look at equation (26), by presenting two extreme cases. In the first case we will let $\Omega \rightarrow 0$, meaning that there is 100% certainty about the views, which transforms (26) into the following:

$$\hat{\Pi} = \Pi + \Sigma P^T[P\Sigma P^T]^{-1}[Q - P\Pi] \quad (28)$$

In this scenario where there is no uncertainty, Ω and τ disappear. τ disappears because the estimated returns will be insensitive to the value of τ. Furthermore, if the investor sets a view for every asset, P becomes invertible, and the following will hold:

$$\widehat{\prod} = P^{-1}Q \tag{29}$$

Equation (29) makes sense as if there is no uncertainty about the views, and then the expected returns should equal the views. On the other hand, we have the scenario in which the investor is not sure about any of the views, such that $\Omega \rightarrow \infty$. Under this scenario, we will obtain the following mean return equation:

$$\widehat{\prod} = \prod \tag{30}$$

The interpretation behind equation (30) is that the expected returns should equal the equilibrium returns given that the confidence of the views is very low when Ω converges toward a high value.

The posterior variance, M, is the uncertainty in the posterior mean estimate, and not the variance of the returns. Thus, it is the variance of the estimate of the posterior mean. To test equation (27) under the two scenarios, we need to rewrite it with help from the Woodbury Matrix Identity to get an alternate formula for the variance:

$$M = \tau\sum - \tau\sum P^T(P\tau\sum P^T + \Omega)^{-1}P\tau\sum \tag{31}$$

In the case of 100% certainty of the views, that is, $\Omega \rightarrow 0$, equation (31) will simply be $M = 0$, which will create the largest departure from the weights in the benchmark portfolio for the assets specified in the views. In the case where the investment views have very low confidence, such that $\Omega \rightarrow \infty$. then equation (31) will converge towards $M = \tau\sum$, i.e. convergence towards the benchmark portfolio with no views. Similarly, if the views are weak, the model does not depart from the prior (Meucci, 2008).

3.2.6 The Impact of τ

The parameter τ has caused great confusion for many authors and practitioners of the BL model due to how the parameter should be expressed and its impact. The starting point will be the definition of Ω from expression (23). However, now the entire covariance matrix of Ω will be used:

$$\Omega = P(\tau\sum)P^T \tag{32}$$

By substituting expression (32) into formula (26), it is possible to obtain a new expression of the mean returns:

$$\widehat{\prod} = \prod + \left(\frac{1}{2}\right)[P^{-1}Q - \prod^{T}] \tag{33}$$

By setting Ω proportional to τ, the latter can be eliminated from equation (33). Hence, τ will be irrelevant for the outcome of the expected returns. However, it does not eliminate τ from the posterior variance formula (27) entirely. It is possible to rewrite Ω in a more general form:

$$\Omega = P(\alpha\tau\sum)P^{T} \tag{34}$$

And rewrite equation (33) to obtain:

$$\widehat{\prod} = \prod + \left(\frac{1}{1+\alpha}\right)[P^{-1}Q - \prod] \tag{35}$$

By substituting equation (32) into the posterior variance formula (27), the following expression will be obtained:

$$M = \left(\frac{1}{2}\right)\tau\sum \tag{36}$$

However, τ is still not eliminated. In the CRM, when setting Ω proportional to $\tau\sum$, the posterior covariance of returns will depend on τ. There are multiple approaches to derive τ. The most common methods are to either use a maximum likelihood estimator or the best quadratic unbiased estimator:

$\tau = \frac{1}{T}$ The maximum likelihood estimator

$\tau = \frac{1}{T-k}$ The best quadratic unbiased estimator

T is the number of samples, and k is the number of assets. This method for calibrating τ assumes that the covariance matrix is estimated from historical data, and the τ value will typically be close to zero. Alternatively, one can also determine τ by setting it as the amount invested in the risk-free given the prior distribution. In this case, the weights of the portfolio invested in the risky assets given prior views are:

$$w = \prod[\delta(1+\tau)\sum]^{-1}$$

Under this method, the weights of $[\frac{1}{1+\tau}]$ allocated to the risky assets will be smaller than suggested by the CAPM. This must hold because the investor, under Bayesian Theorem, is uncertain about the estimate of the prior, and do not want to be fully invested in the risky assets.

3.2.7 Limitations of the Black-Litterman Asset Allocation Model

Like most models, the BL model has its shortcomings and weaknesses. In the original BL model, as well as the extensions, the investor is only required to come up with their views about the returns of the assets. However, there might be difficulties regarding the implementation of the views about the volatility of the assets. Instead, the investor is providing inputs about the expectations of expected returns by relying on the covariance matrix that utilizes historical and back-ward looking data as inputs. The standard, unconstrained BL does not allow the investor to create views regarding the volatility. For example, in periods of low volatility, the investor might believe that future volatility would be higher, and that would be difficult to implement in the model.

The BL model uses variance as the risk measure, which might not be the universal measure of risk. The variance is normally distributed, which means that both the upside and the downside have equal importance. An investor who is more concerned about losses rather than gains may focus on other measures than the variance, as this can provide a false perception of the risk of the portfolio. In that case, the focus on the downside risk may have a greater importance.

Furthermore, the BL model is also sensitive to changes. It is based on the Markowitz mean-variance model, which is sensitive to small estimation errors. Even though the BL model is less sensitive than the mean-variance model, small changes in the inputs can lead to different optimal weightings, though not as extreme as for the mean-variance model. Further, the original BL model does not give the best possible portfolio with the main objective of maximizing the risk-adjusted return, but rather the optimal portfolio based on the views.

4 Data

Our research design applies the BL model to equities on the Nordic stock markets which consists of Oslo (OSE), Stockholm (OMX), Copenhagen (CSE), and Helsinki Stock Exchange (HEX). The Islandic Stock Exchange (ICEX) is excluded from the research due to the limitation of data. The sample of the securities is limited to the ten largest stocks from each stock exchange (see table A1 in the appendix for a full list of the stocks) from January 1997 and held fixed until December 2017. This makes the sample tilt towards large-cap stocks. The constraints are less limited compared to using a sample of small-stocks only. Due to the size effect, fewer analysts follow small-cap stocks than large-cap stocks (Desai et al., 2000; Bauman et al., 1998). The choice of the stocks was due to simplicity and avoiding issues of singularity regarding the inverse optimization that may arise if the stocks are not listed throughout the whole period and complexity of structuring the inputs.

Monthly data were collected from December 1996 to December 2017 from Thomson Reuter's Datastream. The adjusted prices (which adjusts for stock splits, dividends etc.) are used to calculate the historical returns of the stocks. However, data from 1997 to 2001 was mainly used for the estimation of the inputs. The out-of-sample period and start of the back testing is conducted from January 2002 to December 2017. As the time interval is 20 years, this gives a total of 252 monthly observations, which were sufficient for this research design. Furthermore, the stock prices are converted to the local currency, NOK, to avoid dealing with complications regarding currency risk and hedging strategies. However, Haavi and Hansson (1992) found that hedging currency risk between the Nordic stock exchanges did not add significant value over long investment horizons. The method of conversion was directly done in the Datastream software. The data analysis, presentation, and structuring will be managed in the programming language *R*.

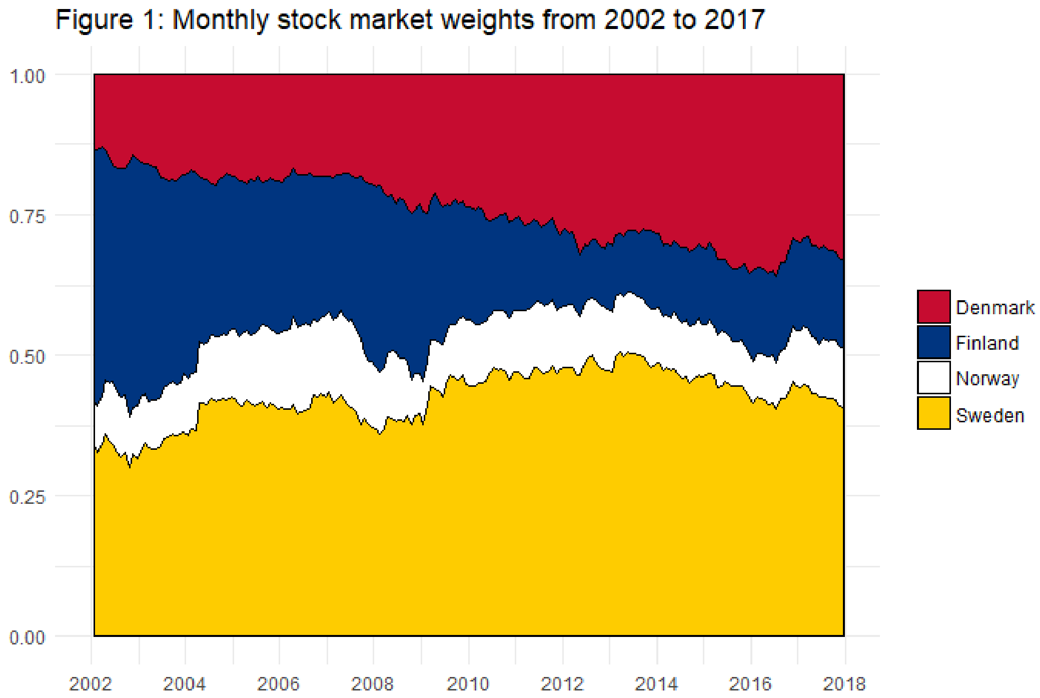

Figure 1 illustrates the country profile and its market weights of the stocks in the portfolio from 2002 to 2017. According to figure 1, the ten largest stocks in the Swedish and Finnish stock market dominate the chosen portfolio with 34% and 44% weighted approximately. The constituents of the portfolio change throughout the investment period as the market capitalization of the Finnish stocks reduce in half, and Danish stocks more than double in size while the Norwegian stocks exhibit little alterations. The Swedish and Finnish ten largest stocks are three to four times larger than the ten largest Norwegian stocks. For example, Ericsson 'B', the largest Swedish stock as of January 2002, with a market capitalization of NOK 318.35 billion, exceeds the total market capitalization of the Norwegian stocks during the same period. This may have some effect regarding the weights of the optimal portfolio based on the consensus analyst recommendations and diversification.

Table 1: Intercountry correlation coefficients for the Nordic portfolios

	Denmark	*Finland*	*Norway*	*Sweden*
Denmark	1.000	0.253	0.521	0.532
Finland		1.000	0.288	0.564
Norway			1.000	0.565
Sweden				1.000

The results of the correlation coefficients between the market value weighted country portfolios in excess of the risk-free rate are exhibited in table 1. If the correlation coefficients between the countries are low, then international diversification may provide gains through reduction of non-systematic risk in the portfolio. The results of the correlation coefficients are far from perfectly correlated, but higher compared to the studies of Haavisto and Hansson (1992) and Booth et al. (1997). However, these authors use different data and time period in their research, which may contribute to the differences in the correlation coefficients found in our data.

To obtain the weights implied by the market portfolio, data for the market capitalization of the stocks were obtained. The market capitalization from Datastream is defined as the share price multiplied by the ordinary shared issued. The 1-month NIBOR is used as the proxy for the risk-free in this context for estimating the inputs, as well as the risk-free rate used throughout this paper. The risk-free rate was converted from annual to monthly rates. The return series, which includes the historical stock returns and risk-free rates, were taken the natural logarithm to induce the normality assumption needed for the BL model. If not explicitly stated otherwise, “returns” will be referred to as the excess returns in what follows.

Datastream standardizes the recommendations from various brokerage firms and systems to numeric investment intervals. The ratings provided by Thomson Reuters are classified as follows: a rating of 1 represents a "strong buy", 2 is a "buy", 3 is a "hold", 4 is a "underperform", and 5 is a "sell". This allows the research design to be flexible for manipulation and adjustments of the recommendations in a research setting.

In total, there are 10,080 data points for each variable, i.e. stock returns, market capitalization and consensus analyst recommendations. However, not all stocks are rated in every month. For example, Orion and Stora Enso are not given any ratings in the sample period, but still included in the portfolio. Total recommendations are 9,170 for the 40 stocks with 5,258 changes from January 1997 to December 2017. Consistent with earlier literature (Stickel, 1995; Elton et al., 1986; Stickel, 1995), “buy”-type ratings are more often recommended compared to “sell”-type ratings. The ratio between “buy”-type (“strong buy” and “buy”) and “sell”-type (“sell” and “underperform”) is 15.4:1. The explanation behind the high ratio might be due the constraint of only large-cap stocks in the sample. About 45% of the recommendations in the sample are “buy”, 52% are “hold”, and 3% are “sell”. Large stocks tend to be less volatile, more profitable than small firms, and have greater coverage. Elton et al. (1986) reports a ratio of 3.5:1 and Stickel (1995) reports ratio of 6:1, a lot lower than the ratio from the stocks in our sample.

In contrast, all stocks listed on the Nordic stock exchanges that have been given a rating between 2002 and 2017 have a ratio of 4.7:1 between “buy”-type and “sell”-type ratings. This is more consistent with earlier literature. About 53% receives a “buy”-type recommendation, 36% receives “hold” recommendations, and 11% receives “sell”-type recommendations. Thus, there might be some selection bias in our sample set due to use of large-cap stocks which might be less prone to unfavorable recommendations from the analysts.

Table 2 displays the descriptive statistics of the ten largest stocks under coverage from the consensus analysts from 2002 to 2017. About 91% of the stocks are under coverage during the sample period and remain stable around the average annually except for 2007 where it drops to 87.5%. The average annual rating was 2.58 and remains stable around the average through the entire investment period. The period with the poorest ratings occurred from 2002 to 2006 where the average rating was 2.65. The period with most favorable ratings was between 2010 and 2012 with the average rating of 2.46. The number of average analysts per covered stock has been on a steady increase from 2002 to 2017, even after the Great Recession.

The consensus rated the stocks modestly with few ratings in “strong buy” and “sell” region. No stocks from 2002 to 2017 were given the most unfavorable rating “sell” (5), and only 0.2% of the ratings were “strong buy”. Similar distributions of “strong buy” and “sell” ratings were observed for the periods of pre- and post-crisis. Instead, more than 95% of the recommendation are distributed amongst “buy” and “hold” on average. Interestingly, there were no considerable increases in “sell”-type ratings after the Great Recession. Furthermore, the consensus was more pessimistic before compared to after the financial crisis of 2007-08.

Table 2: Descriptive statistics of the consensus analyst recommendations on the Nordic stock market

	Average analysts per stock	*Average rating*	*Stocks covered (%)*	*1 (%)*	*2 (%)*	*3 (%)*	*4 (%)*	*5 (%)*	*Buy (%)*	*Hold (%)*	*Sell (%)*
2002	17.5	2.61	90.6	0.2	36.6	60.9	2.3	0.0	33.3	64.6	2.1
2003	17.8	2.70	92.5	0.2	34.5	56.1	9.2	0.0	32.1	59.4	8.5
2004	17.5	2.61	92.5	0.7	38.7	57.4	3.2	0.0	36.5	60.6	2.9
2005	17.5	2.70	92.5	0.2	26.1	70.7	2.9	0.0	24.4	72.9	2.7
2006	17.6	2.64	92.5	0.0	34.7	61.0	4.3	0.0	32.1	64.0	4.0
2007	18.0	2.53	91.5	0.0	44.9	51.2	3.9	0.0	41.0	55.4	3.5
2008	18.4	2.48	90.0	0.2	52.5	44.4	2.8	0.0	47.5	50.0	2.5
2009	19.3	2.69	90.0	0.0	33.6	64.1	2.3	0.0	30.2	67.7	2.1
2010	20.3	2.48	91.7	1.6	50.4	47.6	0.5	0.0	47.7	51.9	0.4
2011	21.2	2.42	92.7	0.2	57.1	42.7	0.0	0.0	53.1	46.9	0.0
2012	22.0	2.49	91.5	0.0	49.2	50.8	0.0	0.0	45.0	55.0	0.0
2013	21.5	2.65	90.0	0.0	33.1	65.5	1.4	0.0	29.8	69	1.2
2014	20.1	2.64	90.0	0.0	33.8	65.0	1.2	0.0	30.4	68.5	1.0
2015	20.3	2.61	90.0	0.0	40.3	56.5	3.2	0.0	36.2	60.8	2.9
2016	20.2	2.64	89.6	0.0	37	61.2	1.9	0.0	33.1	65.2	1.7
2017	19.7	2.69	87.5	0.0	29.3	66.0	4.8	0.0	25.6	70.2	4.2
2002-2007	17.6	2.63	92.0	0.2	35.9	59.6	4.3	0.0	33.2	62.8	4.0
2008-2017	20.3	2.58	90.3	0.2	41.6	56.4	1.8	0.0	37.9	60.5	1.6
2002-2017	19.3	2.60	90.9	0.2	39.5	57.6	2.7	0.0	36.1	61.4	2.5

Second column are the average number of analysts providing recommendations per stocks, the third column states the average rating given by the analysts, and fourth column is the percentage of stocks covered in the investment universe. The individual recommendation distributions are given as "strong buy", "buy", "hold", "underperform", and "sell". Stocks that fall into the sub-portfolios "buy" with recommendation 1 ($1 \leq A \leq 2$), "hold" with recommendation 2 ($2 < A \leq 3$) and "sell" ($A > 3$).

5 Methodology

This section presents the approach of the methodology for the parameters of the BL model using the consensus analyst recommendations and separation into sub-portfolios. The section will go through covariance-variance estimation technique, the choice of τ and risk-aversion coefficient, the views, and comparative benchmark portfolios.

5.1 Constructing the Black-Litterman Model

5.1.1 Covariance Matrix

The choice of covariance matrix estimation is important for fund management and can be essential for forecasting returns (Litterman and Winklemann, 1998). BL assumes that the expected excess returns are proportional to the covariance-matrix of historical returns by a multiple of τ. Black and Litterman (1992) assumed that the returns follow a normal distribution. Thus, we will follow the same assumptions by imposing normal distribution of the returns by calculating the logarithmic returns. The covariance-matrix is estimated based on the rolling 60-month previous log excess returns of the stocks.[3] This method creates a covariance matrix for each month from January 2002 to December 2017 with a total of 192 covariance-variance matrices. The first covariance-matrix in January 2002 is derived from the monthly log excess returns from January 1997 to December 2001. The second covariance matrix in February 2002 is derived from the monthly log excess returns from February 1996 to January 2002, and so on. The covariance matrices in the rolling windows are equally-weighted with no time decaying weighting.

5.1.2 Tau

The scalar τ is inversely related to the weight from the implied equilibrium excess returns. This means that the scalar adjusts the aggressiveness of overweighting or underweighting of the stocks based on the views. A higher scalar implies a greater divergence from equilibrium excess returns. There has been no agreement to how the scalar, τ, should be set. The uncertainty of the mean is lower than the uncertainty of returns. Thus, τ for most application should hold a value close to

[3] Shorter windows cause singular matrixes with determinant of 0 that cannot be used in the optimization problem.

zero (Black and Litterman, 1992; Lee, 2000; Meucci, 2008, Idzorak, 2006). He and Litterman (1999) determine the ratio of the error term to the scalar, ω/τ, to be proportional to the variance of the views, $P\Omega P'$, then the value of the scalar becomes irrelevant for the posterior returns. Increasing the value of τ causes sizable changes in Ω, but would leave posterior returns and the new optimal weights estimates unaffected. However, for consistency with earlier applications, we choose τ equal to 0.05, which is in the range of previous literature (Walters, 2007; Lee, 2000, Blamont & Firoozye, 2003, He & Litterman; 1999).

5.1.3 Risk Aversion

To solve the vector of implied excess returns, the risk aversion coefficient is needed for the investor, which determines how much of the capital is invested in the risk-free and risky assets. Drobetz (2011) assumes the risk aversion of 3 for the Dow Jones STOXX of the European indices, Idzorek (2006) estimates the risk aversion to be 3.07 using U.S. and emerging markets equities, Szpiro and Outreville (1988) estimated the average global risk aversion of 2.89. Following previous literature, we use a risk-aversion of 3. A figure of the BL portfolio with the risk aversions 2, 2.5, and 3 can be found in the appendix (figure A1). The different risk aversions have limited effects on the performance of the BL portfolio, but it is evident that a higher risk aversion does generate slightly higher returns over the period. This might be explained due to the Great Recession.

5.1.4 From Recommendations to Views

This application of the BL model is inspired by the work of He et al. (2013). Investors specify views on expected returns that are blended with prior information. The model provides a quantitative framework for specifying the investor's views into the model to arrive at a new distribution. BL starts from market-weighted implied excess rate of returns, where investor's views and confidence levels adjust from the equilibrium implied returns. In this research, the investor's views are derived from the consensus analyst recommendations aggregated from Thomson Reuter's Datastream. To simplify and make the consensus analyst recommendations applicable in the research design, the rankings are simplified to a value between 1 and 2, which indicates "buy", a value between 2 and 3 is a "hold", and a value greater than 3 is a "sell". The assignment of new analyst recommendations is consistent with Thomson Reuters' rankings.

Furthermore, stocks that are not given an analyst recommendation are allocated a value of 2, i.e. hold, as this provides the stock a weighting implied by the market, or a BL model without views. The argument for assigning hold rating to stocks without consensus recommendations is that it implies a market equilibrium weighting with neither overweight nor underweight. This new method of rating stocks provides views for all stocks in the portfolio.

The stock will be allocated in the consensus sub-portfolios if it receives a rating. If a stock is assigned a "buy" rating continuously over a period, then it will be included in the consensus "buy" portfolio, and only change portfolio if the "buy" recommendation is replaced with "hold" or "sell". The sub-portfolios are weighted according to the relative weighting of the stocks with the same consensus recommendation. The relative weighting will be applied based on the stock's market capitalization. As more stocks receive the same consensus recommendation, the lower will each stock's weight be in the sub-portfolios. In contrary, if only one stock receives a "sell" recommendation, then the "sell" portfolio will be 100% invested in that single stock.

Note the limitations of the consensus analyst recommendations in the optimization problem when the recommendations lag or fail to reflect the expected returns of the asset classes, especially in the short-term. For example, if the consensus may fail to identify sudden increases in the stock prices such as through mergers and acquisitions announcements, obtaining new business contracts, etc. the model might fail to assign the weights for optimal performance if the consensus does not allocate the right recommendation in the right month.

5.1.5 From Views to Q and P

To arrive at the relative views, the recommendations must be converted to stock returns. Through the conversion, we assume relative percentage overperformance (underperformance) for stocks with favorable (unfavorable) recommendations compared to the market portfolio. Tracking the historical returns of the three sub-portfolios allow us to estimate the relative performance of the stocks against the market and use it as a parameter to construct the view and pick matrix. The rolling estimation periods of 60 months are used to estimate the relative performance of the stocks against the market.[4] The results of the rolling estimation windows can be found in table 3. The historic excess returns from the rolling estimation windows are treated as the relative views in the model as over- or underperformance of the stocks relative to the market. For example, the inputs of the views in 2002 (2003) are taken from the estimation period of 1997-2001 (1998-2002). If the relative performance (over the market portfolio) of the "buy"-portfolio was +1% during 1997-2001 (1998-2002), then 1% would be the view of the stocks with a buy recommendation for the months in 2002 (2003). This percentage underperformance or overperformance of the stock relative to the market portfolio is expressed in Q, i.e. the relative view of the stocks. The past performance of the stocks will lead to over- or underweighting of the stocks relative to their counterparts. On the other hand, underperformance of the "buy"-portfolio relative to the market allows the stocks to be subjected to mean-reversal over time.

[4] These rolling estimation periods are derived by first annualizing the returns, and then rolled over fixed 60 months windows.

Table 3: Annualized excess returns of the consensus analyst recommendation portfolios and the market portfolio over 5-year rolling estimation windows

Estimation period (January to December)	*Buy*	*Hold*	*Sell*	*Market*
1997-2001	26.69%	1.44%	-23.23%	27.46%
1998-2002	13.35	-14.80	-45.67	11.34
1999-2003	7.94	-7.17	-35.85	11.02
2000-2004	-7.66	-7.49	-38.20	-3.53
2001-2005	-3.84	2.32	-43.98	-0.93
2002-2006	12.17	8.56	-32.40	9.56
2003-2007	20.19	14.64	-5.73	17.94
2004-2008	8.30	-1.81	-18.65	4.61
2005-2009	8.97	-2.56	-11.01	5.56
2006-2010	5.90	-0.61	-2.60	5.89
2007-2011	-5.24	-9.81	0.15	-3.21
2008-2012	-2.35	-4.32	-2.91	0.64
2009-2013	13.31	10.62	2.37	15.66
2010-2014	12.94	13.28	-1.29	15.93
2011-2015	15.31	7.32	-1.19	12.82
2012-2016	20.94	11.12	2.05	17.05

This table represents the annualized excess returns of the three consensus recommendation portfolios divided into “buy”, “hold”, and “sell” portfolio over 5-year rolling estimation windows. Stocks that fall into the sub-portfolios “buy” with recommendation 1 ($1 \leq A \leq 2$), “hold” with recommendation 2 ($2 < A \leq 3$) and “sell” ($A > 3$).

If the views of the stocks determine the returns to be greater than the returns from the market equilibrium, then the BL portfolio tilts towards the outperforming stock. The expressed views are compared to the implied excess equilibrium returns and establish the expected performance of the portfolios. If the expressed views of abnormal performance are higher (lower) than the implied equilibrium differential, then the views express an overperformance (underperformance).

Thus, the posterior returns are a function of historical performance of the stocks through the views.

$$Q = \begin{bmatrix} R_{p,B} - R_m \\ R_{p,H} - R_m \\ R_{p,S} - R_m \end{bmatrix}$$

Q (1×3 vector) displays the returns for each view. $R_{p,B}$, $R_{p,H}$, and $R_{p,S}$ are the annualized returns of the sub-portfolios, and R_m is the annualized returns of the market. Q is the annualized returns of the sub-portfolios less the annualized returns of the market portfolio. The abnormal returns are then converted back to monthly returns. The views in the Q matches with the assets in the P-matrix, which identifies the stocks in the views. Relative views are applied where the sum of the weights of the views are zero. The views need to be fully invested which implies that the sum of the weights in P are zero (relative view) rather than one (absolute view).

$$P = \begin{bmatrix} W_{p,1} - W_{m,1} & -W_{m,2} & \cdots & -W_{m,k} \\ -W_{m,1} & W_{p,2} - W_{m,2} & \cdots & -W_{m,k} \\ -W_{m,1} & -W_{m,2} & W_{p,3} - W_{m,3} & -W_{m,k} \end{bmatrix}$$

P (3×40 matrix) identifies the weights for each view, where $W_{p,k}$ determines the weighting in each sub-portfolio, and $W_{m,k}$ is the weight of the stock relative to the market portfolio. First row represents stocks with "buy" recommendations, second row represents "hold" recommendations, and third row represents "sell" recommendations. To identify the stocks in the model, P consists of the market capitalization of the stock relative to the market capitalization of other stocks in the same portfolio. For example, if stock 1 at any point in time receives a "buy" recommendation, then the relative weight of the stock is its market capitalization divided by the market capitalization of all the stocks in the "buy" portfolio less its weight in the market portfolio. Similarly, if stock 2 receives a "hold" recommendation, its relative weight is derived from all the stocks in the "hold" portfolio less the market value in the market portfolio. If stock 2 does not receive a "buy" recommendation, then its relative weight will be its weight in the market portfolio, hence $-W_{m,2}$. Assuming only stock 1 receives a "buy" recommendation, then the value of $W_{p,1}$ is 1. Thus, $-(\sum_{i=2}^{k} W_{m,k} - W_{m,1})$ equals $-(1 - W_{m,1})$ and the sum of the row is 0. To achieve the relative performance against the market, the optimization processes take long positions in the stocks of the sub-portfolios and short position in the stocks by their weights in the market portfolio. Similar weighting method was adopted by Drobetz (2001).

The variance of the consensus analyst recommendation portfolio is represented by $P'\Omega^{-1}P$. The confidence of the views, omega, is proportional to the variance of the prior returns following methods applied by He and Litterman (1999). Proportionality of the variance of prior return allows the portfolio to put less weight on the relative performance views and deviation from the equilibrium market weights when uncertainty across asset returns are high, such as during high volatility regimes. When variance levels are low, the confidence of the view vectors have more weight and create greater tilts towards the views.

$$w = (\delta\sum)^{-1}\,\bar{\mu}$$

Pulling the inputs together, a new set of updated, posterior expected returns (1) given the views from the consensus analyst recommendations and the sub-portfolios are obtained. Solving for the updated weights with the posterior returns, we can obtain the optimal weights of the stocks in the portfolio.

$R_{BL} = \sum_{i=1}^{N} w_i\, R_i$Once the new weights are obtained, they are used in the calculation of historical excess returns for back testing of the BL portfolio, which allows comparison through time and against its benchmark portfolios.

5.2 Benchmark Portfolios

The BL model with consensus analyst recommendations are compared against five other benchmark portfolios: market, $1/N$, buy-and-hold, minimum-variance, and mean-variance portfolio. The market-capitalization weighted, or market portfolio would be the main benchmark in this study. Transaction costs and taxes will be ignored for simplicity. For consistency, the covariance matrix is estimated using the historic logarithmic excess returns for the minimum- and mean-variance portfolios using the same method applied to the BL model. Similarly, the benchmark portfolios are given no constraints regarding shorting.

5.2.1 Market Portfolio

The market-capitalization weighted portfolio applies the market capitalization weights for the 40 stocks and the weighting of the stocks change according to the market capitalization for each month. This could be compared to the investor holding the ten largest stocks, weighted by their market capitalization, from each stock exchange without adjusting the weights. This portfolio will act as the market portfolio, the main benchmark, when compared against the BL portfolio.

5.2.2 $1/N$

The equal-weighted, often called $1/N$, portfolio allocates an equal proportion of the weight to the assets: $w_t = \frac{1}{N}$. There is a total of 40 stocks in this investment universe, such that each stock will have a weight of 2.5%. The weights need to be rebalanced every month throughout the whole investment period to hold equal weighting. The equal-weighted portfolio overweight small-cap stocks and underweight large-cap stocks. In general, equal-weighted portfolios have the benefits of increasing the weights to smaller stocks in the portfolio, which may have higher return prospects than large-cap stocks, but also higher risk.

5.2.3 Buy-and-Hold

The weights in the portfolio will be based on the market capitalization of the stocks at the beginning of the investment period and would be held throughout the end. There will be monthly rebalancing to keep the portfolio weights fixed to the beginning weighting. For example, if the market capitalization weight for stock A is 2% in January 2002 (or January 2008), the weight will be 2% until December 2017.

5.2.4 Minimum-Variance

The minimum variance portfolio is based on Markowitz' model, where formula (1) and restriction (2) are being applied. The goal of the portfolio is to minimize the overall variance of the portfolio based on historical returns. The portfolio will produce new weights monthly, which will tend to overweight the low volatility stocks compared to the high volatility stocks. The portfolio intends to minimize the portfolio volatility without sacrificing returns.

5.2.5 Mean-Variance

This portfolio is based on formula (3) and restriction (4). The goal is to maximize the Sharpe ratio based on the historical and expected returns, thus risk is weighted against expected returns. Despite maximizing the Sharpe ratio, this does not mean that the portfolio will necessarily generate high returns, which will depend on the risk taken. The mean of the historical returns of the stocks are used as the proxy for the expected and required rate of returns. Thus, for the first investment period in January 2002, the mean of each of the stocks historical returns from January 1997 to December 2001 were used as the expected return. The mean of the historical returns from January 1997 to December 2001 where used as the required target return, and so on.

6 Performance Analysis and Results

In this section we will present the results and findings from our studies. The performance of the BL portfolio will be evaluated and compared against the benchmark portfolio in terms of raw excess returns and performance measurements. The investment periods are divided into three separate investment periods: pre-financial crisis, post-financial crisis, and overall. The time intervals are from January to December between 2002-2007, 2008-2017, and 2002-2017, respectively. This may provide a better analysis of the performance of the model during the different economic cycles. All else equal, higher performance measurements are better for the performance of the portfolio, unless otherwise stated.

6.1 Performance Analysis

6.1.1 Cumulative Returns

The cumulative returns measure the aggregate gains or losses of an investment over time. Simply the returns of the portfolios are needed to calculate the cumulative returns from the start to the end of the investment period.

$$R_{Cumulative} = \Pi(1 + R_P) - 1$$

where $R_{Cumulative}$ is the excess cumulative returns and R_P is the monthly portfolio returns in excess of the risk-free rate.

An alternative way to compare returns between the portfolios is the annualized returns. While the cumulative returns tell the investor how much value has been added for the investment period, the annualized return reveals how much the investment has gained or lost over specific time periods. The annualized return is the geometric average of the cumulative returns for N years.

$$R_{Annualized} = (1 + R_{Cumulative})^{\frac{12}{N}} - 1$$

6.1.2 Sharpe Ratio

The Sharpe ratio is calculated as the excess returns of the portfolio less the risk-free ($E(r_P) - r_f$) divided by the volatility of the portfolio (σ_P). The standard deviation of the excess returns will be used as the proxy for the volatility. The sample average is taken for the returns of the portfolio and the risk-free rate. The Sharpe ratio allows the investor to measure the reward-to-volatility trade-off. The attraction of the portfolio could be measured by the trade-off between reward and risk to examine if the investor is compensated for the risk taken. If the portfolio is well-diversified, the Sharpe will be close to the benchmark. The Sharpe ratio accounts for the total risk and is appropriate for the risk-adjusted performance of non-diversified portfolios because non-systematic risk is included in the risk measurement. The higher the value of the measure, the better is the portfolio.

$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

6.1.3 Treynor Ratio

The Treynor ratio gives the excess return per unit of risk over the systematic risk, beta, instead of total risk, standard deviation. The beta is measured for the portfolio against a benchmark, which is the market-weighted portfolio of the same stocks used in the portfolios. Treynor tend to be useful for well-diversified portfolio due to the use of the systematic risk. It could be argued that all the portfolios in this study are well-diversified as only a handful of stocks have a correlation higher than 0.7, i.e. stocks in the same sector, which are the stocks in the financial services industry. According to Solnik (1995), a European diversification can be achieved with only 20-40 stocks. The portfolios in our research hold 40 stocks and are enough regarding diversification.

$$T_P = \frac{E(r_P) - r_f}{\beta_P}$$

$E(r_P)$ denotes the expected return of the portfolio, r_f denotes the risk-free rate, and β_P denotes the beta of the portfolio relative to the market portfolio. Thus, the parameter in the numerator is the risk premium, while the denominator is the measure of risk. All risk-averse investors prefer to maximize this value, except for when it is negative. A negative value may indicate a negative beta and non-negative rate of return above the risk-free rate, which may not be considered unfavorable for some investors (Reilly & Brown, 2012).

6.1.4 Information Ratio

The information ratio (IR) is the residual excess returns of the portfolio compared with the residual risk, or the average excess returns per unit of volatility in excess returns. The residual excess return is the return not explained by the benchmark, but other factors, such as choices and skills of the manager or strategy. Similarly, the residual risk is the risk not explained by the benchmark, but risk derived from the manager's portfolio choices. The measure can be used to quantify the information available to the manager compared to the public information available to the market, hence the name. Using this interpretation, the ratio measures the quality of the information discounted by the residual risk in taking the bets on the information. Thus, it is particularly useful for the evaluation of the BL model as the optimal portfolio is formed around the analysts' superior information of the stocks in the views as analysts tend to incorporate private information into their forecasts and recommendations (Chen & Jiang, 2005). IR is useful in determining the performance of the portfolios if the value is added through over- or underweighting of the securities relative to the benchmark portfolio given the same market risk (Goodwin, 1998). In the evaluation, Grinold and Kahn (2000) states that an IR of 0.5 is "good", 0.75 is "very good", and 1.0 is "exceptional". The IR is expressed as:

$$IR = \frac{E(R_P) - E(R_B)}{\sigma(R_P - R_B)}$$

R_P represents the return on the given portfolio, while R_B is the return on the benchmark portfolio. The IR is also commonly written as the residual portfolio return, or alpha (α_P), in the nominator and standard deviation of the residual return, or the tracking error (e_P), in the denominator.

$$IR = \frac{\alpha_P}{\sigma(e_P)}$$

6.1.5 Jensen's Alpha

Jensen's alpha is the difference between the returns of the portfolio in excess of the risk-free rate and the return explained by the market model, or benchmark portfolio. The measure explains the value added from the choices a manager makes in excess of the returns from the benchmark. If the manager can forecast expected returns, or security prices, the intercept, alpha, will be positive (Jensen, 1968). In contrast, if the manager or portfolio is underperforming the benchmark, then the alpha measure will be negative. The term $\beta_P\left(E(R_{Mt}) - R_{ft}\right)$ is the market premium determined by the CAPM, and α_P determines the additional return due to the additional choices made by the portfolio manager or strategy. Jensen's alpha does not allow for comparison across portfolios due to the different level of risks (Amenc & Le Sourd, 2003). The measure should be used to rank the portfolio throughout different periods rather than against the other benchmark portfolios. The measure is calculated by the regression equation:

$$R_{Pt} - R_{ft} = \alpha_P + \beta_P\left(R_{Mt} - R_{ft}\right) + \varepsilon_{Pt}$$

CAPM argues that α_P, or the intercept, would be zero for any portfolio. To make inferences about the portfolios forecasting abilities the standard error of the estimate of the performance measure are obtained. To evaluate the statistical significance of α_P, we run the regression above to obtain the *t-statistic* of the regression. The t-statistic of the alpha is estimated as the value of the alpha divided by its standard error to test if the alpha of the portfolio is statistically different from zero. If the t-statistic is greater than 1.96, then the probability of obtaining the alpha through luck is strictly less than 5%. This means that the alpha is statistically significantly different from zero.

6.2 Empirical Results

6.2.1 Historic Returns

Figure 2: Annualized excess returns of the portfolios divided by periods

In raw annualized excess returns, the BL portfolio exceeds their comparatively benchmark portfolios throughout the entire period and first sub-period (see figure 2 above and table A2 in appendix). For the second subperiod, the BL portfolio underperformed the market, Min-var and the Mean-var portfolio. The BL portfolio experienced the most years with negative returns, a total of six years. Further, the BL portfolio fluctuates the most, which may be a result of the excessive risk taking in the model. Table A3 (from the appendix) illustrates the annualized standard deviation for all the portfolios. It is clear from table A3 that the BL portfolio had the highest volatility overall. Meanwhile the Min-var portfolio produced the lowest volatility for the three periods, consistent with the true intention of the portfolio.

Figure 3: Cumulative returns of the portfolios from 2002 to 2017

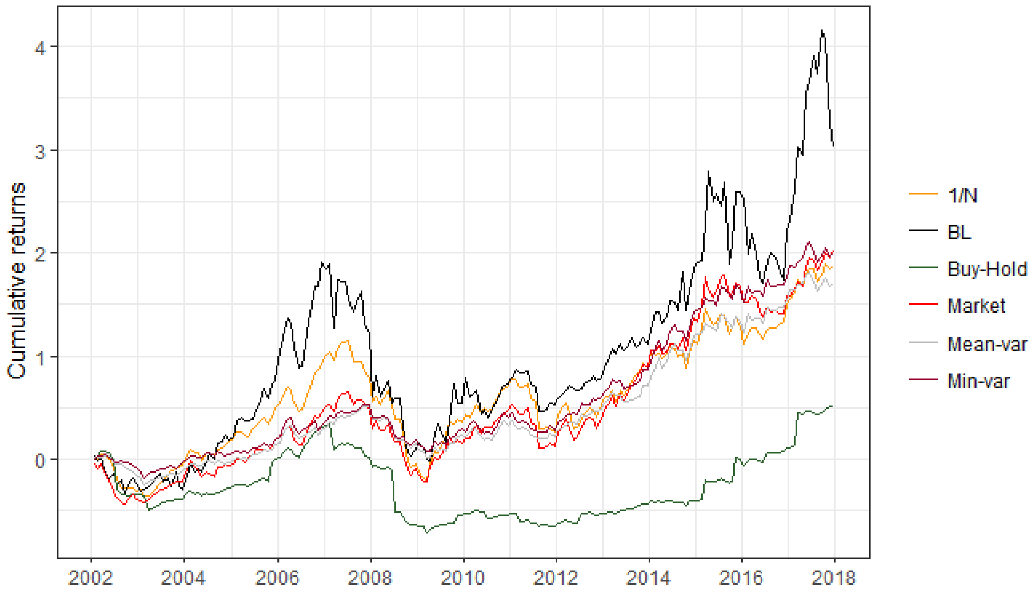

Figure 3 displays the cumulative returns over the entire investment period from January 2002 to December 2017 for BL portfolio against the benchmark portfolios. The BL portfolio yields the highest cumulative return of 303% for the entire period, above the market and comparative benchmark portfolios. The Buy-Hold strategy performs worst with 51% for the entire period, way less than the other strategies. The remaining strategies all lies in the interval 170 to 202%. It is clear from figure 3 that all the portfolios produced negative returns during the first couple of years from 2002 to 2004, but the Buy-Hold portfolio underperformed for most of the period. As figure 3 illustrates, the cumulative returns of the BL portfolio lie above the market and peers in most of the years. Furthermore, $1/N$ portfolio produced similar cumulative returns compared to the comparative benchmark portfolios. This may indicate that simple, naïve diversification models such as the $1/N$ is able to generate returns similar or better than optimal portfolios (DeMiguel, 2007).

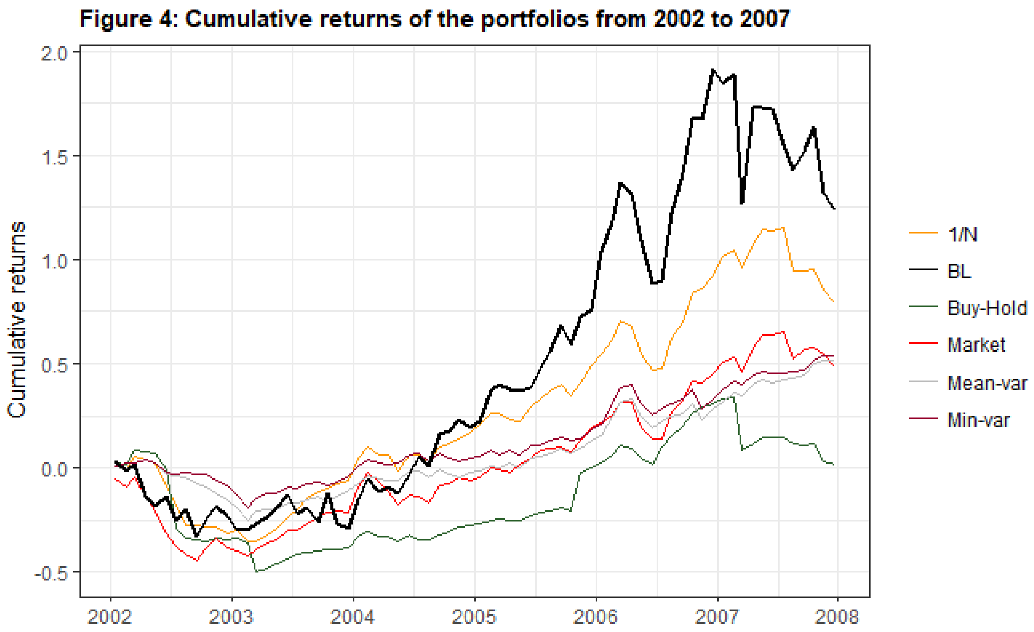

Figure 4 exhibits the first subperiod, or the pre-financial crisis period, from 2002-2007. The period can be considered a bull market, as there were no major recessions during 2002 to mid-2007. The BL portfolio exhibits higher raw excess returns than the market portfolio and its peers, producing a cumulative return of 123% for the period. The Buy-Hold portfolio is still the worst performer with a cumulative return of 2%. Figure 4 also shows that while the Min-var, Mean-var, and the market portfolio generate cumulative returns in the range of 48 to 53%, the $1/N$ portfolio improves, producing a cumulative return of 79%. At least during the first bull run, the BL portfolio with the consensus analyst recommendations performed sufficiently compared to the benchmark portfolios. The results are consistent with the findings of He et al. (2013) in respect of the same investment period.

Figure 5: Cumulative returns of the portfolios from 2008 to 2017

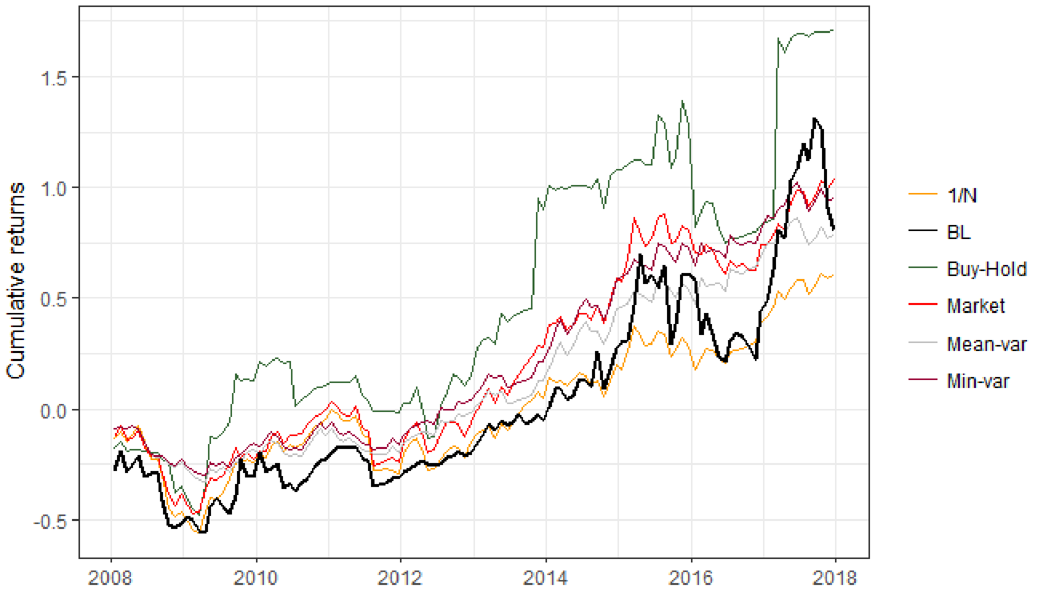

Figure 5 displays the second subperiod from 2008 to 2017 that includes the start of the Great Recession. All the portfolios start off with negative returns for the first few months of 2008 due to the financial crisis. On the other hand, the Buy-Hold portfolio outperformed the other portfolios, returning 170%. The great returns of the Buy-Hold portfolio during the start of 2017 can be explained by multiple stocks achieving more than 10% returns during March 2017. The $1/N$ portfolio is now the worst performer with a cumulative return of 61%. The BL portfolio generates a cumulative return of 80%, performing worse than the market portfolio during the last sub-period.

The optimal BL portfolio calls for high levels of standard deviation in all periods compared to its peers. Overall, the standard deviation was 30.2% compared to the market of 19.0%. The risk is mostly non-systematic as the systematic risk, beta, is 0.93 and less than 1.0, for the overall investment period. This can be explained by the unconstrained nature of the BL model where there were not imposed constraints on shorting.

6.2.2 Ratios and Performance Measures

To measure and evaluate the risk-adjusted returns from the portfolios, we can use the ratios introduced above (see table 4 below for the results). The BL portfolio did not achieve higher risk-adjusted returns than the comparative benchmark portfolios, except for the market portfolio. The total Sharpe ratio across the investment period was 0.30, and only higher than the Buy-Hold portfolio of 0.10.

Prior the financial crisis, the BL model and the Market portfolio achieved annualized Sharpe ratio of 0.46 and 0.36, respectively, but underperformed compared to the other benchmark portfolios. The BL portfolio did well before 2011 when the Sharpe ratio was higher than Min-var portfolio and Mean-var portfolio 7 out of 10 years from 2002 to 2011. The poor risk-adjusted results could be explained by the high volatility and excessive risk-taking of the BL portfolio, as shown by the significantly higher standard deviation employed by the BL portfolio compared to the benchmark portfolios. Overall, the Min-var portfolio produces the highest Sharpe ratios among the comparative portfolios, and the BL portfolio underperforms the market for the entire investment period in terms of the Sharpe ratio.

Similarly, the Treynor ratio of the BL portfolio was lower compared to the Mean-var and Min-var portfolio. However, this time, the BL portfolio achieves a higher total Treynor ratio than the $1/N$ and Buy-Hold portfolios. According to the Treynor ratio, the BL portfolio did not provide the investor with compensation for taking on additional investment risk compared to the Mean-var and Min-var portfolio. Naturally, taking on additional risk would increase the beta, which again reduce the value of the Treynor measure. Furthermore, the values of the Treynor ratio is consistent with the other performance measures in the sense of more positive results before than after the 2007-08 financial crisis. The results of the Treynor ratio confirms with the notion of excessive risk-taking of the BL portfolio.

Beta[5] of the BL portfolio was 0.93 during the overall investment period, close to taking the same systematic risk as the market portfolio. $1/N$, Buy-Hold, Min-var, and Mean-var have a beta of 0.90, 0.74, 0.33, and 0.33, respectively. The information ratio rewards portfolios that take on less risk than the market with a higher information ratio (Goodwin, 1998). Furthermore, the Goodwin found that the top 25% performers achieve an IR of 0.40 of the market-oriented large-cap active money managers and regarded as exceptional according to Grinold and Kahn (1999) before the Great Recession. The BL portfolio produces the highest IR of 1.13 through the first sub-period, and overall. This implies that the BL

[5] Beta is derived from CAPM using formula $\beta_p = \frac{Cov(r_p, r_m)}{Var(r_m)}$, where r_p denotes the excess returns of portfolio p, and r_m denotes the excess returns of the market portfolio.

portfolio achieves higher abnormal return per unit of non-systematic risk with greater diversification compared to holding the market portfolio. During the second subperiod, most of the benchmark portfolios underperform the market with negative values of IR, meaning negative alpha values.

Alpha of the BL portfolio during pre-crisis and post-crisis, and overall were 0.07, -0.008, and 0.02, respectively. Similarly, the other benchmark portfolios decline in their alpha measures after the financial crisis. The test statistic of the BL portfolio exceeds 1.96, the critical value at the 5% significance level, and we can say that the alpha of the BL portfolio is different from zero and statistically significant in the period of 2002-2007, but statistically insignificant for 2008-2017 and overall. This implies that the choices made by the consensus may have investment value for the portfolio, consistent with the IR before the Great Recession. The results of the ratios are consistent through the investment periods in the sense that the BL portfolio performs at its best prior to the 2007-08 financial crisis.

Alternatively, the statistical significance can be tested by performing the two-tailed t-test where the null hypothesis is that Jensen's alpha is zero. The Jensen's alpha is significant if the null hypothesis can be rejected, i.e. the investment strategy or portfolio outperforms the benchmark. Jensen's alpha for the BL portfolio is large and significant at the 5% in the pre-crisis era of 2002-2007. Comparatively, its alpha is the greatest compared to the peers. However, the measures are not so large nor statistically significant in the post-crisis era and the overall investment period. Intuitively, the BL portfolio is generating excess returns more than the required return that might be explained by the reverse optimization of the model and predictability skills of the consensus analyst recommendations.

Even though the Mean-var and Min-var portfolios allow for short-selling as well, they do yield significantly better risk-adjusted returns than the BL portfolio. This can be explained by the fact that these two models have constraints with respect to the risk taking due to how they are constructed. Thus, the two models can provide better risk-adjusted returns, but in terms of raw excess returns the BL portfolio outperforms them.

The BL portfolio performs poorly during and after the 2007-08 financial crisis as it experiences the longest period of drawdown compared to the benchmarks. This might imply that the consensus recommendations were too optimistic regarding the forecasts of the stocks which would cause the BL portfolio to tilt excessively in favor of stocks with buy-type recommendations in alignment with the views. This will cause overweight of stocks that would otherwise would perform poorly, and have a weighting not suggested by the market equilibrium. The average consensus analyst recommendations change before, during, and after the Great Recession. Observing the build-up to the financial crisis from the bottom of post-dotcom crisis of August 2002 to August 2008, the average consensus analyst rating was 2.63. From the top to the bottom of the financial crisis, the average recommendation was 2.42. Accordingly, this suggests that analysts were more bullish during the financial crisis than the build-up to the financial crisis. Furthermore, the average recommendation after the financial crisis between 2008 and 2017 was upgraded to 2.58 (from table 2) suggesting that the consensus may have become more optimistic post- compared to pre-crisis. This may indicate that "buy"-type recommendation should have been "hold"-type, and "hold"-type should have been "sell"-type, as there was no sell (5) and very few underperform (4) recommendations throughout the investment period.

Consistent with Baret et al. (2001) and Mikhail et al. (2004), equity analysts covering the Nordic stock markets are reluctant to use the two negative ratings. The ratio between "buy"-type and "sell"-type recommendations is large. The cost of issuing sell recommendations is greater than buy recommendations causing the latter to be more favorable for the analysts. Pratt (1993) argues that sell recommendations can harm the brokerage firm's investment banking relationships and are discouraged by them, and the flow of information through executives and investment contacts can be cut off from sell recommendations. The conflict of interest is present from unfavorable ratings. Furthermore, sell recommendations are less common and more visible with asymmetric costs with the wrong recommendation compared to favorable ratings. This may suggest that underperforming stocks may not be given the appropriate recommendation or downgrade which causes the model to allocate wrong weights to the stocks.

Table 4: Annualized Sharpe ratio, Treynor ratio, information ratio, and Jensen's alpha for the portfolios divided into pre- and post-crisis, and overall

Year	*BL*	*1/N*	*Buy-Hold*	*Min-var*	*Mean-var*	*Market*
Sharpe ratio						
2002-2007	0.46	0.61	0.01	0.74	0.70	0.36
2008-2017	0.20	0.24	0.36	0.58	0.50	0.39
2002-2017	0.30	0.36	0.10	0.64	0.56	0.38
Treynor ratio						
2002-2007	0.45	0.49	0.02	0.94	0.85	
2008-2017	0.25	0.17	0.39	0.67	0.57	
2002-2017	0.34	0.26	0.12	0.75	0.66	
Information ratio						
2002-2007	1.13	1.10	-1.07	0.13	0.08	
2008-2017	-0.18	-1.37	0.46	-0.10	-0.31	
2002-2017	0.28	-0.12	-0.7	-0.01	-0.16	
Jensen's Alpha						
2002-2007	0.07 *3.67*	0.05 *6.62*	-0.04 *-2.24*	0.06 *7.77*	0.05 *7.20*	
2008-2017	-0.00 *-0.08*	-0.02 *-4.68*	0.04 *1.87*	0.04 *5.31*	0.03 *4.01*	
2002-2017	0.02 *1.17*	0.00 *0.65*	-0.03 *-1.45*	0.05 *6.17*	0.04 *5.11*	

This table represents the comparable ratios and alpha annualized for the different portfolios in the three investment periods. For the Treynor and information ratio, as well as the alpha, the market portfolio is used as the benchmark, and thus does not get a value. To get the annualized Treynor and information ratios, the monthly ratios for the periods are multiplied with the square root of 12. For the annualized Sharpe ratio, the annualized mean excess returns are divided by the annualized monthly standard deviations of the excess returns. To get the annualized alpha, see section 6.1.5. The numbers below Jensen's alpha are the t-statistic. See appendix for the annualized yearly Jensen's alpha, Sharpe, Treynor, and information ratios.

6.2.3 Sub-Portfolios of the Consensus Analyst Recommendations

Figure 6: Cumulative returns of the sub-portfolios from 2002 to 2017

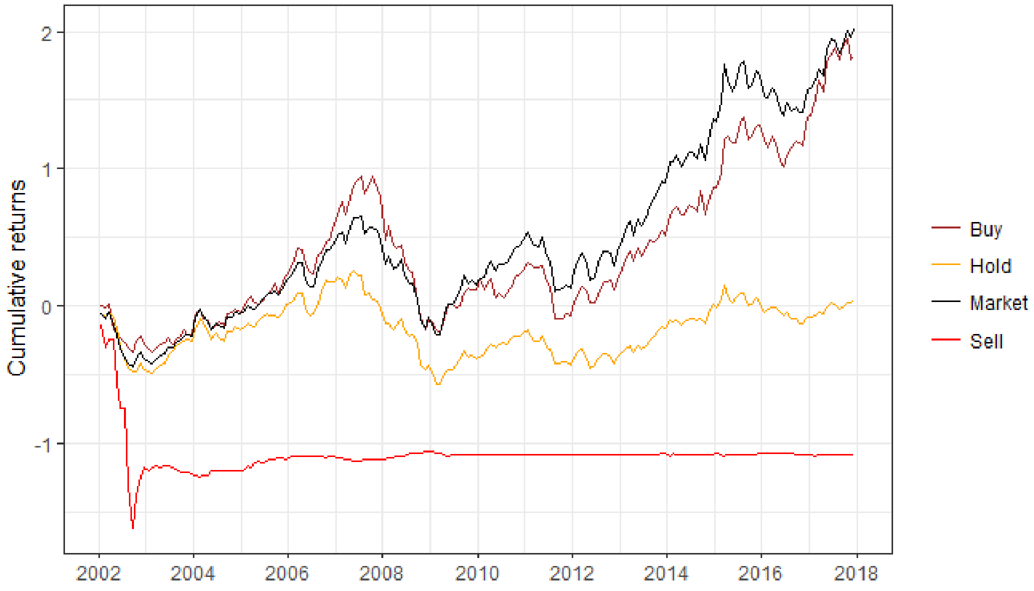

Figure 6 illustrates the three sub-portfolios and the market portfolio for the overall investment period. The sub-portfolios for "buy"-type, "hold"-type, and "sell"-type recommendations are value weighted, where each stock in each portfolio are given a weight according to the peers in with the same type of recommendations. The portfolio with "buy"-type recommendations overperforms the market from 2002 to 2007 and underperforms afterwards, consistent with the performance of the BL portfolio. The "hold" portfolio underperforms the market portfolio in all periods. Stocks with "buy"-type recommendations pulls the average returns of the market portfolio upwards, while the analysts allocate too optimistic ratings for the stocks in the "hold" portfolio. The "sell" portfolio has negative cumulative returns and never recovers. In some periods, stocks are not allocated sell-type recommendations at all, which caused the returns to become flat and never recovers. This is seen in the period between January 2010 and December 2013, which also is the period where the average ratings were the most optimistic and the BL portfolio underperformed the benchmark portfolios. This further supports the argument that analyst recommendations added less economic value to the BL model in the post-crisis compared to the pre-crisis period.

Overall, the BL portfolio outperformed the other portfolios in raw excess returns from 2002 to 2017 and overall, but it comes at the price of excessive risk-taking. The results were consistent with the findings from He et al. (2013) when implemented during the same period. However, the model weakened after the financial crisis when the characteristics of the consensus analyst recommendations changed, and the portfolio underperformed relative to its peers. All the performance measures show that the BL model with the consensus analyst recommendations are not able to generate higher risk-adjusted returns compared to the market and the alternative benchmark portfolios in all periods. One explanation for the higher volatility of the BL portfolio compared to the other portfolios is due to its leveraging because of lack of constraints on the portfolio. However, the consensus analyst recommendations may contribute positively to the construction of an optimal portfolio in a regime of less optimistic view of the market.

7 Conclusion

In this thesis we have implemented the BL model on the Nordic stock markets. The portfolio optimization processes were conducted on the ten largest stocks on each of the Nordic stock markets from January 2002 until December 2017. The consensus analyst recommendations for the stocks are used as input for the "views" in the BL portfolio to divide the stocks into "buy", "hold", and "sell" sub-portfolios. The portfolio of the BL model was compared against benchmark portfolio including the market, $1/N$, buy-and-hold, mean-variance, and minimum-variance portfolio. Performance measures were applied to evaluate the risk-adjusted returns of the portfolio against its peers. The research and back testing of the portfolios were divided into three periods of 2002-2007, 2008-2017 and 2002-2017 to test the performance during different market environments.

In raw excess returns, the BL model outperforms the market and benchmark portfolios overall and before the Great Recession but underperforms in the period of 2008 to 2017 in terms of cumulative and annualized returns. This may be due to non-alignment between the performance of the stocks and the consensus analyst recommendations. After 2008 the average consensus analyst recommendation was higher than before the financial crisis. In regards of risk-adjusted returns, the BL portfolio has mixed results regarding its performance. The performance is by a large extent superior between 2002 and 2007 compared to 2008 and 2017, where Sharpe, Treynor, and information ratio are higher. Comparatively, the risk-adjusted performances are lower against the benchmark portfolios. On the other hand, its information ratio may suggest that there could be information in the analyst recommendations that could assist the decision-making regarding the optimization processes.

In sum, the unconstrained BL portfolio using consensus analyst recommendations can generate high excess returns, but at the cost of high volatility. This is especially clear after recessions, where the BL portfolio tend to underperform the other portfolios, suggesting that the analyst recommendations are too optimistic. However, there is evidence that the consensus analyst recommendation does indeed add investment value for the BL portfolio, as the BL portfolio yield both a positive alpha and information ratio in the performance evaluation.

8 Further research

The BL model was purposely built for asset allocation, and the most common application of the model was with country and industry sector allocations. It might be interesting to test He et al. (2013)'s approach using countries and the aggregated average consensus analyst recommendations as a proxy for the views of the countries. That approach would reduce the number of the assets in the model as well as potential estimation errors.

Furthermore, different choices regarding the parameters could yield different results. The covariance-variance matrix could be estimated using different approaches such as with time-decay or expanding matrix. The confidence of the views (Ω) could be estimated using Idzorek (2014)'s approach with implied confidence of views. It might also be interesting to test for no shorting constraint to inspect whether the constrained BL model would improve the risk-adjusted returns due to lower risk-taking.

References

Amenc, N., & Le Sourd, V. (2005). *Portfolio theory and performance analysis*. John Wiley & Sons.

Andersen, T. G., Bollerslev, T., Diebold, F. X., & Ebens, H. (2001). The distribution of realized stock return volatility. *Journal of Financial Economics*, *61*(1), 43-76.

Bessler, W., Opfer, H., & Wolff, D. (2017). Multi-asset portfolio optimization and out-of-sample performance: an evaluation of Black–Litterman, mean-variance, and naïve diversification approaches. *The European Journal of Finance*, 23(1), 1-30.

Bjerring, J. H., Lakonishok, J., & Vermaelen, T. (1983). Stock prices and financial analysts' recommendations. *Journal of Finance*, 38(1), 187-204.

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial Analysts Journal*, 48(5), 28-43.

Booth, G. G., Martikainen, T., & Tse, Y. (1997). Price and volatility spillovers in Scandinavian stock markets. *Journal of Banking & Finance*, *21*(6), 811-823.

Chen, Q., & Jiang, W. (2005). Analysts' weighting of private and public information. *The Review of Financial Studies*, *19*(1), 319-355.

Datastream. (2018) *Thomson Reuters Datastream*. [Online]. Available at: Subscription Service (Accessed: April 2018)

DeMiguel, V., Garlappi, L., & Uppal, R. (2007). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?. *The Review of Financial Studies*, 22(5), 1915-1953.

Drobetz, W. (2001). How to avoid the pitfalls in portfolio optimization? Putting the Black-Litterman approach at work. *Financial Markets and Portfolio Management*, 15(1), 59-75.

Elton, E. J., Gruber, M. J., & Grossman, S. (1986). Discrete expectational data and portfolio performance. *Journal of Finance*, 41(3), 699-713.

Goodwin, T. H. (1998). The information ratio. *Financial Analysts Journal*, 54(4), 34-43.

Grinold, R. C., & Kahn, R. N. (1999). *Active portfolio management: A quantitative approach for providing superior returns and controlling risk, 2nd edition*. McGraw-Hill.

Haavisto, T., & Hansson, B. (1992). Risk reduction by diversification in the Nordic stock markets. *The Scandinavian Journal of Economics*, 94(4), 581-588.

He, G., & Litterman, R. (1999). The intuition behind Black-Litterman model portfolios. *Goldman Sachs Investment Management Research.* Publication. New York, December, 1-27

He, P. W., Grant, A., & Fabre, J. (2013). Economic value of analyst recommendations in Australia: an application of the Black–Litterman asset allocation model. *Accounting & Finance*, *53*(2), 441-470.

Idzorek, T. M. (2005). A step-by-step guide through the Black-Litterman model, incorporating user specified confidence levels. *Chicago: Ibbotson Associates*, 1-32.

Jensen, M. C. (1968). The performance of mutual funds in the period 1945–1964. *Journal of Finance*, 23(2), 389-416.

Lee, W. (2000). *Advanced theory and methodology of tactical asset allocation* (Vol. 65). John Wiley & Sons.

Litterman, R. (2003). *Modern Investment Management: An Equilibrium Approach*. Quantitative Resources Group, Goldman Sachs Asset Management. New Jersey: John Wiley & Sons.

Litterman, R., & Winkelmann, K. (1998). Estimating covariance matrices. *Risk Management Series*, Goldman Sachs.

Markowitz, H. M. (1999). The early history of portfolio theory: 1600–1960. *Financial Analysts Journal*, *55*(4), 5-16.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance, 7*(1), 77-91.

Meucci, A. (2008). The Black-Litterman approach: original model and extensions. *The Encyclopedia of Quantitative Finance*, Wiley, 2010

Reilly, F. K., & Brown, K. C. (2011). *Investment analysis and portfolio management*. Cengage Learning.

Satchell, S., & Scowcroft, A. (2000). A demystification of the Black–Litterman model: Managing quantitative and traditional portfolio construction. *Journal of Asset Management*, 1(2), 138-150.

Solnik, B. H. (1995). Why not diversify internationally rather than domestically?. *Financial Analysts Journal*, 51(1), 89-94.

Walters, J. (2014). The Black-Litterman model in detail. Boston University

Womack, K. L. (1996). Do brokerage analysts' recommendations have investment value?. *Journal of Finance*, 51(1), 137-167.

Appendix

Figure A1: Cumulative returns of the BL portfolio with risk aversions of 2, 2.5, and 3

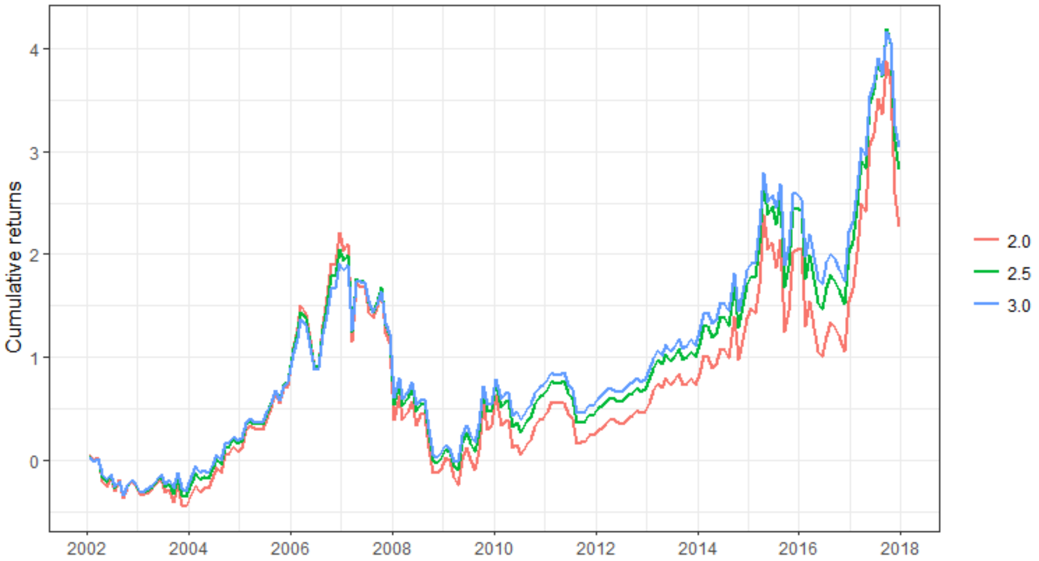

Table A1: Company names and tickers

Company	*Ticker*
DNB	DNB
NORSK HYDRO	NHY
ORKLA	ORK
STOREBRAND	STB
ATEA	ATEA
SCHIBSTED 'A'	SCHA
BONHEUR	BON
TOMRA SYSTEMS	TOM
PETROLEUM GEO SERVICES	PGS
STOLT-NIELSEN	SIN
NORDEA BANK	NDA
VOLVO 'B'	VOLVB
SANDVIK	SAND
SWEDBANK 'A'	SWEDA

ERICSSON 'B'	ERICB
SVENSKA HANDBKN. 'A'	SHBA
HENNES & MAURITZ 'B'	HMB
SEB 'A'	SEBA
INVESTOR 'B'	INVEB
SKANSKA 'B'	SKAB
NOVO NORDISK 'B'	NOVOB
DANSKE BANK	DANSKE
COLOPLAST 'B'	COLOB
CARLSBERG 'B'	CARLB
AP MOELLERMAERSK 'B'	MAERSKB
TDC	TDC
WILLIAM DEMANT HLDG.	WDH
KOBENHAVNS LUFTHAVNE	KBHL
FLSMIDTH & CO. 'B'	FLS
CARLSBERG 'A'	CARLA
NOKIA	NOKIA
SAMPO 'A'	SAMPO
UPMKYMMENE	UPM
METSO	METSO
STORA ENSO 'A'	STEAV
KEMIRA	KEMIRA
OUTOKUMPU 'A'	OUT1V
RAISIO	RAIVV
ORION 'A'	ORNBV
STORA ENSO 'R'	STERV

The table represents the different company names of the stocks that were used in the thesis, and their respective tickers. Some of the companies have several share groups, where the letters ‘A’, ‘B’, and ‘R’ represents which of the share groups were used.

Table A2: Annualized returns of the BL and benchmark portfolios

Year	*BL*	*1/N*	*Buy-Hold*	*Min-var*	*Mean-var*	*Market*
2002	-23.3%	-31.1%	-34.7%	-8.4%	-15.4%	-38.3%
2003	-7.2	36.7	-6.0	5.4	5.2	26.8
2004	67.8	24.2	17.7	8.7	9.4	19.9
2005	47.5	27.6	38.9	13.1	16.5	26.7
2006	65.4	28.6	30.3	12.2	13.1	21.9
2007	-23.3	-6.5	-22.2	15.3	18.0	2.2
2008	-51.2	-46.8	-64.1	-22.4	-23.8	-38.3
2009	42.3	42.6	25.5	9.2	7.4	25.6
2010	12.0	25.7	1.3	7.7	7.5	28.7
2011	-11.7	-25.6	-24.3	-8.0	-8.7	-23.6
2012	16.6	19.8	33.0	24.3	22.4	23.4
2013	18.6	23.5	20.6	16.7	14.8	35.9
2014	34.2	14.0	6.9	30.0	28.6	24.5
2015	26.3	6.8	65.8	9.2	6.2	13.6
2016	-10.5	8.3	10.5	4.5	9.0	-3.7
2017	25.3	16.1	37.7	8.1	6.0	16.9
2002-2007	14.3	10.2	0.3	7.4	7.2	6.8
2008-2017	6.1	4.9	10.5	6.9	5.9	7.4
2002-2017	9.1	6.8	2.6	7.1	6.4	7.1

This table represents the annualized (from monthly) excess returns for all the portfolios. See section 6.1.1 for an explanation as to how to annualize the returns. The first 16 rows show the annualized yearly excess returns, while the three last rows show the annualized excess returns for three different entire periods.

Table A3: Annualized standard deviation of the BL and benchmark portfolios

Year	*BL*	*1/N*	*Buy-Hold*	*Min-var*	*Mean-var*	*Market*
2002	33.4%	18.6%	31.1%	6.5%	7.3%	25.0%
2003	33.9	14.0	24.2	12.6	12.4	11.8
2004	29.2	17.6	13.0	9.4	9.4	24.2
2005	17.1	11.1	23.1	7.0	7.0	9.0
2006	30.8	17.7	12.8	15.7	15.3	18.3
2007	34.8	15.2	21.2	5.7	5.5	15.2
2008	42.6	27.8	44.4	13.2	12.9	25.2
2009	45.5	29.6	29.8	13.9	15.2	27.2
2010	27.5	12.0	17.3	12.8	12.7	11.6
2011	17.4	18.9	15.1	9.8	9.3	20.0
2012	7.1	23.6	26.8	9.0	9.1	22.8
2013	12.7	13.1	9.0	9.6	9.7	13.5
2014	25.7	15.9	12.7	14.1	14.4	15.3
2015	36.4	16.5	40.9	10.9	10.9	17.4
2016	29.6	13.0	12.8	11.4	12.4	12.4
2017	29.2	9.3	27.7	9.0	8.7	9.9
2002-2007	31.0	16.9	22.4	10.0	10.3	19.0
2008-2017	29.8	20.0	29.4	11.8	12.0	19.0
2002-2017	30.2	18.8	25.7	11.1	11.3	19.0

This table represents the annualized (from monthly) standard deviations for all the portfolios. To get the annualized standard deviations, the monthly standard deviations are multiplied with the square root of 12. The first 16 rows show the annualized yearly standard deviations, while the three last rows show the annualized standard deviations for three different periods of before and after the 2007-08 financial crisis, and overall.

Table A4: Annualized yearly Treynor ratio for the different portfolios

Year	*BL*	*1/N*	*Buy-Hold*	*Min-var*	*Mean-var*
2002	-0.336	-0.708	-0.644	-5.625	-11.816
2003	-0.047	0.499	0.242	0.072	0.070
2004	0.625	0.354	0.360	0.318	0.341
2005	0.439	0.288	0.301	0.486	0.523
2006	0.435	0.304	0.604	0.204	0.223
2007	-0.143	-0.068	-0.261	0.666	0.788
2008	-0.339	-0.461	-0.858	-0.621	-0.669
2009	-27.806	0.404	0.489	0.521	0.350
2010	0.072	0.271	0.013	0.202	0.192
2011	-0.146	-0.275	-0.580	-0.338	-0.390
2012	0.783	0.196	0.387	1.260	1.097
2013	0.258	0.251	0.354	0.345	0.296
2014	0.244	0.141	0.096	0.436	0.411
2015	0.283	0.075	0.390	0.215	0.141
2016	-0.064	0.089	0.142	0.085	0.188
2017	0.132	0.196	0.454	0.127	0.101

This table represents the annualized yearly (from monthly) Treynor ratio for the different portfolios. The market portfolio is used as the benchmark when computing the Treynor ratio, therefore it is not listed with its own column. Excessively high Treynor ratios are explained by low betas which cause the values to scale up.

Table A5: Annualized yearly Information ratio for all the portfolios

Year	*BL*	*1/N*	*Buy-Hold*	*Min-var*	*Mean-var*
2002	0.503	0.348	0.117	1.174	0.888
2003	-0.834	0.295	-0.773	-0.751	-0.762
2004	1.131	0.134	-0.077	-0.366	-0.342
2005	0.668	0.029	0.445	-0.464	-0.349
2006	1.347	0.309	0.367	-0.349	-0.322
2007	-0.513	-0.266	-0.614	0.532	0.647
2008	-0.233	-0.188	-0.504	0.568	0.518
2009	0.323	0.386	-0.003	-0.596	-0.635
2010	-0.517	-0.131	-1.181	-1.143	-1.153
2011	0.494	-0.074	-0.024	0.692	0.652
2012	-0.263	-0.099	0.230	0.032	-0.040
2013	-0.705	-0.498	-0.654	-1.016	-1.113
2014	0.238	-0.339	-0.629	0.180	0.131
2015	0.354	-0.298	1.667	-0.177	-0.292
2016	-0.167	0.467	0.519	0.289	0.446
2017	0.191	-0.032	0.730	-0.335	-0.418

This table represents the annualized yearly (from monthly) information ratio for all the portfolios. The market portfolio is uses as the benchmark and is therefore not included.

Table A6: Annualized yearly Jensen's alpha

Year	*BL*	*1/N*	*Buy-Hold*	*Min-var*	*Mean-var*
2002	0.008	-0.013	-0.010	-0.006	-0.013
2003	-0.033	0.012	0.002	-0.010	-0.010
2004	0.028	0.007	0.005	0.002	0.003
2005	0.012	0.001	0.003	0.005	0.006
2006	0.019	0.005	0.013	-0.000	0.000
2007	-0.021	-0.007	-0.021	0.011	0.013
2008	0.005	-0.010	-0.044	-0.006	-0.008
2009	0.037	0.010	0.011	0.004	0.002
2010	-0.023	-0.000	-0.017	-0.001	-0.001
2011	0.007	-0.003	-0.013	-0.001	-0.002
2012	0.008	-0.002	0.009	0.014	0.013
2013	-0.004	-0.006	0.000	0.000	-0.001
2014	0.000	-0.007	-0.007	0.009	0.008
2015	0.013	-0.004	0.028	0.002	0.000
2016	-0.001	0.009	0.010	0.005	0.009
2017	-0.003	0.001	0.018	-0.001	-0.002

This table represents the annualized (from monthly) yearly alphas for all the different portfolios. The market portfolio is used as the benchmark portfolio for all the portfolios and is therefore not represented in its own column.