Asset Pricing with Multiple Assets and Goods

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Acknowledgments

Contents

Introduction

This dissertation consists of three papers; 'Asset Prices and Real Exchange Rates with Deep Habits', 'Financial Market Completeness in Multi-Good Economies' and 'Correlations'. The rest of the section is organized as follows. I first discuss the common feature of the papers, namely agent heterogeneity and multiple risky assets. I then briefly discuss the main results in each of the papers.

One of the main topics in finance is to understand the behavior of asset prices. Important questions are how the equilibrium compensation for risk is determined, what the important risk factors are and how these evolve over time. A key concept is that in any equilibrium, prices should be free of arbitrage. However, in most instances one can not directly use the concept of arbitrage and one must turn to other equilibrium concepts. This involves modeling the demand and supply of risky asset in a general equilibrium framework. This will in turn link equilibrium stock prices to consumption of individuals. Early work on consumption based equilibrium asset pricing models (CCAPM) linked the equilibrium excess return (the return on the asset minus the return on a risk-free bond) to the covariance between consumption growth and stock returns ([Breeden 1979],[Lucas 1978], [Cox *et al.* 1985]). Underlying these models is the assumption that the there is a representative agent that maximize expected utility of lifetime consumption. The agent is assumed to dislike risk (risk averse) and prefer consumption today over consumption tomorrow (impatient). I depart from the standard consumption based asset pricing in two ways. Firstly, I consider economies with multiple risky assets in positive net supply. Secondly, I model multiple agents that are heterogeneous. I will now elaborate on these two departures from the basic model.

1.1 Multiple Risky Asset

In the standard consumption based model the aggregate stock market is modeled as a claim to the aggregate consumption.[1] This gives important insights into the

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[1] The basic model is a pure exchange economy with no investments or labor income. In this setup consumption and dividends are equated.

behavior of the returns on the market, but is not very informative on the cross-section of returns. Extending the model to allow for multiple risky asset allows for the study of the cross-section of returns. Each risky asset is a claim to a risky output stream.[2] Models with multiple dividend streams (Lucas trees) can be divided into two groups. The first group consists of models where the cash flows are perfect substitutes ([Cochrane *et al.* 2008], [Santos & Veronesi 2006], [Menzly *et al.* 2004], [Bansal *et al.* 2005]). The second group considers models where the Lucas trees are less than perfect substitutes ([Cole & Obstfeld 1991],[Cass & Pavlova 2004], [Zapatero 1995], [Pavlova & Rigobon 2007], [Serrat 2001]). I will refer to the latter case as economies with multiple goods.[3] In general, models with multiple risky assets give raise to different dynamics of the market price of risk, both at the aggregate level and at the individual security level. In a pure exchange economy with inelastic supply of risky assets, the required return for holding a particular asset will in general depend on the output share of the asset. This is the argument put forward by [Cochrane *et al.* 2008].

1.2 Heterogeneous agents

I depart from the representative agent setup and allow for heterogeneity. Heterogeneity can take many forms (information, beliefs, risk aversion, taste, idiosyncratic income shocks, time preferences etc.). I will focus on heterogeneity in risk aversion and taste. Heterogeneity in risk aversion will imply that as agents optimally share consumption risk, the dominating agent will be different in different states of the world. The less risk averse agents will optimally choose a more volatile consumption profile than the more risk averse. Heterogeneity in taste is related to situations in which there are different consumption goods that are less than perfect substitutes. In such situations agents might have different preferences over the goods. Similarly as the case with heterogeneity in risk aversion, heterogeneity in taste will give raise to trade in the goods to optimally share the risk.

1.3 Asset Prices and Real Exchange Rates with Deep Habits

Real exchange rates and asset prices are too volatile compared to fundamentals according to standard utility functions. Moreover, if calibrated to

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[2]I will use output and dividends interchangeably. In the pure exchange economy these two quantities are the same.

[3]Strictly speaking the case of multiple goods nests both classes of models, as perfect substitutes is only a special case.

match the equity premium the volatility of the real exchange rate is too high. [Campbell & Cochrane 1999] shows that a model with external habit formation can successfully explain the equity premium and the excess volatility of stock returns. The mechanism for generating the results is a slow moving subsistence point (external habit). [Ravn *et al.* 2006] extends the external habit formation to a multiple good setting and label it deep habits. In this paper I consider a two country - two good model with deep habits. Habit formation increases the volatility of the marginal rate of substitution. This in turns leads to higher volatility of both stock returns and the real exchange rate. I can match the equity premium, the volatility of the real exchange rate and the failure of the uncovered interest rate parity. The equity premium is predominately driven by the risk aversion, while the real exchange rate is driven by the elasticity of substitution between the home country good and the foreign country good. The failure of the uncovered interest rate parity is matched because of the high volatility of the exchange rate risk premium and the negative covariance between the interest rate differential and the risk premium on the real exchange rate. In an extension of the model I consider heterogeneity of the home country and the foreign country agents. The agents are assumed to have home bias in consumption. I show that home bias in consumption leads to home bias in portfolios. Just as the homogeneous agent economy, the economy with heterogeneous agents can match the equity premium and the volatility of the real exchange rate.

1.4 Financial Market Completeness in Multi-Good Economies

In this paper we study how market completeness depends on the utility function of the representative agent in the economy. A market is said to be complete if any contingent claim can be replicated by a set of basic securities. For the market to be complete there must be a set of basic securities that spans the entire uncertainty in the economy. A basic example is a situation where there are two states of nature (rain and sun) and two securities. Security one pays off one if there is rain and zero otherwise, while security two pays off one if there is sun and zero otherwise. In this case the market is complete because any other security can be synthetically constructed as combinations of the two basic securities. In this paper we consider a pure exchange economy with multiple goods. The utility function of the representative agent is defined over each good. There are as many basic securities (stocks) as the number of goods in the economy, and each stock is a claim to future dividends in one of the goods. The stock price processes are determined in equilibrium. We show that even though the risky dividend stream spans the entire uncertainty, the endogenous determined stock price processes might not, and thus rendering the market incomplete. Moreover, we show that the completeness of the market crucially depend on the utility function of the representative agent. The main result establishes a

sufficient condition for market completeness that only depends on the primitives of the economy. We also establish a sufficient condition for market incompleteness, and show that market completeness can depend on the choice of numeraire good. Finally we show that in a market with heterogeneous taste the market can be complete even though the individual utility functions are within the class that leads to incomplete markets if that agent was the sole agent in the economy.

1.5 Correlations

One of the most fundamental concepts in finance is diversification. If agents are risk averse, they should diversify their risky positions. In order to make optimal portfolio choice, understanding the dependency structure of the risky assets is essential. In a Gaussian world, the key concept that captures the dependency is the correlations matrix. If risky assets are less than perfectly correlated there is room for diversification benefits, and the Mean-Variance analysis has thought us how to optimize our portfolio in terms of maximizing the expected return given a certain level of variance. However, to operationalize this one will have to estimate the correlations between assets. One of the main issues one is facing is that the correlations might be stochastic, and thus using the average correlation will not be satisfactory. There is a huge empirical literature documenting several empirical stylized facts about stock return correlations. Correlations are time varying and stochastic, and tend to be high during recessions. Moreover, in times of high market volatility the correlation between stock returns are typically higher than in less turbulent times. Frequent explanations for these empirical stylized facts have been that investors are panicking or herding, thus they behave in an irrational way. Only a few papers aim at explaining the stochastic behavior of correlations using equilibrium models with rational expectations.[4] In this paper we propose an explanation for countercyclical behavior of correlations and the relation between volatility and correlations. Moreover, we establish a connection between trading volume, correlations and volatility. We consider an economy with multiple dividend streams and agents that are heterogeneous in risk aversion. When agents are heterogeneous in risk aversion, then the least risk averse agent will dominate in good states while the most risk averse agent will dominate in bad states. The intuition for this is that the least risk averse agent optimally has a more volatile consumption profile, and consequently both the upside and the downside is greater than for the least risk averse agent. The optimal risk sharing of the agents leads to a time varying risk aversion for the representative agent. In bad states of the world the risk aversion is high and in good states it is low. In certain parts of the state space the wealth shifts between the two agents are particularly

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[4]   Exceptions   are   [Chue 2005],   [Pavlova & Rigobon 2008]   and [Ribeiro & Veronesi 2002].

volatile. The high volatility of the relative wealth of the agents leads to high volatility of the representative agents risk aversion. This in turns leads to high volatility of the market price of risk, and consequently also high volatility of the discount rates in the economy. This is an economy wide effect that will impact the discount rates for all risky assets. As the volatility of the discount rates are high, so is the volatility and the correlation between stock returns. The higher correlation is a consequence of the fact that the risk aversion is a common factor in all assets' discount rates. We show that the model can deliver changes in correlations and average stock return volatility over the business cycle that are in line with the data.

Asset Prices and Real Exchange Rates with Deep Habits

Abstract

I study a two country - two good pure exchange economy with deep habits that jointly explains the volatility of the real exchange rate, equity premiums, levels of risk free rates and that reproduces the uncovered interest rate (UIP) puzzle. While both the volatility of the real exchange rate and the equity premium depend on the habit formation, the magnitudes are governed by different parameters. The equity premium depends mainly on the risk aversion while the real exchange rate depends on the elasticity of substitution between the home good and the foreign good. In an extension of the model I allow for preference heterogeneity of the home and the foreign representative agents. I solve for optimal portfolios and show that consumption home bias leads to portfolio home bias.

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2.1 Introduction

I study the effect of deep habits[2] on asset prices and on exchange rates in a [Lucas 1982] two country - two good pure exchange economy. Instead of forming habits over an aggregate consumption basket, the representative agent forms habit over individual good varieties. My model with deep habits replicates the volatility of real exchange rates, the failure of the uncovered interest parity, the equity premium and the level of the risk free rate.

With standard CRRA preferences, the risk aversion needed to match the equity premium results in a too high volatility of the real exchange rate. [Backus *et al.* 2001] show in a complete market setting, that the growth of the exchange rate equals the difference between the log stochastic discount factor in the foreign country and the home country. To be consistent with the observed exchange rate volatility, the stochastic discount factors in the home and the foreign country must be highly correlated ([Brandt *et al.* 2006]).

I match the volatility of real exchange rates by using habit formation at the country good level combined with a non separable utility function over the home and the foreign good. A non separable utility function allows for separating the elasticity of substitution between the home and the foreign good from the risk aversion. Habit formation increases the volatility of the marginal rate of substitution between home and foreign goods, and thus the volatility of the real exchange rate. Habit formation also increases the volatility of the stochastic discount factor, and allows for matching the expected excess return on the stock market. However, while the equity premium mainly depends on risk aversion, the volatility of the real exchange rate depends on the elasticity of substitution between home good and the foreign good.

The uncovered interest rate parity (UIP) states that the expected change in exchange rates equals the interest rate differential. Hence, countries with high interest rates are expected to experience depreciating exchanges rates relative to low interest rate countries.[3] However, empirical evidence shows that high interest rate countries experience appreciating rather than a depreciating exchange rates (see [Hansen & Hodrick 1980], [Fama 1984], [Backus *et al.* 2001]). To reproduce the failure of UIP under rational expectations, the risk premium for holding exchange rate risk must be negatively correlated with the interest differential and exhibit higher variance. In my model with deep habits the market price of risk is countercyclical.

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[2][Ravn *et al.* 2006] are the first to study deep habits in a macro setting.

[3]According to the UIP a regression of interest differentials on the exchange rate changes should yield a slope coefficient of one.

In times when the surplus consumption ratio[4] in the home good is lower than the foreign good, the representative agent is reluctant to scale back on consumption of domestic goods, and therefore requires a positive premium on the exchange rate. The risk free rates depend on the time discount factor, an intertemporal smoothing motive and a precautionary savings motive. The intertemporal smoothing motive is high in times when consumption is close to the habit level. In times when consumption is close to the habit level the volatility of the habit adjusted consumption is high and so is the precautionary savings motive. The interest rate differential is the difference between the intertemporal smoothing motive and the precautionary savings motive in the home and the foreign good. The difference between the intertemporal smoothing motive depends on the elasticity of substitution between the home good and the foreign good, while the difference between the precautionary savings motive depends on both the elasticity of substitution and the risk aversion. If risk aversion is high compared to the elasticity of substitution, then the interest rate differential will be low in times when the surplus consumption ratio in the home good is lower than the surplus consumption ratio in the foreign good. The interest rate differential will then move in the opposite direction as the exchange rate risk premium. Since the latter is more volatile than the former, the model reproduces the UIP puzzle.

My model is related to several recent papers that study the joint behavior of exchange rates and asset prices. [Colacito & Croce 2008] study a two-country version of the long run risk model of [Bansal & Yaron 2004]. When the long-run component in consumption growth is perfectly correlated across the home country and the foreign country, then the volatility of the real exchange rate and the equity premium is matched simultaneously. Their model, however, cannot match the failure of the UIP as it generates constant market price of risk. [Bansal & Shaliastovich 2008] also study a two-country long-run risk model. When the home country and foreign country consumption volatility is stochastic, then the model reproduces the failure of the expectation hypothesis in both the bond market and the exchange rate market. [Verdelhan 2008] uses the model of [Campbell & Cochrane 1999] to study the failure of the UIP. The model replicates the failure of the UIP, the equity premium and the risk free rate, although at the expense of too high real exchange rate volatility.[5] Common to the models by [Colacito & Croce 2008], [Bansal & Shaliastovich 2008], and [Verdelhan 2008] is that they all specify a separate pure exchange economy for the home country and the foreign country. Consumption in the two countries is exogenously specified. In a closed pure exchange economy, the required assumption for

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[4]The surplus consumption ratio is given by the consumption in excess of the habit level divided by consumption (see [Campbell & Cochrane 1999]).

[5]In the base case model the volatility of the real exchange rate is 42% but the data shows only an average exchange rate volatility of about 12%. [Verdelhan 2008] shows that including trade costs can reduce the volatility of the real exchange rate.

such a consumption allocation to hold is that the consumption good is non tradable or that there is complete home bias. In my model the countries produce different goods that are less than perfect substitutes. Home and Foreign goods are tradable, and preferences are homogeneous.

My model is also related to [Moore & Roche 2008]. They consider a [Lucas 1982] economy with separable power utility for the foreign and the home consumption good. They only examine the properties of the exchange rate, and do not take into account how the model fairs on asset pricing moments. Their model matches several features of the real exchange rate, but cannot jointly match the equity premium and the exchange rate volatility due to the assumption of separable power utility. The models of [Moore & Roche 2008] and [Verdelhan 2008] yield the same properties for the real exchange rate and for asset pricing moments. In Verdelhan (2008), the assumption is that the representative agent in each country only cares about domestic consumption (complete home bias with standard habit formation). In contrast, the model of [Moore & Roche 2008] assumes that all goods are traded and that preferences are homogeneous.[6]

In the final section of the paper I depart from the representative agent setup and allow for heterogeneity of the home and the foreign agents. The agents are assumed to have home bias for their domestically produced consumption good. Home bias in consumption causes the agents to optimally hold different portfolios. In terms of the asset pricing moments and the real exchange rate volatility the model delivers similar results as the homogeneous agent economy.

The rest of the paper is organized as follows. Section 2.2 describes the model and the equilibrium. Section 2.3 numerically examines the properties of equilibrium, and illustrates how the model matches the volatility of the real exchange rates, the equity premium, the risk free rate and the failure of the UIP. In section 2.4 I extend the basic model to include heterogeneous agents. Section 2.5 concludes. Appendix A.1 derives the equilibrium for the homogeneous agent economy, while Appendix A.2 derives the equilibrium in the heterogeneous agent economy. Appendix A.3 and A.4 deals with the Malliavin derivatives. Appendix A.5 presents the numerical method used to solve for equilibrium. Finally, Appendix A.6 discuss how the choice of numaraire impacts the equilibrium.

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[6]My model nests [Moore & Roche 2008] and [Verdelhan 2008] on the preference side. If the elasticity of substitution between the home good and the foreign good is the reciprocal of the risk aversion then my model collapses to a separable power utility over the two goods. The economic interpretation of my model is closer to [Moore & Roche 2008] as they also consider a multiple good setting with deep habits.

2.2   The Model

My model is an extension of the [Lucas 1982] two country model to include deep habits. I only focus on real quantities and therefore do not include nominal quantities.[7]

2.2.1   The Economy

I consider a continuous time pure exchange economy over the time span $[0, T]$. The uncertainty is represented by a filtered probability space $(\Omega, \Im, \{\Im_t\}, P)$, on which is defined a two-dimensional Brownian motion $B = (B_1, B_2)$. In the following all stochastic processes are assumed to be progressively measurable and all equalities are assumed to hold a.s. Stochastic differential equations are assumed to have solutions without stating the regularity conditions.

There are two countries in the world economy. Each country produces its own perishable consumption good. Output of each good follow

$$\frac{dC_i(t)}{C_i(t)} = \mu_{C_i}(t)dt + \sigma_{C_i}^{\top} dB(t) \tag{2.1}$$

where

$$d\mu_{C_i}(t) = \alpha_i \left(\overline{\mu}_{C_i} - \mu_{\delta_i}(t)\right) dt + \nu_i^{\top} dB(t) \tag{2.2}$$

for $i = (H, F)$. Here H denotes the home country and F denotes the foreign country. The diffusion coefficients are two-dimensional vectors. In this way I allow output in countries to be correlated. The expected growth rate is a mean reverting process possibly correlated with output.

A representative agent maximizes lifetime expected utility over consumption of the two goods[8]

$$E \left[\int_0^T u\left(C_H(t), C_F(t), X_H(t), X_F(t), t\right) dt \right]$$

---

[7]As my goal is to study the real exchange rate I do not employ cash in advance as in [Lucas 1982]. The real exchange rate would not change if I include cash in advance.

[8]In the original setup of [Lucas 1982] there is a representative agent in both the home country and the foreign country with equal endowments. This results in a perfect pooling equilibrium, where each representative agent consumes half the aggregate output of the two goods. I directly model the preferences of the representative agent. The resulting equilibrium is the same as the perfect pooling in [Lucas 1982].

where

$$u\left(C_H(t), C_F(t), X_H(t), X_F(t), t\right) = \frac{e^{-\rho t}}{1 - \gamma} Z\left(C_H(t), C_F(t), X_H(t), X_F(t)\right)^{1-\gamma} \quad (2.3)$$

and where

$$Z\left(C_H(t), C_F(t), X_H(t), X_F(t)\right) = \left((C_H(t) - X_H(t))^\beta + (C_F(t) - X_F(t))^\beta\right)^{\frac{1}{\beta}}. \quad (2.4)$$

The above utility function is motivated by [Ravn *et al.* 2006], where C_i is the optimal consumption of goods from country i, and X_i is the habit level of the same good. Instead of forming habits over an aggregate consumption bundle, the representative agent forms habits over each country good variety. The representative agent does not take into account the habit level in his optimization and thus the habit is external. Utility is defined as a standard power utility function over the composite good Z,[9] with at time discount factor of ρ. Z captures the degree of substitutability between the two goods in the economy. The functional form is a constant elasticity of substitution (CES) aggregator over the habit adjusted consumption. When $\beta = 1$ the goods are perfect substitutes. For $\beta = 0$ the Cobb-Douglas utility function over the habit adjusted consumption of the two goods is obtained.[10]

As in [Campbell & Cochrane 1999], I define the surplus consumption ratio for good $i = H, F$ as

$$s_i(t) = \frac{C_i(t) - X_i(t)}{C_i(t)} \quad (2.5)$$

and assume that s_i follows

$$ds_i(t) = \phi_i\left(\overline{s_i} - s_i(t)\right)dt + s_i(t)\lambda_i\left(s_i(t)\right)\sigma_{C_i}^\top dB(t) \quad (2.6)$$

where

$$\lambda_i\left(s_i(t)\right) = \sqrt{\frac{1 - \overline{s_i}}{\overline{s_i}}}\sqrt{\frac{1 - s_i(t)}{s_i(t)}}. \quad (2.7)$$

The variable s_i is a mean reverting process with long run mean $\overline{s_i}$ and speed of mean reversion of ϕ_i. The process is locally perfectly correlated with output shocks to good $i = H, F$. To understand the dynamics of the surplus consumption ratio, consider the case when $X_i(t)$ is an exponential weighted average of past consumption of good i[11]

$$X_i(t) = X_i(0)e^{-\alpha t} + \alpha \int_0^t e^{-\alpha(t-u)} C_i(u)du. \quad (2.8)$$

---

[9]Risk aversion in a multiple good setting with deep habits is not trivial, however I will frequently refer to γ as the risk aversion.

[10]The elasticity of substitution between the home good and the foreign good, η, is related to β by $\eta = \frac{1}{1-\beta}$.

[11][Constantinides 1990] and [Detemple & Zapatero 1991] model habits as an exponential weighted average of past consumption.

An application of Ito's lemma on $s_i(t) = \frac{C_i(t) - X_i(t)}{C_i(t)}$ yields

$$
ds_i(t) = \left(\mu_{C_i}(t) - \alpha - \sigma_{C_i}^\top \sigma_{C_i} \right) \left(\frac{\mu_{C_i}(t) - \sigma_{C_i}^\top \sigma_{C_i}}{\mu_{C_i}(t) - \alpha - \sigma_{C_i}^\top \sigma_{C_i}} - s_i(t) \right)
$$
$$
+ \left(1 - s_i(t) \right) \sigma_{C_i}^\top dB(t). \tag{2.9}
$$

The surplus consumption ratio is a mean reverting process, locally perfectly correlated with output shocks to good i. However, nothing prevents the process from turning negative. To bound the surplus consumptions away from zero, I use the representation in (2.6). The functional form of the sensitivity function, λ_i, differs from [Campbell & Cochrane 1999] and follows [Aydemir 2008]. The process is guaranteed to stay within the boundaries of $[0, 1]$ for a large set of parameters values.[12]

Define the habit adjusted consumption as Q_H and Q_F with

$$
Q_H(t) = C_H(t) s_H(t) \tag{2.10}
$$
$$
Q_F(t) = C_F(t) s_F(t). \tag{2.11}
$$

An application of Ito's lemma gives the process followed by the habit adjusted consumption and the composite good

$$
dQ_i(t) = Q_i(t) \left(\mu_{Q_i}(t)(t)dt + \sigma_{Q_i}(t)^\top dB(t) \right) \tag{2.12}
$$
$$
dZ(t) = Z(t) \left(\mu_Z(t)dt + \sigma_Z(t)^\top dB(t) \right) \tag{2.13}
$$

where

$$
\mu_{Q_i}(t) = \mu_{C_i}(t) + \phi_i \left(\frac{\overline{s_i}}{s_i(t)} - 1 \right) + \lambda_i \left(s_i(t) \right) \sigma_{C_i}^\top \sigma_{C_i} \tag{2.14}
$$
$$
\sigma_{Q_i}(t) = \left(1 + \lambda_i \left(s_i(t) \right) \right) \sigma_{C_i} \tag{2.15}
$$

and where

$$
\mu_z(t) = s_\beta(t)\mu_{Q_i}(t) + \left(1 - s_\beta(t) \right) \mu_{Q_i}(t)
$$
$$
- \frac{1}{2} s_\beta(t) \left(1 - s_\beta(t) \right) \left(1 - \beta \right)
$$
$$
\left(\sigma_{Q_H}(t)^\top \sigma_{Q_H}(t) + \sigma_{Q_F}(t)^\top \sigma_{Q_F}(t) - 2\sigma_{Q_H}(t)^\top \sigma_{Q_F}(t) \right) \tag{2.16}
$$
$$
\sigma_Z(t) = s_\beta(t)\sigma_{Q_H}(t) + \left(1 - s_\beta(t) \right) \sigma_{Q_F}(t) \tag{2.17}
$$

and where

$$
s_\beta = \frac{Q_H(t)^\beta}{Q_H(t)^\beta + Q_F(t)^\beta}. \tag{2.18}
$$

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[12]To guarantee that s stays within the boundaries one requires that $a > 1$ and $b > 1$ with $a = \frac{2\phi_i \overline{s}_i^2}{\sigma_{C_i}^T \sigma_{C_i}(1 - \overline{s}_i)}$ and $b = \frac{2\phi_i \overline{s}_i}{\sigma_{C_i}^T \sigma_{C_i}}$ (see [Aydemir 2008] for details).

In times when the surplus consumption ratio is low, then the sensitivity function $\lambda_i\left(s_i(t)\right)$ is high, and so is the volatility of the habit adjusted consumption.

Investment opportunities consists of a bond in zero net supply paying out in the home good, a bond in zero net supply paying out in the foreign good and stock markets in the home country and the foreign country.[13] Stocks are in unit supply and represent claims to each country's respective output stream. The bond price dynamics are given by

$$dB_H(t) = r_H(t)B_H(t)dt \tag{2.19}$$

$$dB_F(t) = r_F(t)B_F(t)dt. \tag{2.20}$$

with $B_H = 1$ and $B_F = 1$.

The real exchange rate follows

$$de(t) = e(t)\left(\mu_e(t)dt + \sigma_e(t)^\top dB(t)\right). \tag{2.21}$$

Stock price dynamics measured in terms of the home good are given by

$$dS_H(t) + C_H(t)dt = S_H(t)\left(\mu_H(t)dt + \sigma_H(t)^\top dB(t)\right) \tag{2.22}$$

$$dS_F(t) + e(t)C_F(t)dt = S_F(t)\left(\mu_F(t)dt + \sigma_F(t)^\top dB(t)\right). \tag{2.23}$$

Coefficients for bond prices, stock priceses and the real exchange rate are determined in equilibrium.

The equilibrium state price density process, $\xi(t)$, follows

$$d\xi(t) = \xi(t)\left(-r_H(t)dt - \theta(t)^\top dB(t)\right) \tag{2.24}$$

with $\xi(0) = 1$ and where $\theta(t)$ denotes the market price of risk given by

$$\theta(t) = \sigma(t)^{-1}\left(\mu(t) - r_H(t)I\right) \tag{2.25}$$

where σ is a 4×4 matrix containing the stock price diffusion coefficients, μ is a vector of drift rates and I is a vector of ones.

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[13]One of the two bonds are redundant since we have two sources of uncertainty and four assets. However, to address the UIP I calculate both the home and the foreign risk free rates.

2.2.2 Equilibrium

To derive equilibrium, I use standard martingale techniques (see [Cox & Huang 1989], [Karatzas *et al.* 1990]). In the following I will take the view of the home country and measure all quantities in terms of the home country good. Equilibrium is characterized by a price system $(r_H, r_F, \mu_H, \mu_F, \mu_e, \sigma_H, \sigma_F, \sigma_e)$ such that the consumption profile is optimal and all markets clear for $t \in [0, T]$. All proofs are relegated to Appendix A.1.

The first proposition characterizes the equilibrium risk free rate in the home country, the foreign country and the market price of risk.

Proposition 1. *The equilibrium risk free rate in country $i = H, F$ is given by*

$$
\begin{aligned}
r_i(t) \;=\; & \rho + (\gamma + \beta - 1)\,\mu_Z(t) + (1 - \beta)\,\mu_{Q_i}(t) \\
& - \frac{1}{2}\,(\beta + \gamma)\,(\gamma + \beta - 1)\,\sigma_Z(t)^\top \sigma_Z(t) \\
& - \frac{1}{2}\,(\beta - 1)\,(\beta - 2)\,\sigma_{Q_i}(t)^\top \sigma_{Q_i}(t) \\
& + (\gamma + \beta - 1)\,(\beta - 1)\,\sigma_Z(t)^\top \sigma_{Q_i}(t).
\end{aligned} \tag{2.26}
$$

The equilibrium market price of risk is given by

$$
\theta(t) = (\gamma + \beta - 1)\,\sigma_Z(t) + (1 - \beta)\,\sigma_{Q_H}(t). \tag{2.27}
$$

From (2.27) we see that the market price of risk depends on the volatility of the composite good, Z, and the volatility of the habit adjusted consumption of the numeraire good, Q_H. When the elasticity of substitution is high, the market price of risk is mainly driven by the composite good. The next two remarks illustrates two extremes, one in which the risk free rate and the market price of risk only depends on the composite good and one where the risk free rate and the market price of risk are solely driven by the numeraire good.

Remark 1. *When the home good and the foreign good are perfect substitutes, the risk free rates take the form*

$$
r_i(t) \;=\; \rho + \gamma \mu_Z(t) - \frac{1}{2}\gamma\,(\gamma + 1)\,\sigma_Z(t)^\top \sigma_Z(t). \tag{2.28}
$$

The market price of risk is

$$
\theta(t) = \gamma \sigma_Z(t). \tag{2.29}
$$

Remark 2. *When the elasticity of substitution, η, is equal to $\frac{1}{\gamma}$, the utility function is separable in the two goods and the risk free rates are*

$$
r_i(t) \;=\; \rho + \gamma \mu_{Q_i}(t) - \frac{1}{2}\gamma\,(\gamma + 1)\,\sigma_{Q_i}(t)^\top \sigma_{Q_i}(t). \tag{2.30}
$$

The market price of risk is

$$
\theta(t) = \gamma \sigma_{Q_H}(t). \tag{2.31}
$$

Remark 1 and Remark 2 show that we obtain the standard power utility case over the habit adjusted consumption for certain values of the elasticity of substitution. In the general case of Proposition 1 the interest rate depends on the time preference, the intertemporal smoothing motive and the precautionary savings motive. The intertemporal smoothing motive for the risk free rate of country i depends on the expected growth of the composite good Z and the growth of the habit adjusted consumption of good i. Similarly, the precautionary savings motive for the risk free rate of country i depends on the variance of the habit adjusted consumption of the good of country i, the variance of the composite good Z and the covariance between the habit adjusted consumption of good i and the composite good Z. The intertemporal smoothing motive works in the usual way; in times when the surplus consumption ratio is low, the expected growth of the habit adjusted consumption is high, and the representative agent's demand for bonds is low. The precautionary savings motive is high when surplus consumption ratios are low because the effective risk aversion is high. When utility is non separable over the home good and the foreign good, the risk free rate in the home (foreign) country depends not only the home (foreign) good surplus consumption ratio, but also on the foreign (home) surplus consumption ratio. In times when the surplus consumption ratio is low in the home country compared to the foreign country, the variance of the habit adjusted consumption of the home good and the covariance of the home good with the composite good are both high, and results in a higher precautionary savings motive in the home good than for the foreign good.

The interest differential is given by

$$
\begin{aligned}
r_H(t) - r_F(t) \quad = \quad & (1 - \beta)\left(\mu_{Q_H}(t) - \mu_{Q_F}(t)\right) \\
& - \frac{1}{2}(\beta - 1)(\beta - 2)\left(\sigma_{Q_H}(t)^\top \sigma_{Q_H}(t) - \sigma_{Q_F}(t)^\top \sigma_{Q_F}(t)\right) \\
& + (\gamma + \beta - 1)(\beta - 1)\sigma_Z(t)^\top \left(\sigma_{Q_H}(t) - \sigma_{Q_F}(t)\right).
\end{aligned}
\tag{2.32}
$$

The difference in the intertemporal smoothing motive depends on the elasticity of substitution. The risk aversion and the drift of the composite good do not enter. The difference between the precautionary savings motive depends on the risk aversion via the covariance with the composite good and the elasticity of substitution.

Remark 3. *If expected consumption growth in both countries is constant and the habit formation is turned off, then the interest rate differential is*

$$
\begin{aligned}
r_H(t) - r_F(t) \quad = \quad & \bar{r} + (\gamma + \beta - 1)(\beta - 1) \\
& \left(s_\beta(t)\sigma_{C_H} + (1 - s_\beta(t))\sigma_{C_F}\right)^\top \left(\sigma_{C_H} - \sigma_{C_F}\right)
\end{aligned}
\tag{2.33}
$$

where

$$
s_\beta(t) \quad = \quad \frac{C_H(t)^\beta}{C_H(t)^\beta + C_F(t)^\beta}
\tag{2.34}
$$

and where

$$\bar{r} = (1 - \beta)) \left(\mu_{C_H} - \mu_{C_F} \right) - \frac{1}{2} \left(\beta - 1 \right) \left(\beta - 2 \right) \left(\sigma_{C_H}^\top \sigma_{C_H} - \sigma_{C_F}^\top \sigma_{C_F} \right) \qquad (2.35)$$

is a constant.

In the special case of Remark 3 the interest differential will increase when there is a positive shock to the home good and decrease when there is a positive shock to the foreign good given that $\gamma + \beta > 1$. The reason for this is that the home good has a higher covariance with the composite good after a positive shock. The only variation in the interest differential stems from the time varying covariance between the countries' goods and the composite good. In the case with habit formation, the variation is no longer only driven by the differences in the precautionary savings motives but also by the difference in the intertemporal smoothing motive. These two move in opposite directions because of the mean reversion in the surplus consumption ratios. In times when the surplus consumption ratio is low, the expected growth of the habit adjusted consumption will be high. This pushes the interest rate up due to the increased incentive to borrow. The difference in the precautionary savings motive will behave in a similar way as in Remark 3. The reason for the increased comovement is no longer an increase in s_β, but is due to higher volatility of the habit adjusted consumption.

The next proposition characterizes the equilibrium real exchange rate and the dynamics of the real exchange rate.

Proposition 2. *The equilibrium real exchange rate is*

$$e(t) = \left(\frac{C_H(t)}{C_F(t)} \right)^{1-\beta} \left(\frac{s_H(t)}{s_F(t)} \right)^{1-\beta} . \qquad (2.36)$$

The real exchange rate follows

$$de(t) = e(t) \left(\mu_e(t) dt + \sigma_e(t)^\top dB(t) \right) \qquad (2.37)$$

where

$$\begin{aligned} \mu_e(t) \;\; &= \;\; (1 - \beta) \left(\mu_{Q_H}(t) - \mu_{Q_F}(t) + \sigma_{Q_F}(t)^\top \left(\sigma_{Q_F}(t) - \sigma_{Q_H}(t) \right) \right) \\ &\quad + \frac{1}{2} \beta \left(\beta - 1 \right) \left(\sigma_{Q_H}(t) - \sigma_{Q_F}(t) \right)^\top \left(\sigma_{Q_H}(t) - \sigma_{Q_F}(t) \right) \end{aligned} \qquad (2.38)$$

and where

$$\sigma_e(t) = (1 - \beta) \left(\sigma_{Q_H}(t) - \sigma_{Q_F}(t) \right) . \qquad (2.39)$$

Remark 4. *The expected growth of the real exchange rate can be decomposed into the interest rate differential and a risk premium. The risk premium is $\theta(t)^\top \sigma_e(t)$.*

Remark 5. *If expected consumption growth in both countries is constant and the habit formation is turned off, the exchange rate is*

$$e(t) = \left(\frac{C_H(t)}{C_F(t)} \right)^{1-\beta}. \tag{2.40}$$

The expected growth rate $\mu_e(t) = \overline{\mu_e}$ is constant. The diffusion coefficients are also constant and given by

$$\sigma_e = (1 - \beta)(\sigma_{C_H} - \sigma_{C_F}). \tag{2.41}$$

From Remark 5 we see that we either need a high volatility of the consumption of the home and the foreign good or a low elasticity of substitution to match the observed real exchange rate volatility. Without the habit formation the real exchange rate follows a random walk and there is no predictability. As we can see from Proposition 2, the exchange rate volatility depends on the surplus consumption ratios. The surplus consumption ratios are much more volatile than consumption and thus help matching the real exchange rate volatility. Note that the volatility of the real exchange rate is independent of the risk aversion. From Proposition 2 we have that the market price of risk in a model with deep habits is stochastic, and we therefore obtain a time varying exchange rate risk premium. This is necessary to match the UIP puzzle. In parameterizations in which the risk premium and the risk free rate differential are negatively correlated and the volatility of the former is higher, I can match the UIP puzzle. Note that in the case of Remark 3 and Remark 5 the interest rate differential is stochastic while the expected exchange rate growth in constant. This implies that changes in the interest differential are perfectly offset by changes in the risk premium. This case results in a slope coefficient of zero for an UIP regression.

The next proposition characterizes the equilibrium stock price diffusion matrix. I follow [Gallmeyer 2002] and apply the Clark-Ocone formula from Malliavin Calculus to obtain explicit formulas for the stock price diffusion coefficients.

Proposition 3. *The equilibrium stock return diffusion coefficients are given by*

$$\sigma_H(t) = \theta(t) + \frac{E_t \left[\int_t^T \xi(s) C_H(s) \left(D_t \ln \xi(s) + D_t \ln C_H(s) \right) ds \right]}{E_t \left[\int_t^T \xi(s) C_H(s) ds \right]} \tag{2.42}$$

and

$$\sigma_F(t) = \theta(t) + \frac{E_t \left[\int_t^T \xi(s) e(s) C_F(s) \left(D_t \ln \xi(s) + D_t \ln e(s) + D_t \ln C_F(s) \right) ds \right]}{E_t \left[\int_t^T \xi(s) C_F(s) ds \right]}. \tag{2.43}$$

In Proposition 3, D_t refers to the Malliavin derivative. The stock price diffusion matrix for the home country depends on the market price of risk and a ratio of expectations involving Malliavin derivatives. Consider the Malliavin derivative in the integrand

$$\xi(s)C_H(s)\left(D_t \ln \xi(s) + D_t \ln C_H(s)\right). \tag{2.44}$$

This Malliavin derivative captures the response to a small change at time t to the state price density and the output of the home good at future time $s > t$. For the foreign stock price diffusion coefficient we must also consider the response of the real exchange rate at time s to a change at times $t < s$. Given the risk-free rate and the market price of risk from Proposition 1 and the diffusion coefficients from Proposition 3 we can calculate the drift rates of the stock price processes as follows

$$\mu(t) = r_H(t)I + \sigma(t)\theta(t). \tag{2.45}$$

This completes the description of the stock price process in the home and the foreign market.

2.3 Analysis and Numerical Results

In this section I numerically study the properties of the equilibrium. The base case scenario is an economy with a time horizon of 50 years. I calibrate the model to US and UK data.[14] Table 2.1 summarizes the model parameters. For the risk aversion I use a coefficient of five. This is higher than in the standard external habit literature where the typical value is two, but equal to the value used by [van Binsbergen 2007]. I set the steady state value of the habit level to 0.15 and the persistence of the habit level to 0.05. Time discount factor is 0.135 and chosen to match the level of the risk free rate. For the parameters of the consumption processes I calibrate my model to the average of the US and UK GDP data. The output processes have an expected growth rate of 2.4% and a standard deviation of 2.3%. For the expected growth of consumption I use a speed of mean reversion of 0.1 and a standard deviation of 0.00115. The persistence of the expected growth is less than what is typically used in the long run literature. I assume that the correlation between the expected consumption growth of the US and UK goods is zero. This contrast with Bansal and Shaliastovich (2007) and with Colcaito and Croce (2007) who use a nearly perfect correlation between expected consumption growth in the two countries.

Table 2.2 summarize the key moments in the baseline calibration. The model excess returns are 5.1% in the US and 5.7% in the UK.[15] The corresponding values

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[14]The data is from Datastream and covers the period 1970-2008. For the financial data I use the total return index from Datastream.

[15]The returns are measured in US dollar.

Table 2.1: **Model Parameters — Baseline Calibration.** The table summarizes the model parameters for the baseline calibration. The calibration is symmetric in terms of the home good and the foreign good.

Parameter	Value
Risk aversion	5
Elasticity of substitution	1.33
Time preference	0.135
Steady state habit level	0.15
Speed of mean reversion habit growth	0.05
Average consumption growth	0.024
Standard error of consumption growth	0.023
Cross-country correlation of consumption growth	0.24
Speed of mean reversion for consumption growth	0.1
Volatility of expected consumption growth	0.00115
Correlation between US and UK expected consumption growth	0

in the data are 5.9% and 6.9% respectively. The model implied risk free rate is 1.6% in both countries compared to 1.3% for US and 1.7% for the UK data. As we can see, the model is able to resolve the risk free rate puzzle. The standard deviation of the risk free rate is somewhat high compared to the data. This feature is typical for habit formation models. The model implied correlation between US and UK returns is 0.78 compared to 0.56 in the data. The standard deviation of the real exchange rate is 0.142 compared to 0.104 in the data. The autocorrelation of the model implied exchange rate is close to what we see in the data.

2.3.1 The volatility of the real exchange rate

Figure 2.1 shows the volatility of the real exchange rate in the baseline calibration as we change the surplus consumption ratio in the home good and the foreign good. There is a literature documenting stochastic volatility in exchange rates,[16] and we can see that the model is able to generate this feature endogenously. To understand the dynamics of the exchange rate volatility consider the diffusion coefficients

$$\sigma_e(t) = (1 - \beta) \left((1 + \lambda \left(s_H(t) \right)) \sigma_{C_H} - (1 + \lambda \left(s_F(t) \right)) \sigma_{C_F} \right). \qquad (2.46)$$

From the above expression we see that the variance of the exchange rate depends

---

[16]see [Poon & Granger 2003] for a review.

Figure 2.1: **Standard Deviation of the Real Exchange Rate.** The figure shows the standard deviation of the real exchange rate as a function of the surplus consumption ratio in the home good and the foreign good. The economy is parameterized as in the baseline calibration (see Table 2.1).
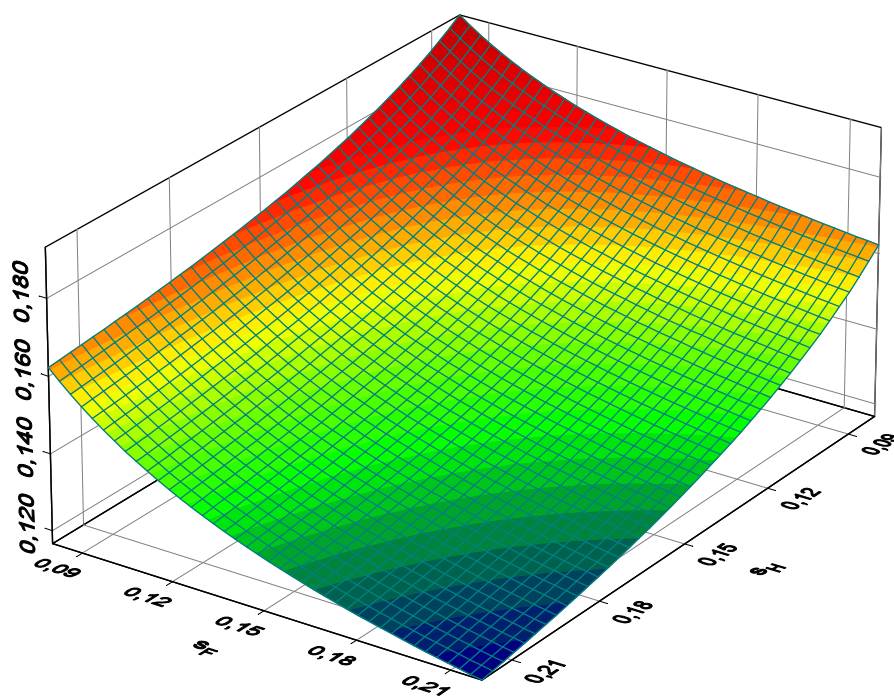
Table 2.2: **Key Moments — Baseline Calibration.** The table shows the calibrated moments and the corresponding values in the data.

	Data		Model	
	US	UK	US	UK
Excess return	0.059	0.069	0.051	0.057
Average risk free rate	0.013	0.017	0.016	0.016
Standard deviation market	0.154	0.197	0.085	0.144
Standard deviation risk free rate	0.011	0.021	0.034	0.034
Correlation US and UK market	0.560		0.780	
Standard deviation real exchange rate	0.104		0.142	
Autocorrelation real exchange rate growth	0.091		0.123	

on the volatility of the home and the foreign good, the elasticity of substitution and the sensitivity function for the habit level in the home and the foreign good. Consider the case when the current consumption of the home good is close to the habit level. In this case, the representative agent is very reluctant to scale back on consumption of the home good. This makes the effective elasticity of substitution between the home good and the foreign good volatile. Small changes to output of any of the two goods results in large changes to the relative price (exchange rate).

Table 2.3 shows a GARCH(1,1) model for the dollar-pound monthly exchange rate series. To compare this to my model, I simulate 5000 months of exchange rate data. As we can see from the table, both the data and the model produce highly persistent exchange rate volatility. The persistence is captured by the sum of the ARCH(1) and the GARCH(1) term. For the data this is 0.975 and for the model it is 0.993.

To shed further light on the relation between the surplus consumption ratios and the volatility of the real exchange rate I back out the surplus consumption ratios from the data. I assume that both the US and UK surplus consumption ratios are in their steady states at the beginning of the sample (1970 Q1). I then use the dynamics of the surplus consumption ratios and the realized shocks to GDP to back out the values of the surplus consumption ratios. Figure 2.2 shows the implied surplus consumption ratios and the time series of standard deviations of the real exchange rate estimated from the GARCH(1,1). As we can see from the figure the low volatility in the beginning and end of the sample are accompanied by high surplus consumption ratios and the high volatility in the early eighties and mid nineties are accompanied by low surplus consumption ratios as the model predicts. Figure 2.3 plots the model implied real exchange rate and the realized real exchange rate volatilities. As we can see from the plot, the model implied and the realized

Table 2.3: **GARCH(1,1).** The table shows the coefficients and the t-values for the GARCH(1,1) estimation. The data column is estimated using monthly real exchange rate data. The model column is estimated on 5000 months of simulated data using parameters from the baseline calibration.

Variable	Data		Model	
	coef	t-value	coef	t-value
C	3.27E-05	2.9413	8.77E-06	3.0937
ARCH(1)	0.128412	4.0224	0.024146	6.0023
GARCH(1)	0.846658	23.6986	0.968866	187.4996

exchange rate volatility are similar except for the late eighties, where the model underestimates the real exchange rate volatility. The correlation between the two series is 0.45. Note that only real GDP is used in order to calculate the real exchange rate volatility.

2.3.2 The uncovered interest rate parity puzzle

Figure 2.4 shows the interest rates, the interest rate differential and the excess return on the real exchange rate. For the baseline calibration the interest rate is high in times when the surplus consumption ratios are low. This contrasts with [Verdelhan 2008] where pro-cyclical interest rates are necessary for matching the UIP puzzle. The precautionary savings motive reacts less than the intertemporal smoothing motive to shocks to the surplus consumption ratios. The interest differential is increasing in foreign surplus consumption ratio and decreasing in the home surplus consumption ratio. The interest rate differential is therefore pro-cyclical in the difference between the home and the foreign surplus consumption ratio. The counter-cyclical interest rate combined with pro-cyclical interest rate differential is driven by the difference between the elasticity of substitution and the risk aversion. While the risk aversion is important for the intertemporal smoothing motive for the interest rate, it does not affect the difference in the intertemporal smoothing motive between the home country and the foreign country. The excess return on the real exchange rate is decreasing in the home surplus consumption ratio and increasing in the foreign surplus consumption ratio. In times when the home good is close the habit level the representative agent is very risk averse to shocks to the home good. The exchange rate is highly correlated with shocks to the home good, and consequently the representative agent requires a large risk premium for holding exchange rate risk. Comparing the figure of the interest differential and the excess return

Figure 2.2: **Implied Surplus Consumption Ratios and the Realized Real Exchange Rate.** The figure shows the implied surplus consumption ratios calculated using GDP data and the realized real exchange rate volatility estimated with GARCH(1,1).
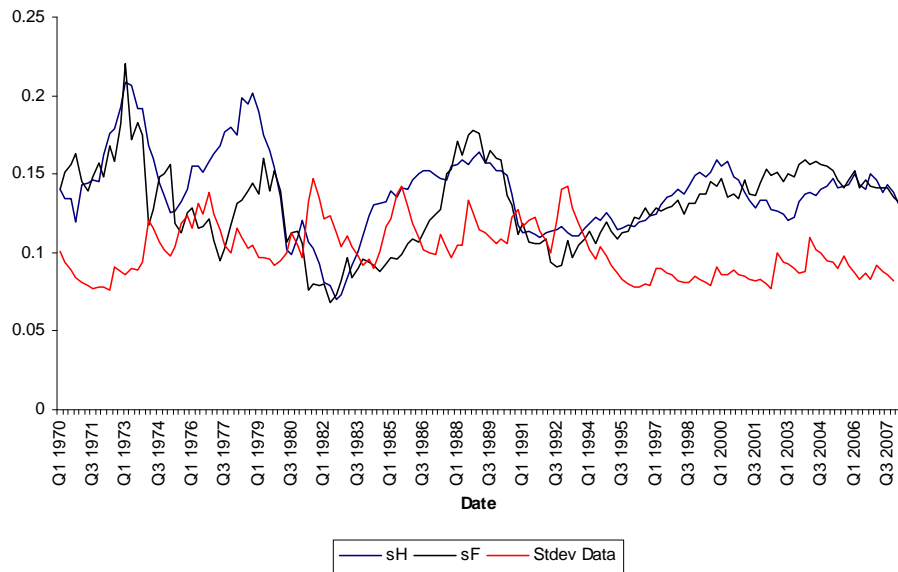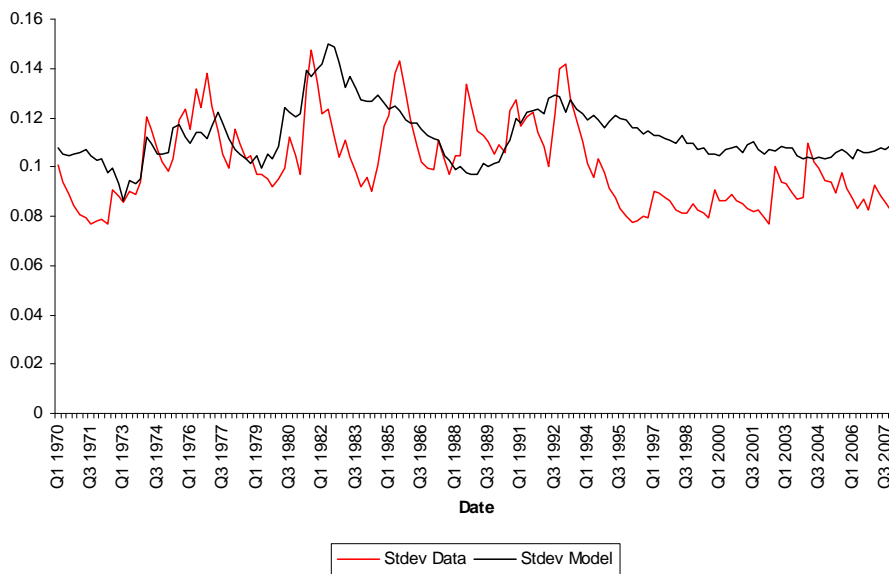
Figure 2.3: **Model Implied and Realized Real Exchange Rate Volatility.** The figure shows the model implied and the realized real exchange rate volatility. The model implied real exchange rate volatility is estimated using the formula for the real exchange rate volatility and the estimated surplus consumption ratios.

on the exchange rate, we see that they move in opposite directions. The negative correlation between the interest differential and the excess return on real exchange rate is necessary to match the failure of the UIP. To reproduce the empirical stylized fact that high interest rate countries exhibit appreciating currencies, the volatility of the excess real exchange rate return must be higher than the interest rate differential. From the figure we can see that the baseline calibration is able to generate this feature. The range of the interest differential is small compared to the range of the exchange rate premium.

2.4 Extension to Heterogeneous Agents

In this section I extend the basic model to include heterogeneous agents. The home and the foreign representative agent have home bias for their own domestically produced consumption good. The agents solve

$$\max_{C_H^j, C_F^j, \pi^j, \varphi^j} E\left[\int_0^T u_j\left(C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t), t\right) dt\right] \quad (2.47)$$

where

$$u_j\left(C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t), t\right) = \frac{e^{-\rho t}}{1-\gamma} Z_j\left(C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t)\right)^{1-\gamma} \quad (2.48)$$

and

$$Z_j\left(C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t)\right) = \left(\begin{array}{c} \lambda_j^{1-\beta}\left(C_H^j(t) - X_H^j(t)\right)^{\beta} \\ (1-\lambda_j)^{1-\beta}\left(C_F^j(t) - X_F^j(t)\right)^{\beta} \end{array}\right)^{\frac{1}{\beta}} \quad (2.49)$$

subject to

$$dW^j(t) = \varphi_H^j(t)\frac{dB_H(t)}{B_H(t)} + \varphi_F^j(t)\frac{dB_F(t)}{B_F(t)} + \pi_H^j(t)\frac{dS_H(t) + C_H(t)dt}{S_H(t)} \quad (2.50)$$

$$+\pi_F^j(t)\frac{dS_F(t) + P_F(t)C_F(t)dt}{S_F(t)} - P(T)^{\top}C^j(t)dt \quad (2.51)$$

for $j = H, F$, where $\pi^j(t) = \left(\pi_H^j(t), \pi_F^j(t)\right)$ is the vector of amounts held in the stocks by agent j. $W^j(0) > 0$ with $W^j(0) = \pi^j(0)^{\top}1$, i.e, the agents are endowed with initial shares in the stocks. $P(t) = (P_H(t), P_F(t))^{\top}$ is the vector of the country specific good prices. Note that I use the home country as numeraire good so that $P_H(t) = 1$ for all t. φ_i^j denotes the amount invested by agent j in the bond of country i. φ_j denotes the vector of bond positions for agent j. There is home bias if $\lambda_H > \lambda_F$. Next I define equilibrium.

Definition 1. *Equilibrium is a collection of allocations* $\left(C_H^j, C_F^j, \varphi_H^j, \varphi_F^j, \pi_H^j, \pi_F^j\right)$ *for* $j = H, F$, *and a price system* $\left(\mu, \mu_{P_F}, \sigma, \sigma_{P_F}, r_H, r_F\right)$, *such that*

Figure 2.4: **The Risk Free Rates, Interest Rate Differential and the Excess Return on the Real Exchange Rate.** The figure shows the risk free rate in the home country, the foreign country, the interest differential and the excess return on the real exchange rate as a function of the surplus consumption ratio in the home good and the foreign good.
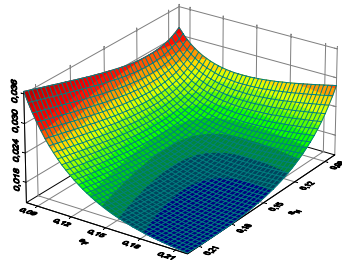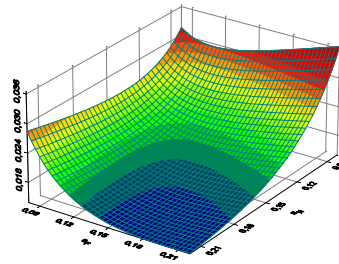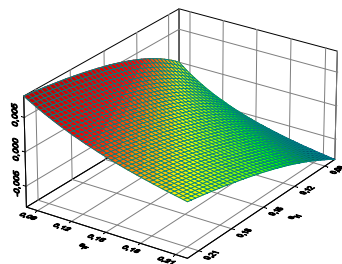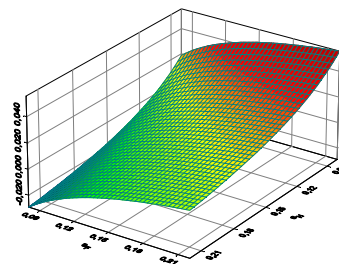
$\left(C_H^j, C_F^j, \varphi_H^j, \varphi_F^j, \pi_H^j, \pi_F^j\right)$ *are optimal solutions to agent j's optimization problem and good and financial markets clear*

$$C_H^H(t) + C_H^F(t) = C_H(t) \qquad (2.52)$$

$$C_F^H(t) + C_F^F(t) = C_F(t) \qquad (2.53)$$

$$\pi_H^H(t) + \pi_H^F(t) = S_H(t) \qquad (2.54)$$

$$\pi_F^H(t) + \pi_F^F(t) = S_F(t) \qquad (2.55)$$

$$\varphi_H^H(t) + \varphi_H^F(t) = 0 \qquad (2.56)$$

$$\varphi_F^H(t) + \varphi_F^F(t) = 0 \qquad (2.57)$$

for $t \in [0, T]$.

Under complete markets[17] we can solve the corresponding social planner problem state by state and time by time

$$U\left(C_H(t), C_F(t), X_H(t), X_F(t)\right) = \max_{\substack{C_H^H(t), C_H^F(t) \\ C_F^H(t), C_F^F(t)}}$$

$$\left\{ \begin{array}{l} a u_H\left(C_H^H(t), C_F^H(t), X_H^H(t), X_F^H(t), t\right) + \\ (1-a) u_F\left(C_F^H(t), C_F^F(t), X_H^F(t), X_F^F(t), t\right) \end{array} \right\}$$

$$s.t$$

$$C_H^H(t) + C_H^F(t) = C_H(t)$$

$$C_F^H(t) + C_F^F(t) = C_F(t)$$

where a is the weight in the social planner problem with a one-to-one mapping with initial wealth distribution. By the implicit function theorem it can be shown that the habit adjusted consumption of the two goods for the home country representative agent are given by

$$Q_H^H(t) = C_H^H(t) - X_H^H(t)$$

$$= g_H^H\left(C_H(t) - X_H(t), C_F(t) - X_F(t)\right) \qquad (2.58)$$

$$Q_F^H(t) = C_F^H(t) - X_F^H(t)$$

$$= g_F^H\left(C_H(t) - X_H(t), C_F(t) - X_F(t)\right). \qquad (2.59)$$

Note that the utility function of the representative agent only depends on aggregate habits for each good. We can therefore model the surplus consumption ratios as in

---

[17]There are four securities and two Brownian motions, consequently the market is potentially complete. However, as market completeness depends on the endogenous stock price diffusion coefficients the market completeness must be verified after calculating the stock price diffusion matrix.

the case with only a representative agent. To determine the optimal consumption profiles I need to make assumptions about the individual habit levels. I set the habit levels such that

$$X_H^H(t) = w_H X_H(t) \qquad (2.60)$$

$$X_H^F(t) = (1 - w_H) X_H(t) \qquad (2.61)$$

$$X_F^H(t) = w_F X_F(t) \qquad (2.62)$$

$$X_F^F(t) = (1 - w_F) X_F(t) \qquad (2.63)$$

for some constants $w_H, w_F \in (0, 1)$. The optimal consumption allocations are given by

$$C_H^H(t) = Q_H^H(t) + w_H X_H(t) \qquad (2.64)$$

$$C_H^F(t) = C_H(t) - C_H^H(t) \qquad (2.65)$$

$$C_H^F(t) = Q_H^F(t) + w_F X_F(t) \qquad (2.66)$$

$$C_F^F(t) = C_H(t) - C_F^H(t) \qquad (2.67)$$

Note that $Q_j^i(t)$ only depends on the aggregate habit level. From the above we can see that the consumption for the home agent in the home (foreign) good is increasing (decreasing) in w_H. This introduces another potential channel for home bias. w_i works as a level factor for the subsistence point. It is important to note that w_i does not impact the equilibrium state price density or the equilibrium real exchange rate, as the marginal utility only depends on the habit adjusted consumption. However, this form of home bias differs from the home bias introduced by λ_i.

In a complete market setup the marginal utilities of the two agents are co-linear. We can therefore use the marginal utility of the home country agent evaluated at optimal consumption as the state price density. The equilibrium relative price of the foreign good measured in terms of the home good is

$$
\begin{aligned}
P_F(t) &= \frac{\frac{\partial u_i\left(C_H^H(t), C_F^H(t), X_H^H(t), X_F^H(t), t\right)}{\partial C_F^H(t)}}{\frac{\partial u_i\left(C_H^H(t), C_F^H(t), X_H^H(t), X_F^H(t), t\right)}{\partial C_H^H(t)}} \\
&= \left(\frac{1 - \lambda_j}{\lambda_j}\right)^{1-\beta} \left(\frac{g_H^H\left(C_H(t) - X_H(t), C_F(t) - X_F(t)\right)}{g_F^H\left(C_H(t) - X_H(t), C_F(t) - X_F(t)\right)}\right)^{1-\beta} \\
&= \left(\frac{1 - \lambda_j}{\lambda_j}\right)^{1-\beta} \left(\frac{Q_H^H(t)}{Q_F^H(t)}\right)^{1-\beta}. \qquad (2.68)
\end{aligned}
$$

An application of Ito's lemma yields

$$\frac{dP_F(t)}{P_F(t)} = \mu_{P_F}(t)dt + \sigma_{P_F}(t)^\top dB(t) \qquad (2.69)$$

where the expressions for $\mu_{P_F}(t)$ and $\sigma_{P_F}(t)$ can be found in Appendix A.2. To derive the real exchange rate I need to define the price index in the home and the foreign country. I use the consumption based price index consistent with constant elasticity of substitution utility function (see [Obstfeld & Rogoff 1996])

$$P^H(t) = \left(\lambda_H + (1-\lambda_H)\,P_F(t)^{\beta}\right)^{\frac{1}{\beta}} \tag{2.70}$$

$$P^F(t) = \left(\lambda_F + (1-\lambda_F)\,P_F(t)^{\beta}\right)^{\frac{1}{\beta}}. \tag{2.71}$$

The next proposition describes the process followed by the price indexes.[18]

Proposition 4. *If equilibrium exists, then the price index in the home country and the foreign country follow*

$$dP^H(t) = P^H(t)\left(\mu_{PH}(t)dt + \sigma_{PH}(t)^{\top}dB(t)\right) \tag{2.72}$$

$$dP^F(t) = P^F(t)\left(\mu_{PF}(t)dt + \sigma_{PF}(t)^{\top}dB(t)\right) \tag{2.73}$$

where

$$\mu_{PH}(t) = (1-\lambda_H)\left(\frac{P_F(t)}{P^H(t)}\right)\mu_{P_F}(t) \tag{2.74}$$

$$+\frac{1}{2}\,(1-\beta)\,(1-\lambda_H)\left(\frac{P_F(t)}{P^H(t)}\right)\left(\left(\frac{P_F(t)}{P^H(t)}\right)(1-\lambda_H)-1\right)$$

$$\sigma_{P_F}(t)^{\top}\sigma_{P_F}(t)$$

$$\mu_{PF}(t) = (1-\lambda_F)\left(\frac{P_F(t)}{P^F(t)}\right)\mu_{P_F}(t) \tag{2.75}$$

$$+\frac{1}{2}\,(1-\beta)\,(1-\lambda_F)\left(\frac{P_F(t)}{P^F(t)}\right)\left(\left(\frac{P_F(t)}{P^F(t)}\right)(1-\lambda_F)-1\right)$$

$$\sigma_{P_F}(t)^{\top}\sigma_{P_F}(t)$$

and

$$\sigma_{PH}(t) = (1-\lambda_H)\left(\frac{P_F(t)}{P^H(t)}\right)\sigma_{P_F}(t) \tag{2.76}$$

$$\sigma_{PF}(t) = (1-\lambda_H)\left(\frac{P_F(t)}{P^F(t)}\right)\sigma_{P_F}(t). \tag{2.77}$$

The real exchange rate is given by the ratio of the foreign consumption price

---

[18]I use subscripts for prices that are good related and superscripts for countries price indexes.

index to the home consumption price index

$$
\begin{aligned}
e(t) &= \frac{P^F(t)}{P^H(t)} \\
&= \left(\frac{\lambda_F + (1 - \lambda_F) P_F(t)^\beta}{\lambda_H + (1 - \lambda_H) P_F(t)^\beta} \right)^{\frac{1}{\beta}}.
\end{aligned}
\tag{2.78}
$$

The next proposition describes the process followed by the real exchange rate.

Proposition 5. *If equilibrium exists, then the real exchange rate follows*

$$
\frac{de(t)}{e(t)} = \mu_e(t)dt + \sigma_e(t)^\top dB(t)
\tag{2.79}
$$

where

$$
\mu_e(t) = \mu_{PF}(t) - \mu_{PH}(t) - \sigma_{PH}(t)^\top \sigma_e(t)
\tag{2.80}
$$

and

$$
\sigma_e(t) = \sigma_{PF}(t) - \sigma_{PH}(t).
\tag{2.81}
$$

The stock prices and the individual wealth of the home and the foreign agent are[19]

$$
\begin{aligned}
S_H(t) &= E_t \left[\int_t^T \frac{\xi(u)}{\xi(t)} C_H(u) du \right] \tag{2.82} \\
S_F(t) &= E_t \left[\int_t^T \frac{\xi(u)}{\xi(t)} P_F(u) C_F(u) du \right] \tag{2.83} \\
W^H(t) &= E_t \left[\int_t^T \frac{\xi(u)}{\xi(t)} \left(C_H^H(u) + P_F(u) C_F^H(u) \right) du \right] \tag{2.84} \\
W^F(t) &= E_t \left[\int_t^T \frac{\xi(u)}{\xi(t)} \left(C_H^F(u) + P_F(u) C_F^F(u) \right) du \right] \tag{2.85}
\end{aligned}
$$

The stock price diffusion coefficients can be found by application of the Clark-Ocone theorem.

Proposition 6. *If equilibrium exits, then the stock price diffusion coefficients are*

---

[19]The stock prices are valued in terms of the home country good.

given by

$$\sigma_H(t) = \theta(t) + \frac{E_t\left[\int_t^T \xi(u)C_H(u)\left(D_t\ln\xi(u) + D_t\ln C_H(u)\right)\right]}{E_t\left[\int_t^T \xi(u)C_H(u)du\right]} \tag{2.86}$$

$$\sigma_F(t) = \theta(t) + \frac{E_t\left[\int_t^T \xi(u)P_F(u)C_F(u)\left(D_t\ln\xi(u)\right)\right]}{E_t\left[\int_t^T \xi(u)P_F(u)C_F(u)du\right]} +$$

$$\frac{E_t\left[\int_t^T \xi(u)P_F(u)C_F(u)\left(D_t\ln P_F(u) + D_t\ln C_F(u)\right)\right]}{E_t\left[\int_t^T \xi(u)P_F(u)C_F(u)du\right]}. \tag{2.87}$$

The next proposition characterizes the optimal portfolio of agent $j = H, F$.

Proposition 7. *If equilibrium exists, then the optimal portfolio policies are given by*

$$\pi^H(t) = \left(\sigma(t)^\top\right)^{-1}\left(W^H(t)\theta(t) + \frac{\Psi^H(t)}{\xi(t)}\right) \tag{2.88}$$

$$\pi^F(t) = \left(\sigma(t)^\top\right)^{-1}\left(W^F(t)\theta(t) + \frac{\Psi^F(t)}{\xi(t)}\right) \tag{2.89}$$

where

$$\Psi^H(t) = E_t\left[\int_t^T D_t\left(\xi(u)\left(C_H^H(u) + P_F(u)C_F^H(u)\right)\right)du\right] \tag{2.90}$$

$$\Psi^F(t) = E_t\left[\int_t^T D_t\left(\xi(u)\left(C_H^F(u) + P_F(u)C_F^F(u)\right)\right)du\right]. \tag{2.91}$$

2.4.1   Calibration and Analysis

In this section I calibrate the model with heterogeneous agents. Instead of using the home good as numeraire I will use the composite basket corresponding to the home country's price index. Appendix A.6 illustrates how the equilibrium quantities change as I change the numeraire.

Table 2.4 summarizes the model parameters. I set the risk aversion to four. The reason for the lower risk aversion compared to the homogeneous agent case is the volatility induced by the heterogeneity of the agents. The elasticity of substitution between the home good and the foreign good is two. The time preference parameter is set to 0.105 to match the level of the risk-free rate. λ_i and ω_i are calibrated so that $Import_i/GDP_i = 0.15$ for $i = H, F$. This is motivated by [Backus *et al.* 1994]. Note that the interpretation of the parameter is different since the utility function is defined over the habit adjusted consumption. The steady state habit level is 0.15

for both the home good and the foreign good. Similarly, the speed of mean reversion of the surplus consumption ratio is 0.0525 for both goods. The output processes are calibrated as in the homogeneous agent except that the volatility of the expected growth is zero.

Table 2.4: **Model Parameters — Heterogeneous Agent Economy.** The table summarizes the model parameters for the heterogeneous agent economy. The calibration is symmetric in terms of goods and agents.

Parameter	Value
Risk aversion	4
Elasticity of substitution	2
Time preference	0.105
Home Bias in Utility (Lambda)	0.85
Home Bias in Habit (Omega)	0.85
Steady state habit level	0.15
Speed of mean reversion habit growth	0.0525
Average consumption growth	0.024
Standard error of consumption growth	0.025
Cross-country correlation of consumption growth	0.24
Speed of mean reversion for consumption growth	0
Volatility of expected consumption growth	0
Correlation between US and UK expected consumption growth	0

Table 2.5 reports the main characteristics of the asset pricing moments. The model implies excess return is 5.3% for the US and 5.9% for the UK. The risk-free rate is 1.6% in both countries. Note that the volatility of the risk-free rate is lower in the heterogeneous agent economy than in the homogeneous agent economy. The correlation between returns is 0.74 compared to 0.56 in the data. Finally, we see that the standard deviation of the real exchange rate in the model is 11.6% compared to 10.4% in the data. The optimal portfolios are symmetric in the steady state. The home (foreign) agent holds 0.89 (0.11) of the home (foreign) stock market. For US investors the fraction is 0.883 in the data.[20] As we can see the model replicates the home bias in portfolios.

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[20]See [Obstfeld & Rogoff 2000].

Table 2.5: **Key Moments — Heterogeneous Agent Economy.** The table shows the calibrated moments and the corresponding values in the data.

	Data		Model	
	US	UK	US	UK
Excess return	0.059	0.069	0.053	0.059
Average risk free rate	0.013	0.017	0.016	0.016
Standard deviation market	0.154	0.197	0.11	0.155
Standard deviation risk free rate	0.011	0.021	0.02	0.02
Correlation US and UK market	0.560		0.740	
Standard deviation real exchange rate	0.104		0.116	
Autocorrelation real exchange rate growth	0.091		0.123	

2.5 Conclusion

I study the effect of deep habits in a two country-two good pure exchange economy. Deep habits allow for jointly matching the real exchange rate volatility, the equity premium, the risk free rate and the failure of the uncovered interest rate parity. Habit formation makes the effective elasticity of substitution between the home country good and the foreign country good stochastic. In times when the current consumption is close to the habit level, the volatility of effective elasticity of substitution is high and consequently the volatility of the real exchange rate is high. The uncovered interest rate puzzle is reproduced because of the highly volatile exchange rate premium. In an extension of the model I illustrate that when agents have home bias in consumption, then the agents have home bias in the portfolios.

Financial Market Completeness in Multi-Good Economies

Abstract

In this paper, we study financial market completeness and incompleteness in economies with multiple goods. We provide, in the form of a non-linear partial differential equation, sufficient conditions for completeness of the financial market. Completeness requires invertibility of the commodity diffusion matrix, which we assume, and invertibility of a matrix containing non-linear functions of marginal rates of substitutions at one point in time. We also provide, in the form of a linear partial differential equation, sufficient conditions for incompleteness of the financial market. This partial differential equation is linear in marginal utilities. We emphasize that financial market completeness depends on the choice of numeraire. Heterogeneity in taste restores completeness of financial markets when individual preferences exhibit unit elasticity of substitution —i.e., when preferences satisfy our proposition— except when preferences are log-linear.

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Keywords: Financial Market Completeness; Financial Market Incompleteness;

Multi-Good Economies; Non-Separable Utility Functions; Unit Elasticity of Substitution

JEL Classification: G10; G11

3.1 Introduction

In this paper, we investigate the determinants of financial market completeness in potentially complete economies with multiple goods. We use the term *potentially complete* as implying the invertibility of the output diffusion matrix. Our main theorem provides, in the form of a non-linear partial differential equation, sufficient conditions for completeness of the financial market in a setting similar to [Anderson & Raimondo 2008a], i.e., output at the terminal date are lumps and analyticity —a function locally given by a convergent power series— of the primitive functions of the economy. Financial market completeness requires invertibility of the commodity diffusion matrix and invertibility of a matrix containing non-linear functions of marginal rates of substitutions at one point in time. If the partial differential equation does not hold, then the matrix containing non-linear functions of marginal rates of substitutions can be inverted.

In addition, we provide sufficient conditions in the form of a proposition for incompleteness of the financial market with multiple goods in a general setting with non-negative Ito processes. This partial differential equation is linear in marginal utilities. In the two good case, the partial differential equation implies unit elasticity of substitution. A recent work closely related to this paper is [Berrada *et al.* 2007]. They study Pareto efficient no-trade equilibria and also discuss settings with incomplete financial markets. Their Proposition 2, which deals with the implementation of portfolio autarky in no-trade equilibria, includes the two-good version of our proposition.

Importantly, market completeness depends on the choice of numeraire as changing the numeraire implies that the original risk-free asset is non-tradable under the new numeraire. Therefore, changing the numeraire may move an economy from complete to incomplete. Even if the theorem and the proposition are not satisfied under two numeraires, one of the numeraires may imply complete financial markets while the other numeraire may imply incomplete financial markets. When the financial market is incomplete because the linear partial differential equation in marginal utilities is satisfied, numeraire invariance holds.[3]

---

[3]We use the term *numeraire invariance* to mean that the market does not go from complete to incomplete when changing the numeraire.

Financial market completeness matters only if agents are heterogeneous. Our approach, first studying the contingent market economy and then deriving from equilibrium price processes conditions for dynamic completeness, naturally extends to economies with heterogeneous agents as it is always possible to construct a representative agent in complete markets. This is crucial as our theorem helps to avoid financial market incompleteness.

An interesting question is whether heterogeneity can restore completeness of financial markets when individual preferences exhibit unit elasticity of substitution, i.e., preferences satisfy our proposition. As for economies with aggregate agents, risk aversion and beliefs do not matter for financial market completeness and, thus, belief heterogeneity, or heterogeneity in risk aversion, do not matter, either, if agents have homogeneous taste. Therefore, learning is also irrelevant for completeness if agents have homogeneous taste. However, if agents have different taste, then allocations are efficient and financial markets are complete, except with log-linear preferences. This result extends the model of [Cass & Pavlova 2004] with multiple trees and log-linear preferences. They show that allocations are Pareto efficient although the financial market is incomplete.

Our paper is motivated by [Serrat 2001], who employs a non-separable utility function to study portfolio choice in an international setting with traded and non-traded goods. [Kollmann 2006], however, proves that the asset price variance matrix in [Serrat 2001] cannot be inverted. Hence, equity prices are colinear and portfolio policies are indeterminate. The result in [Kollmann 2006] emerges naturally from our proposition. Preferences in the model by [Serrat 2001] are such that a composition of the goods shows unit elasticity of substitution and, thus, financial markets are incomplete.

The outline of the paper is as follows: Section 3.2 presents a stylized example with Cobb-Douglas preferences resulting in colinear stock prices. Section 4.2 describes the economic environment. Section 3.4 contains our main theorem and further results on financial market incompleteness as well as numeraire invariance. Agent heterogeneity is addressed in Section 3.5. Section 4.5 sets forth our conclusions. Appendix B.1 contains various examples, including [Serrat 2001], that lead to incomplete financial markets. Appendix B.2 contains Malliavin derivatives used throughout the paper.

3.2 An Example with Cobb-Douglas Preferences

Consider a representative agent with non-separable Cobb-Douglas utility function

$$U(X_1, X_2) = X_1^\alpha X_2^\beta \qquad (3.1)$$

with $\alpha + \beta < 1$, defined over terminal wealth, X. Assume that output follows Geometric Brownian motion, that is,

$$d\delta_i(t) = \delta_i(t)(\gamma_i dt + \lambda_i dW_i(t)) \qquad (3.2)$$

with $\delta_i(0) > 0$ for $i = 1, 2$. The drift rates and diffusion coefficients are given and assume positive values. The $2-$dimensional Brownian motion, W, satisfies the usual assumptions. We also assume invertibility of $\lambda \in R^{2 \times 2}$. Hence, the market is "potentially complete." The maximization problem of the representative agent is given by

$$\max_{X_1, X_2} E\left[X_1^\alpha(T) X_2^\beta(T)\right] \qquad (3.3)$$

$$s.t.$$

$$E\left[\xi(T)(X_1(T) + P(T)X_2(T))\right] \leq X_1(0) + P(0)X_2(0)$$

where ξ represents the state price density and P denotes the price of the second consumption good (in units of the first consumption good). First order conditions (FOC) are given by

$$\alpha X_1(T)^{\alpha-1} X_2(T)^\beta = \xi(T) \qquad (3.4)$$
$$\beta X_1(T)^\alpha X_2(T)^{\beta-1} = \xi(T)P(T).$$

From the FOCs, we see that in equilibrium, the relative price of commodity 1 and commodity 2 at T is given by

$$P(T) = \frac{\beta}{\alpha}\left[\frac{\delta_1(T)}{\delta_2(T)}\right]. \qquad (3.5)$$

Hence, the nominal value of output 2 at T is given by

$$P(T)\delta_2(T) = \frac{\beta}{\alpha}\left[\frac{\delta_1(T)}{\delta_2(T)}\right]\delta_2(T) = \frac{\beta}{\alpha}\delta_1(T). \qquad (3.6)$$

This confirms our intuition that in equilibrium, *nominal* proportions between output (consumption) goods are constant.

Market clearing and the above first order conditions imply the following expressions for stock prices

$$S_1(t) = \alpha \frac{E_t\left[\delta_1(T)^\alpha \delta_2(T)^\beta\right]}{E_t\left[\delta_1(T)^{\alpha-1}\delta_2(T)^\beta\right]} \qquad (3.7)$$

and

$$S_2(t) = \frac{\beta}{\alpha}S_1(t). \tag{3.8}$$

As $S_2(t)$ is given by a constant times $S_1(t)$, the market is incomplete. The example above highlights, that preferences —for instance the Cobb-Douglas utility— which have received great attention not only in the economics literature, but also in the finance literature, somewhat surprisingly imply incompleteness of the financial market. It is, therefore, important to investigate conditions yielding complete financial markets in economies with multi-goods.

3.3   The Economy

This section presents the economic environment. We consider a pure exchange economy over a finite time span $[0, T]$ equipped with a measure ν. As in [Anderson & Raimondo 2008a], consumption and output on $[0, T)$ are flows; consumption and output at T are lumps. The measure ν agrees with Lebesgue measure on $[0, T)$ with $\nu(T) = 1$. The uncertainty is represented by a filtered probability space $\left(\Omega, \Im, P, \{\Im_t\}_{t \geq 0}\right)$, on which is defined a $N-$dimensional Brownian motion $W = (W_1, ..., W_N)$. In the following, all stochastic processes are assumed to be progressively measurable and all equalities are assumed to hold a.s. Further, stochastic differential equations are assumed to have solutions without stating the regularity conditions. $D_t F = (D_{1t}F, D_{2t}F, ..., D_{Nt}F)^\top$ denotes the Malliavin derivative[4] of the random variable F. The explicit calculation of the Malliavin derivatives can be found in Appendix B.2.

Output follows

$$d\delta_i(t) = \delta_i(t)(\gamma_i dt + \lambda_i dW_i(t)) \tag{3.9}$$

where $\delta_i(0) > 0$ are given, for $i = 1, 2, ..., N$. Here, γ_i and λ_i denote growth rates and diffusion coefficients in R and R^N, respectively. We assume that $\lambda \in R^{N \times N}$ is invertible. This implies that the market is "potentially complete." Output at T follows exactly the same dynamics as stated in Equation (3.9).[5] In the Subsection 3.4.2, we relax the assumptions on the constant coefficients of the output process.

The economy is populated by an aggregate agent with utility function defined over the N goods. The maximization problem of the agent is given by

$$\max_C E\left[\int_0^T e^{-\rho s}u(C(s))ds + e^{-\rho T}u(C(T))\right] \tag{3.10}$$

$$s.t.$$

---

[4] For details on Malliavin derivatives, see [Detemple & Zapatero 1991], [Detemple *et al.* 2003a], [Nualart 1995a], and the references therein.

[5] [Anderson & Raimondo 2008a] allow for different dynamics of output at intermediate dates and at terminal date.

$$E\left[\int_0^T \xi(s)P(s)^\top C(s)ds + \xi(T)P(T)^\top C(T)\right] \le X(0)$$

where $\rho > 0$, u is a classical time-additive VNM utility function, ξ stands for the state price density, P denotes the vector of prices, $C = \{c_1, c_2, ..., c_N\}$ denotes the vector containing the N consumption goods, and X represents wealth. Above, it is assumed that the utility function satisfies the standard assumptions. It is also assumed that the utility function for intertemporal consumption is identical to the utility function at terminal date.[6] Unless otherwise stated, consumption good one is used as numeraire and the price of the first consumption good is normalized to one, i.e., $P_1(t) = 1$ for all t.

Definition 2. *A function is real analytic if, at every point in its domain, there exists a power series which converges to the function on an open set containing the point.*

The primitives of the economy are assumed to be real analytic functions of time and the current value of the Brownian motion for $t \in [0, T]$.[7] This means that the utility function in Equation (3.10) is real analytic. Note that most conventional utility functions and all utility functions considered in this work are real analytic, see [Anderson & Raimondo 2008a].

Definition 3. *We define I to be the $N \times N$ identity matrix. For a vector x, with $x \in R^N$, we define I_x to represent a $N \times N$ dimensional matrix with x_i as element (i, i) and zero elsewhere.*

The endogenous commodity price evolves according to

$$dP(t) = I_P(t)\left(\mu_P(t)dt + \sigma_P(t)dW(t)\right) \tag{3.11}$$

where μ_P and σ_P denote growth rates and diffusion coefficients in R^N and $R^{N \times N}$, respectively.

Definition 4. *We define the N-dimensional nominal output process as*

$$\widetilde{\delta}(t) = I_P(t)\delta(t). \tag{3.12}$$

---

[6][Anderson & Raimondo 2008a] allow for different utility functions for intertemporal consumption and for terminal consumption.

[7]The assumption that δ_i is a geometric Brownian motion ensures analyticity. We can generalize this by assuming that instead of following the process in Equation (3.9) the output is a real analytic function of time and the Brownian motions, i.e, $\delta_i : [0, T] \times R^N \to R$ is real analytic on $[0, T] \times R^N$.

There are N stocks, each representing a claim to its respective output process. In equilibrium, stock prices are given by

$$S_i(t) = \frac{1}{\xi(t)} E_t \left[\int_t^T \xi(s) P_i(s) \delta_i(s) ds + \xi(T) P_i(T) \delta_i(T) \right] \qquad (3.13)$$

for $i = 1, 2, ..., N$. Consequently, stock price dynamics follow

$$dG(t) = dS(t) + I_{\tilde{\delta}}(t) dt = I_S(t) \left(\mu(t) dt + \sigma(t) dW(t) \right). \qquad (3.14)$$

The diffusion term $\sigma(t)$ denotes a $N \times N$ matrix with the $i'th$ row given by $\sigma_i(t)^\top$. Both the drift rates and the diffusion terms in Equation (3.14) represent endogenous quantities.

The aggregate agent also has access to a locally risk-free asset in zero net supply paying out in the first good, with dynamics

$$dB(t) = r(t) B(t) dt \qquad (3.15)$$

with $B(0) = 1$. The risk-free rate, r also represents an endogenous quantity.

The equilibrium state price density process, $\xi(t)$, follows

$$d\xi(t) = \xi(t) \left(-r(t) dt - \theta(t)^\top dW(t) \right) \qquad (3.16)$$

where $\xi(0) = 1$ and where $\theta(t)$ denotes the market price of risk.

Now, for the market to be complete the following equation must have a unique solution

$$
\begin{aligned}
\sigma(t)\theta(t) &= \mu(t) - r(t) \qquad (3.17) \\
\theta(t) &= \sigma(t)^{-1}(\mu(t) - r(t)).
\end{aligned}
$$

This means that $\sigma(t)$ must be invertible.

Remark 6. *For the market to be arbitrage free, $\sigma(t)\theta(t) = \mu(t) - r(t)$ must have a solution. However, it is not required that the solution be unique.*

Remark 7. *We define financial market completeness as a case when $\sigma(t)$ is invertible for almost all (a.a.) (t, ω) in $[0, T] \times \Omega$.*

Now we are ready to raise the main research question: When will $\sigma(t)$ be invertible? We cannot, however, answer this important question by directly analyzing conditions on $\sigma(t)$ as this requires a functional for the asset price. Therefore, we follow [Anderson & Raimondo 2008a] and derive conditions on the primitives of the economy that guarantee invertibility of $\sigma(t)$ for a.a. (t, ω) in $[0, T] \times \Omega$.

Next we define equilibrium, when it exists, to study the contingent market economy, and then to derive from equilibrium price processes the conditions for dynamic completeness.

Definition 5. *Equilibrium is a price system* $(\mu, \mu_P, \sigma, \sigma_P, r)$, *such that* (C, π) *represents the optimal solution to the optimization problem of the aggregate agent plus good as well as financial markets clear*

$$C(t) = \delta(t) \tag{3.18}$$

$$\pi(t) = S(t) \tag{3.19}$$

$$\pi_0(t) = 0 \tag{3.20}$$

for $t \in [0, T]$ *where* $\pi(t) = (\pi_1(t), ..., \pi_N(t))$ *is the vector of amounts held in the stocks and* $\pi_0(t)$ *is the amount held in the bond market.*

The next proposition characterizes the equilibrium state price density and the commodity price vector.

Proposition 8. *The equilibrium state price density is*

$$\frac{\xi(t)}{\xi(0)} = e^{-\rho t} \frac{\frac{\partial u(\delta(t))}{\partial \delta_1(t)}}{\frac{\partial u(\delta(0))}{\partial \delta_1(t)}}. \tag{3.21}$$

Moreover, the equilibrium commodity price vector is

$$P(t) = \frac{\nabla u(\delta(t))}{\frac{\partial u(\delta(t))}{\partial \delta_1(t)}}. \tag{3.22}$$

The following two propositions represent important building blocks in the sequel. The first proposition defines the commodity price diffusion coefficient matrix. The second proposition defines the *nominal* output diffusion coefficient matrix.

Proposition 9. *The commodity price diffusion coefficients,* σ_P, *are given by*

$$\sigma_P(t) = \varepsilon(t)\lambda \tag{3.23}$$

where $\varepsilon(t)$ *is a* $N \times N$ *matrix with element* (i, j) *given by*

$$\varepsilon_{i,j}(t) = \delta_j(t) \frac{\partial \ln MRS_{i,1}(t)}{\partial \delta_j} \tag{3.24}$$

where $MRS_{i,1}(t) = \frac{\frac{\partial u(\delta(t))}{\partial \delta_i}}{\frac{\partial u(\delta(t))}{\partial \delta_1}}$ *is the marginal rate of substitution. Moreover, the price diffusion coefficients are analytic functions of time and the Brownian motion processes on* $[0, T] \times R^N$.

Proof. In equilibrium, the commodity price vector $P(t)$ is given by

$$
\begin{aligned}
P(t) &= \frac{\nabla u(\delta(t))}{\frac{\partial u(\delta(t))}{\partial \delta_1(t)}} \\
&= [MRS_{1,1}(t), ..., MRS_{N,1}(t)]^\top.
\end{aligned}
\tag{3.25}
$$

Applying Ito's lemma to $P(t)$ yields the proposition. Recall that the marginal utility of good i is an analytic function of time and the N-dimensional Brownian motion. Since the marginal utility is bounded away from zero and the ratio of two analytic functions is analytic, the price vector is also an analytic function. Further, the derivative of an analytic function is analytic so the commodity price diffusion coefficients are analytic. □

Proposition 10. *The diffusion coefficients of nominal output are given by*

$$\sigma_{\tilde{\delta}}(t) = \lambda + \sigma_P(t). \tag{3.26}$$

Moreover, the nominal consumption diffusion coefficients are analytic functions of time and the Brownian motion at time t in $[0,T] \times R^N$.

Proof. The proposition follows directly from applying Ito's lemma to the nominal consumption process. Analyticity follows from the fact that consumption and commodity prices are both analytic functions. Nominal consumption is the product of the two, and thus analytic. Moreover, the derivative of the nominal consumption is analytic. □

The next proposition characterizes the equilibrium market price of risk and the risk free rate.

Proposition 11. *The equilibrium market price of risk, $\theta(t)$, is*

$$\theta(t) = -\sum_{i=1}^{N} \frac{\frac{\partial^2 u(\delta(t))}{\partial \delta_1 \partial \delta_i}}{\frac{\partial u(\delta(t))}{\partial \delta_1}} \delta_i(t) \lambda_i \tag{3.27}$$

The equilibrium risk free rate, $r(t)$, is

$$r(t) = \rho - \sum_{i=1}^{N} \frac{\frac{\partial^2 u(\delta(t))}{\partial \delta_1 \partial \delta_i}}{\frac{\partial u(\delta(t))}{\partial \delta_1}} \delta_i(t) \gamma_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\frac{\partial^3 u(\delta(t))}{\partial \delta_1 \partial \delta_i \partial \delta_j}}{\frac{\partial u(\delta(t))}{\partial \delta_1}} \delta_i(t) \delta_j(t) \lambda_i^{\top} \lambda_j \tag{3.28}$$

Proof. This follows from applying Ito's lemma to (3.21) and from matching the drift and diffusion terms with (3.16). □

Proposition 12. *The stock price diffusion coefficients, $\sigma(t)$, are given by*

$$
\begin{aligned}
\sigma_{ij}(t) &= \theta_j(t) + \frac{E_t \left[\int_0^T \xi(s) P_i(s) \delta_i(s) \chi_{ij}(t,s) ds \right]}{E_t \left[\int_t^T \xi(s) P_i(s) \delta_i(s) ds + \xi(T) P_i(T) \delta_i(T) \right]} \\
&\quad + \frac{E_t \left[\xi(T) P_i(T) \delta_i(T) \chi_{ij}(t,T) \right]}{E_t \left[\int_t^T \xi(s) P_i(s) \delta_i(s) ds + \xi(T) P_i(T) \delta_i(T) \right]}
\end{aligned} \tag{3.29}
$$

where

$$\chi_{ij}(t,u) = D_{jt} \ln \xi(u) + D_{jt} \ln P_i(u) + D_{jt} \ln \delta_i(u) \tag{3.30}$$

for $i,j = 1, 2, ..., N$

Proof. In equilibrium,

$$\xi(t)G_i(t) = E_t \left[\int_0^T \xi(s)P_i(s)\delta_i(s)ds + \xi(T)P_i(T)\delta_i(T) \right]. \tag{3.31}$$

We note that the above represents a martingale. After applying Ito's lemma on the left hand side and the Clark-Ocone theorem on the right hand side, we obtain the following expression for the diffusion terms

$$\sigma_{ij}(t) = \theta_j(t) + \frac{E_t \left[\int_t^T D_{jt} \left(\xi(s)P_i(s)\delta_i(s) \right) ds + D_{jt} \left(\xi(T)P_i(T)\delta_i(T) \right) \right]}{E_t \left[\int_t^T \xi(s)P_i(s)\delta_i(s)ds + \xi(T)P_i(T)\delta_i(T) \right]}. \tag{3.32}$$

Solving for the Malliavin derivatives explicitly leads to the expression in the proposition. □

The drift of the stock prices can be found by solving (3.17) for $\mu(t)$. This completes the description of equilibrium.

3.4 On the Invertibility and on the Non-Invertibility of the Asset Return Diffusion Matrix

This section first extends the results on market completeness in [Anderson & Raimondo 2008a] to the multi-good case. Next, we extend the results on market incompleteness in [Berrada *et al.* 2007] from the 2-good case to the N-good case. Further, we show that numeraire invariance does not hold; that is, a market can be complete under one numeraire and incomplete under another numeraire.

3.4.1 Complete Financial Markets

Theorem 1. *The market will be complete if*

$$I + \varepsilon(T, \omega) \tag{3.33}$$

is invertible for some $\omega \in \Omega$.

Proof. According to [Anderson & Raimondo 2008a] the market will be complete if there exists an $\omega \in \Omega$ such that the output diffusion matrix is invertible. In our multi-good setting, we must consider the nominal output diffusion matrix

$$(I + \varepsilon(T, \omega))\,\lambda. \tag{3.34}$$

By definition, λ is invertible, and thus the nominal diffusion matrix will be invertible if $I + \varepsilon(T, \omega)$ is invertible. The determinant is a product of the elements of $(I + \varepsilon(T, \omega))\,\lambda$ and is therefore analytic. This implies that if there is an $\omega \in \Omega$ such that the determinant is non-zero, then the determinant attains zero only on a measure zero. $\qquad \square$

The key to the proof lies in the non-zero measure on the terminal date combined with analyticity. Since an analytic function is either always zero or zero on a measure zero it is sufficient to find a single point where the determinant of the nominal consumption diffusion matrix is non-zero.

Remark 8. *Note that $I + \varepsilon(T, \omega)$ is completely determined by exogenous quantities. We have thus reduced the problem of determining whether the endogenous stock price diffusion matrix is invertible to a condition only depending on the primitives in the economy.*

Remark 9. *The fact that market completeness is guaranteed if $I + \varepsilon(T)$ is invertible at one point makes the verification of completeness easy even in a situation where the utility of the representative agent is not known in closed form. This might happen in economies with multiple agents with heterogeneous preferences (see Section 3.5). Using standard aggregation techniques, one can easily check if the resulting market is complete by numerically solving for the utility function of the representative agent and applying the theorem.*

Corollary 1. *In the two good case, the market will be complete if the utility function does not satisfy the following partial differential equation (PDE)*

$$\delta_2(t)\frac{\partial \ln MRS_{2,1}(t)}{\partial \delta_2} + 1 = 0. \tag{3.35}$$

Proof. This follows from applying the theorem to the two good case. $\qquad \square$

Remark 10. *The N-good case can also be stated as a PDE. A sufficient condition for market completeness is that $\det(I + \varepsilon(T)) = 0$ does not have a solution.*

3.4.2   Incomplete Financial Markets

Here, the n-dimensional output processes is assumed to be a nonnegative Ito process. We continue to assume that $\lambda \in R^{N \times N}$ is invertible, but drop the requirements that consumption and output at T are lumps and that primitive functions are real analytic.

The following two propositions extend the 2-good case in [Berrada *et al.* 2007] to the N-good case. The first proposition provides sufficient conditions for non-invertible $\sigma(t)$. We include between the two propositions a definition of utility functions. The second proposition states that any utility function satisfying the definition also satisfies the first proposition.

Proposition 13. *The diffusion matrix $\sigma(t)$ is non-invertible if there exists a non-trivial solution $a \in R^N$ to*

$$a_1 \frac{\partial u\left(\delta(t)\right)}{\partial \delta_1} \delta_1(t) + a_2 \frac{\partial u\left(\delta(t)\right)}{\partial \delta_2} \delta_2(t) + ... + a_N \frac{\partial u\left(\delta(t)\right)}{\partial \delta_N} \delta_N(t) = 0 \qquad (3.36)$$

for all $t \in [0, T]$.

Proof. Assume that there exists a solution to the following equation

$$a_1 \frac{\partial u\left(\delta(t)\right)}{\partial \delta_1} \delta_1(t) + a_2 \frac{\partial u\left(\delta(t)\right)}{\partial \delta_2} \delta_2(t) + ... + a_N \frac{\partial u\left(\delta(t)\right)}{\partial \delta_N} \delta_N(t) = 0. \qquad (3.37)$$

Integrate from 0 to T to get

$$\int_0^T (a_1 \frac{\partial u\left(\delta(s)\right)}{\partial \delta_1} \delta_1(s) + a_2 \frac{\partial u\left(\delta(s)\right)}{\partial \delta_2} \delta_2(s) + ... + a_N \frac{\partial u\left(\delta(s)\right)}{\partial \delta_N} \delta_N(s)) ds = 0. \quad (3.38)$$

Take conditional expectation on both sides

$$E_t \int_0^T (a_1 \frac{\partial u\left(\delta(s)\right)}{\partial \delta_1} \delta_1(s) + a_2 \frac{\partial u\left(\delta(s)\right)}{\partial \delta_2} \delta_2(s) + ... + a_N \frac{\partial u\left(\delta(s)\right)}{\partial \delta_N} \delta_N(s)) ds = 0. \quad (3.39)$$

By dividing by $\xi(t)$ and comparing with the pricing formula in Equation (3.13), we can infer that the following equation is satisfied

$$a_1 G_1(t) + a_2 G_2(t) + ... + a_N G_N(t) = 0 \qquad (3.40)$$

for all t. Hence, the gain processes are linearly dependent and the financial market is incomplete. $\qquad \square$

Remark 11. *For the two good case, Proposition (13) corresponds to a constraint on the elasticity of substitution. If the elasticity of substitution equals one, then the market is incomplete. This is the argument put forward in [Berrada et al. 2007]. In the N-good case, a sufficient condition for market incompleteness is that there are at least two composite goods that have unit elasticity of substitution. Note that actual composition of composite goods can take complicated forms, which we clarify in the definition below.*

Definition 6. *A utility function* $u : R^N \to R$ *will be defined to be in* U_{IC}^K, *where IC stands for incompleteness, if it has a representation*

$$u(x_1, ..., x_N) = \varphi(Z) \tag{3.41}$$

where $\varphi : R^K \to R$, $K < N$, $Z = (Z_1, Z_2, ..., Z_K)$ *with*

$$Z_i = \left(\sum_{j=1}^{N} d_{1ji} x_j^{b_{1i}} \right)^{a_{1i}} \times \left(\sum_{j=1}^{N} d_{2ji} x_j^{b_{2i}} \right)^{a_{2i}} \times ... \times \left(\sum_{j=1}^{N} d_{Nji} x_j^{b_{Ni}} \right)^{a_{Ni}} \tag{3.42}$$

with $a_i, b_i, d_{ji} \in R^N$, *where* $i = 1, ..., N$ *and where* $j = 1, ..., N$.

Proposition 14. *If* $u \in U_{IC}^N$, $\sigma(t)$ *is non-invertible.*

Proof. Any utility function $u \in U_{IC}^K$ satisfies the partial differential equation in Proposition 13. $\square$

To sum up, Proposition 13 provides sufficient conditions for the financial market to be incomplete in a setting more general than that of [Anderson & Raimondo 2008a].

3.4.3 Numeraire Invariance

In this subsection, we show that the choice of numeraire good might matter for financial market completeness and for incompleteness. Numeraire invariance fails since choosing a different good as numeraire also changes the assets available to the investors. The risk-free asset under one numeraire is a non-tradable asset under another numeraire. The next proposition states how changing the numeraire alters equilibrium.

Proposition 15. *The relation between equilibrium Sharp ratios under two different numeraires (A and B) is given by*

$$\theta_B(t) = \theta_A(t) - \sigma_{p_A}(t). \tag{3.43}$$

Furthermore, the relation between risk free rates is given by

$$r_B(t) = r_A(t) - \mu_{p_A}(t) + \theta_A^T(t) \sigma_{p_A}(t). \tag{3.44}$$

Stock price diffusion coefficients under two numeraires relate via

$$\sigma_B(t) = \sigma_A(t) - \sigma_{p_A}(t) \tag{3.45}$$

where the subscript A (B) denotes equilibrium quantities in the economy with A (B) as numeraire.

Proof. Stochastic discount factors in economy A and in economy B relate via commodity prices as follows

$$\xi_B(t) = P_A(t)\xi_A(t). \tag{3.46}$$

$\xi_A(t)$, $\xi_B(t)$ and $P_A(t)$ evolve accordingly to

$$d\xi_A(t) = \xi_A(t)\left(-r_A(t)dt - \theta_A(t)^\top dW(t)\right) \tag{3.47}$$

$$d\xi_B(t) = \xi_B(t)\left(-r_B(t)dt - \theta_B(t)^\top dW(t)\right) \tag{3.48}$$

$$dP_A(t) = P_A(t)\left(\mu_{P_A}(t)dt + \sigma_{P_A}(t)^\top dW(t)\right). \tag{3.49}$$

Applying Ito's lemma to the right hand side of Equation (3.46) leads to

$$\frac{dP_A(t)\xi_A(t)}{P_A(t)\xi_A(t)} = \mu_{P_A\xi_A}dt - \sigma_{P_A\xi_A}^\top dW(t). \tag{3.50}$$

where

$$\mu_{P_A\xi_A}(t) = -r_A(t) + \mu_{P_A}(t) - \theta_A(t)^\top\sigma_{P_A}(t) \tag{3.51}$$

and

$$\sigma_{P_A\xi_A}(t) = \theta_A(t) - \sigma_{P_A}(t). \tag{3.52}$$

Matching drift and diffusion terms yields

$$\theta_B(t) = \theta_A(t) - \sigma_{P_A}(t) \tag{3.53}$$

$$r_B(t) = r_A(t) - \mu_{P_A}(t) + \theta_A(t)^\top\sigma_{P_A}(t). \tag{3.54}$$

To prove that the relation between the stock price diffusion coefficients is as in (3.45) note that the only term that changes when changing the numeraire is the Sharp ratio. Using the relation between the Sharp ratios proves the relation. □

Proposition 16. *If the utility function satisfies Proposition 13, then the market is incomplete under all numeraires.*

Proof. This follows directly from noting that the PDE in Proposition 13 does not depend on the choice of numeraire. □

Proposition 17. *Consider the case of two goods. Then the following conditions are equivalent*

1. *The utility function satisfies Proposition 13.*

2. *The utility function does not satisfy Theorem 1 under any of the two numeraires.*

Proof. We first show that (1) implies (2). We then show that (2) implies (1). First, note that by definition $MRS_{2,1} = \frac{1}{MRS_{1,2}}$ holds. Now, assume that the utility function satisfies Proposition 13, which then implies

$$a\frac{\partial u(\delta(t))}{\partial \delta_1}\delta_1(t) + \frac{\partial u(\delta(t))}{\partial \delta_2}\delta_2(t) = 0. \tag{3.55}$$

Rearranging the above yields

$$MRS_{2,1}(t) = b\left(\frac{\delta_1(t)}{\delta_2(t)}\right). \tag{3.56}$$

After taking the log of the MRS above, we obtain

$$\ln MRS_{2,1}(t) = \ln b + \ln \delta_1(t) - \ln \delta_2(t). \tag{3.57}$$

Applying Corollary 1 leads to

$$\frac{\partial \ln MRS_{2,1}(t)}{\partial \delta_2}\delta_2(t) = -\frac{1}{\delta_2}\delta_2(t) = -1. \tag{3.58}$$

Now we perform the above steps once more for the case when good two serves as numeraire

$$\ln MRS_{1,2}(t) = -\ln b + \ln \delta_2(t) - \ln \delta_1(t) \tag{3.59}$$

$$\frac{\partial \ln MRS_{1,2}(t)}{\partial \delta_1}\delta_1(t) = -\frac{1}{\delta_1(t)}\delta_1(t) = -1. \tag{3.60}$$

This proves that (1) implies (2). Now assume that the utility function does not satisfy Corollary 1 under any of the two numeraires. When good one serves as numeraire, we have

$$\frac{\partial \ln MRS_{2,1}}{\partial \delta_2} = -\frac{1}{\delta_2}. \tag{3.61}$$

Solving the above PDE yields

$$\ln MRS_{2,1} = \ln f_1(\delta_1) - \ln \delta_2 \tag{3.62}$$

which we rewrite as follows

$$MRS_{2,1} = \frac{f_1(\delta_1)}{\delta_2}. \tag{3.63}$$

Next, consider the case when we choose the second good as numeraire. We then have

$$\frac{\partial \ln MRS_{1,2}}{\partial \delta_1} = -\frac{1}{\delta_1}. \tag{3.64}$$

Solving the above PDE results in

$$\ln MRS_{1,2} = \ln f_2(\delta_2) - \ln \delta_1 \tag{3.65}$$

or, alternatively, in

$$MRS_{1,2} = \frac{f_2(\delta_2)}{\delta_1}. \tag{3.66}$$

Using Equation (3.63) and Equation (3.66) implies that $f_1(x) = bx$ and that $f_2(x) = \frac{1}{b}x$. Thus we have

$$MRS_{2,1} = b\left(\frac{\delta_1(t)}{\delta_2(t)}\right). \tag{3.67}$$

Rearranging yields,

$$a\frac{\partial u(\delta(t))}{\partial \delta_1}\delta_1(t) + \frac{\partial u(\delta(t))}{\partial \delta_2}\delta_2(t) = 0 \tag{3.68}$$

which satisfies Proposition 13. This concludes the proof. $\square$

The next two examples further illustrate how numeraire invariance fails with multi-goods. In the first example, the utility function includes a subsistence point. This is similar to the setup in [Ait-Sahalia *et al.* 2004] with a negative subsistence point for luxury goods. The second example includes preference shocks in the spirit of [Pavlova & Rigobon 2007].

3.4.3.1   Subsistence Point

Consider the following utility function defined over the two goods c_1 and c_2

$$u(c_1, c_2) = log(c_1 + b) + log(c_2) \tag{3.69}$$

where $b > 0$.

First, consider the case in which the first good serves as numeraire. The relative price of the second good in terms of the first good is given by

$$P(t) = \frac{\delta_1(t) + b}{\delta_2(t)}. \tag{3.70}$$

Applying Corollary 1, we have

$$\delta_2(t)\frac{\partial \ln MRS_{2,1}(t)}{\partial \delta_2} + 1 = -\frac{\delta_2(t)}{\delta_2(t)} + 1 = 0. \tag{3.71}$$

Hence, the market is not guaranteed to be complete when the first good is the numeraire. In fact, if δ_1 and δ_2 evolve as uncorrelated geometric Brownian motions, we then obtain the following expressions for stock prices

$$S_1(t) = (\delta_1(t) + b)E_t\left[\int_t^T \frac{\delta_1(u)}{\delta_1(u) + b}du + \frac{\delta_1(T)}{\delta_1(T) + b}\right] \tag{3.72}$$

and

$$S_2(t) = (\delta_1(t) + b). \tag{3.73}$$

As δ_1 is only driven by one of the two Brownian motions, the market must be incomplete. Next consider the case in which the second good is the numeraire. The relative price of the first good in terms of the second good is given by

$$P(t) = \frac{\delta_2(t)}{\delta_1(t) + b}. \tag{3.74}$$

Again applying Corollary 1, we have

$$\delta_1(t)\frac{\partial \ln MRS_{1,2}(t)}{\partial \delta_1} + 1 = -\frac{\delta_1(t)}{\delta_1(t)+b} + 1 \neq 0. \tag{3.75}$$

Here the utility function does not satisfy the partial differential equation. Thus the market is complete.

The above example also illustrates that unit elasticity of substitution is only a sufficient condition for market incompleteness. When the first good is used a numeraire, the market might be incomplete even though the elasticity of substitution differs from one.

3.4.3.2 Preference Shocks

Consider the following utility function defined over the two goods c_1 and c_2

$$u(c_1, c_2) = \ln c_1(t) + \eta(t) \ln c_2(t). \tag{3.76}$$

Above $\eta(t)$ serves as a preference shock. The preference shock is an analytic function of time and the Brownian motions.[8] Moreover, we assume that the preference shock is perfectly correlated with the output of consumption good 2 and that

$$\eta(0) = 1 \tag{3.77}$$

and

$$E_t[\eta(s)] = \eta(t) \tag{3.78}$$

for $s > t$. To simplify expositions further, we assume that output of the two goods are uncorrelated. Note that with preference shocks we cannot directly use Corollary 1 as it does not allow for preference shocks.[9] Let us employ good one as numeraire. Calculating stock prices leads to

$$\begin{aligned} S_1(t) &= \delta_1(t)E_t\left[\int_t^T \frac{\delta_1(s)}{\delta_1(s)}ds + \frac{\delta_1(T)}{\delta_1(T)}\right] \\ &= \delta_1(t) \end{aligned} \tag{3.79}$$

---

[8]In the setup in Section 3.4.1, we do not allow for preference shocks. However, Theorem 1 can easily be extended to allow for preference shocks as long as the preference shocks are analytic functions of the current value of the Brownian motions and time. [Anderson & Raimondo 2008a] allows for such state dependencies.

[9]Corollary 1 may be modified to incorporate preference shocks. For this, one must verify that the nominal dividend diffusion matrix is invertible as the relative price depends on the preference shock. In the example, we obtain closed form solutions for stock prices, and thus choose not to provide a modified version of Corollary 1.

and to

$$
\begin{aligned}
S_2(t) &= \delta_1(t) E_t\left[\int_t^T \eta(s)\frac{\delta_2(s)}{\delta_2(s)}ds + \eta(T)\frac{\delta_2(T)}{\delta_2(T)}\right] \\
&= \delta_1(t)\eta(t)\left(1 + T - t\right).
\end{aligned}
\tag{3.80}
$$

Using the fact that the two output processes are uncorrelated and that the preference shock is perfectly correlated with the output of good 2, we can conclude that financial markets are complete. Next, assume that good 2 serves as the numeraire. Calculating the stock prices we get

$$
\begin{aligned}
S_1(t) &= \frac{\delta_2(t)}{\eta(t)} E_t\left[\int_t^T \frac{\delta_1(s)}{\delta_1(s)}ds + \frac{\delta_1(T)}{\delta_1(T)}\right] \\
&= \frac{\delta_2(t)}{\eta(t)}
\end{aligned}
\tag{3.81}
$$

and

$$
\begin{aligned}
S_2(t) &= \frac{\delta_2(t)}{\eta(t)} E_t\left[\int_t^T \eta(s)\frac{\delta_2(s)}{\delta_2(s)}ds + \eta(T)\frac{\delta_2(T)}{\delta_2(T)}\right] \\
&= \delta_2(t)\left(1 + T - t\right).
\end{aligned}
$$

From the correlation structure, we notice that the market is incomplete when good 2 is the numeraire.

3.5   Heterogeneous Agents

In this section, we consider an economy with J agents, $j \in \{1, 2, ..., J\}$. As in Section 3.3, we assume that there are N goods and that agent j has VNM utility function $u_j : R_{++}^N \to R$. The utility function is assumed to be real analytic, $\nabla u_j \in R_{++}^N$ and $\nabla^2 u_j$ is negative definite.[10] Moreover, we assume that the multi-dimensional Inada conditions hold. These conditions ensure that there exists a solution to the central planner problem and that the solution is unique.

Agent j's maximization problem is

$$
\max_{C^j, \pi^j} E\left[\int_0^T e^{-\rho s} u_j(C^j(s))ds + e^{-\rho T} u_j(C^j(T))\right]
\tag{3.82}
$$

$$
s.t.
$$

$$
dX^j(t) = X^j(t)r(t)dt - P(T)^\top C^j(T)dt + \pi^j(t)^\top \left(\mu(t) - r(t)1\right)dt + \pi^j(t)^\top \sigma(t)dW(t)
$$

where $\pi^j(t) = \left(\pi_1^j(t), \pi_2^j(t), ..., \pi_N^j(t)\right)$ is the vector of amounts held in the stocks by agent j. $X^j(0) > 0$ with $X^j(0) = \pi^j(0)^\top 1$, i.e, the agents are endowed with initial shares in the stocks.

---

[10]∇ denotes the gradient and ∇^2 denotes the Hessian.

Definition 7. *Equilibrium is a collection of allocations $\left(C^j, \pi^j\right)$ for $j = 1, 2, ..., J$, and a price system $(\mu, \mu_P, \sigma, \sigma_P, r)$, such that $\left(C^j, \pi^j\right)$ are optimal solutions to agent j's optimization problem and good and financial markets clear*

$$\sum_j C^j(t) = \delta(t) \tag{3.83}$$

$$\sum_j \pi^j(t) = S(t) \tag{3.84}$$

$$\sum_j \pi_0^j = 0 \tag{3.85}$$

for $t \in [0, T]$ where $\pi_0^j(t)$ is the amount held in the bond market.

Define the representative agents utility function as

$$u(\delta; a) = \max_{\sum_j C^j = \delta} \sum_j a_j u_j(C^j). \tag{3.86}$$

In Equation (3.86) the utility weights, a are solutions to

$$E\left[\int_0^T \xi(s; a) P(s; a)^\top C^j(s; a) ds + \xi(T; a) P^\top(T; a) C^j(T; a)\right] = X^j(0) \tag{3.87}$$

where the above is evaluated at the optimal solution for $j = 1, ..., J$. According to the analytic implicit function theorem (see [Anderson & Raimondo 2008a]), the utility function of the representative agent is real analytic. To ensure that the market is complete, the utility function of the representative agent must satisfy Theorem 1. Instead of considering the dynamics optimization in Equation (3.82), we can solve the static optimization of the representative agent

$$\max_C E\left[\int_0^T e^{-\rho s} u(C(s); a) ds + e^{-\rho T} u(C(T); a)\right] \tag{3.88}$$

$$s.t.$$

$$E\left[\int_0^T \xi(s) P(s)^\top C(s) ds + \xi(T) P(T)^\top C(T)\right] \leq X(0).$$

Note that this is equivalent to the optimization in Equation (3.10), but with the utility derived from Equation (3.86). Given that the utility function in Equation (3.86) satisfies Theorem 1, the equilibrium characterization follows from Section 4.2.

The optimal consumption profiles for agent $j = 1, ..., J$ are found via the optimization in Equation (3.86). Since markets are complete, the optimal portfolios can be found by using the approach in [Cox & Huang 1989]. The next proposition characterizes the optimal portfolio policies.

Proposition 18. *The optimal portfolio policy of agent j, π^j, is given by*

$$\pi^j(t) = \left(\sigma(t)^T\right)^{-1}\left(X^j(t)\theta(t) + \frac{\psi^j(t)}{\xi(t)}\right) \tag{3.89}$$

where

$$\psi^j(t) = \left(\psi_1^j(t), \psi_2^j(t), ..., \psi_N^j(t)\right)^{\top} \tag{3.90}$$

with

$$\begin{aligned}
\psi_i^j(t) &= E_t\left[\int_t^T \sum_{k=1}^N \left(\xi(s)P_k(s)c_k^j(s)\Pi_{ijk}(t,s)\right)ds\right] \\
&+ E_t\left[\sum_{k=1}^N \left(\xi(T)P_k(T)c_k^j(T)\Pi_{ijk}(t,T)\right)\right]
\end{aligned} \tag{3.91}$$

where

$$\Pi_{ijk}(t,u) = D_{it}\ln\xi(u) + D_{it}\ln P_k(u) + D_{it}\ln c_k^j(u) \tag{3.92}$$

for $j = 1, .., J$ and $i = 1, 2, ..., N$.

Proof. According to [Cox & Huang 1989] the optimal portfolio policy is given by

$$\pi^j(t) = \left(\sigma(t)^T\right)^{-1}\left(X_j(t)\theta(t) + \frac{\psi^j(t)}{\xi(t)}\right) \tag{3.93}$$

where $\psi^j(t)$ is the diffusion coefficient of $\xi(t)X^i(t) = E_t\left[\int_0^T \xi(s)P(s)^{\top}C^j(s)ds + \xi(T)P(T)^{\top}C^j(T)\right]$. Applying the Clark-Ocone theorem and solving for the Malliavin derivatives explicitly yields the expression above. $\quad\square$

3.5.1 Unit Elasticity of Substitution Preferences

In this subsection, we restrict our attention to the case with only two types of agents. To simplify the exposition further, we consider the case of two goods. Agents have utility functions with heterogeneity in taste given by

$$u_j(c_1^j, c_2^j) = \varphi_j\left(c_1^j\left(c_2^j\right)^{\alpha_j}\right). \tag{3.94}$$

We choose α_j and $\varphi_j(\cdot)$ such that u_j satisfies the conditions in Section 3.5. Note that the utility function in Equation (3.94) exhibits unit elasticity of substitution.

3.5.1.1 Portfolio Policies

Assume that $\alpha_1 \neq \alpha_2$ and that $\varphi_j(x) \neq ln(x)$. Before we provide a proposition for optimal portfolios, we introduce several auxiliary results. Assume that the utility of the representative agent is non-separable in δ_1 and δ_2. This can be expressed as

$$\frac{\partial^2 u(\delta(t); a)}{\partial\delta_1\partial\delta_2} \neq 0. \tag{3.95}$$

Let the optimal consumption profiles of agent 1 and agent 2 be given by

$$c_1^1(t) \quad = \quad f_1^1(\delta_1(t), \delta_2(t)) \tag{3.96}$$

$$c_1^2(t) \quad = \quad \delta_1(t) - f_1^1(\delta_1(t), \delta_2(t)) \tag{3.97}$$

$$c_2^1(t) \quad = \quad f_2^1(\delta_1(t), \delta_2(t)) \tag{3.98}$$

$$c_2^2(t) \quad = \quad \delta_2(t) - f_2^1(\delta_1(t), \delta_2(t)) \tag{3.99}$$

where f is a functional mapping of aggregate dividends onto optimal consumption.

The sharing rule above is separable if

$$c_1^1(t) \quad = \quad f_1^1(\delta_1(t)) \tag{3.100}$$

$$c_2^1(t) \quad = \quad f_2^1(\delta_2(t)) \tag{3.101}$$

i.e., optimal consumption of the first agent of the first good (second good) is only a function of aggregate output of the first good (second good). From the FOCs, we see that

$$\frac{\partial u(\delta(t); a)}{\partial \delta_1} = a_1 \frac{\partial u_1(c^1(t))}{\partial c_1^1}. \tag{3.102}$$

Assume that the sharing rule is separable. We then obtain the following result (see [Hara 2006])

Proposition 19. *If the representative agent's utility function is non-separable and the sharing rules are separable, then*

$$f_1^1(\delta_1) \quad = \quad A\delta_1 \tag{3.103}$$

$$f_2^1(\delta_2) \quad = \quad A\delta_2 \tag{3.104}$$

for $A \in R_{++}$.

Proof. The FOCs imply the following

$$\frac{\partial u(\delta(t); a)}{\partial \delta_1} \quad = \quad a_1 \frac{\partial u_1(c^1(t))}{\partial c_1^1} \tag{3.105}$$

$$\frac{\partial u(\delta(t); a)}{\partial \delta_2} \quad = \quad a_1 \frac{\partial u_1(c^1(t))}{\partial c_2^1}. \tag{3.106}$$

Differentiate the above with respect to δ_2 (δ_1) under the assumption of separable sharing rules thus

$$\frac{\partial^2 u(\delta(t); a)}{\partial \delta_1 \partial \delta_2} \quad = \quad a_1 \frac{\partial u_1(c^1(t))}{\partial c_1^1 \partial c_2^1} \frac{d}{d\delta_1} f_1^1(\delta_1) \tag{3.107}$$

$$\frac{\partial^2 u(\delta(t); a)}{\partial \delta_2 \partial \delta_1} \quad = \quad a_1 \frac{\partial u_1(c^1(t))}{\partial c_2^1 \partial c_1^1} \frac{d}{d\delta_2} f_2^1(\delta_2). \tag{3.108}$$

As $\frac{\partial^2 u(\delta(t);a)}{\partial \delta_1 \partial \delta_2} = \frac{\partial^2 u(\delta(t);a)}{\partial \delta_2 \partial \delta_1} \neq 0$ and $\frac{\partial u_1(c^1(t))}{\partial c_1^1 \partial c_2^1} = \frac{\partial u_1(c^1(t))}{\partial c_2^1 \partial c_1^1} \neq 0$, which together with non-separability of the representative agent's utility function implies that

$\frac{d}{d\delta_1} f_1^1(\delta_1) = \frac{d}{d\delta_2} f_2^1(\delta_2)$. As this must hold for all δ_1 and δ_2, we obtain the following result

$$f_1^1(\delta_1) \quad = \quad A\delta_1 \tag{3.109}$$

$$f_2^1(\delta_2) \quad = \quad A\delta_2. \tag{3.110}$$

This ends the proof. $\hfill\square$

The next proposition shows that the utility function of the representative agent is non-separable.

Proposition 20. *Consider the utility function in Equation (3.94) with $\alpha_1 \neq \alpha_2$ and with $\varphi_j(x) \neq \ln(x)$; then the utility function of the representative agent is non-separable in δ_1 and δ_2.*

Proof. A necessary condition for the utility function of the representative agent to be separable is

$$\frac{\partial u(\delta(t))}{\partial \delta_1} = g(\delta_1(t)) \tag{3.111}$$

for some function g, i.e., the partial derivative with respect to the first good only depends on the first good. In equilibrium, the following relation holds

$$\frac{\partial u(\delta(t))}{\partial \delta_1} = a_1 \frac{\partial u_1(c^1(t))}{\partial c_1^1}. \tag{3.112}$$

Using the sharing rules and the utility function in (3.94), we obtain

$$\frac{\partial u(\delta(t))}{\partial \delta_1} = \varphi_1\left(f_1^1(\delta_1(t), \delta_2(t)) f_2^1(\delta_1(t), \delta_2(t))^{\alpha_1}\right) f_2^1(\delta_1(t), \delta_2(t))^{\alpha_1}. \tag{3.113}$$

This above function depends only on δ_1 if φ_1 is given by $\varphi_1(x) = \ln(x)$.

$\hfill\square$

The next proposition deals with linear sharing rules.

Proposition 21. *If $\alpha_1 \neq \alpha_2$ and if $\varphi_j(x) \neq \ln(x)$, then equilibrium cannot be implemented by linear sharing rules.*

Proof. Consider that sharing rules are linear; then the following holds

$$a_1 \varphi_1\left(A\delta_1(t)(B\delta_2(t))^{\alpha_1}\right)(B\delta_2(t))^{\alpha_1} \quad = \quad a_1 \varphi_1\left((1-A)\delta_1(t)((1-B)\delta_2(t))^{\alpha_2}\right)$$
$$((1-B)\delta_2(t))^{\alpha_2} \tag{3.114}$$

for some positive constants A and B. Since dividends are less than perfectly correlated, the above cannot hold unless agents have log-linear preferences or homogeneous taste. $\hfill\square$

Proposition 22. *The optimal portfolio of agent 1 is given by*

$$\pi_1^1(t) = \frac{1 - B_2}{B_1 - B_2} S_1(t) \tag{3.115}$$

$$\pi_2^1(t) = -\frac{B_2}{B_1 - B_2} S_2(t) \tag{3.116}$$

with $B_i = \frac{1}{(1+\alpha_i)}$.

Proof. The strategy of the proof is as follows

1. Prove that $\left| corr_t \left(X^1(t), X^2(t) \right) \right| < 1$.

2. Prove that portfolio policies takes the form as in Proposition 22.

3. Prove that this implies that $\left| corr_t \left(S_1(t), S_2(t) \right) \right| < 1$.

To prove (1): From the FOC it follows that

$$P(t) = \alpha_1 \left(\frac{c_1^1}{c_2^1} \right) \tag{3.117}$$

and that

$$\alpha_1 \left(\frac{c_1^1}{c_2^1} \right) = \alpha_2 \left(\frac{c_1^2}{c_2^2} \right). \tag{3.118}$$

The wealth of agent 1 is given by

$$X^1(t) = \frac{E_t \left[\int_0^T \left(\xi(s)c_1^1(s) + \xi(s)P(s)c_2^1(s) \right) ds + \xi(T)c_1^1(T) + \xi(T)P(T)c_2^1(T) \right]}{\xi(t)}. \tag{3.119}$$

Using Equation (3.117), we can rewrite wealth as

$$X^1(t) = \frac{A_1 E_t \left[\int_0^T \xi(s)c_1^1(s)ds + \xi(T)c_1^1(T) \right]}{\xi(t)} \tag{3.120}$$

with $A_1 = (1 + \alpha_1)$. Similarly, we can write the wealth of agent 2 as

$$X^2(t) = \frac{A_2 E_t \left[\int_0^T \xi(s)c_1^2(s)ds + \xi(T)c_1^2(T) \right]}{\xi(t)} \tag{3.121}$$

with $A_2 = (1 + \alpha_2)$. Next, we show that X^1 and X^2 are linearly independent. To this end we use the [Anderson & Raimondo 2008a] technique. Let

$$dc_1^1(t) = \phi_1(t)dt + \Sigma_1(t)^T dW(t) \tag{3.122}$$
$$dc_1^2(t) = \phi_2(t)dt + \Sigma_2(t)^T dW(t) \tag{3.123}$$

and

$$\Sigma(t) = \left[\begin{array}{c} \Sigma_1(t)^T \\ \Sigma_1(t)^T \end{array} \right] = \left[\begin{array}{cc} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{array} \right]. \tag{3.124}$$

If there exists an $\omega \in \Omega$ such that $\det(\Sigma(T, \omega)) \neq 0$, then $X^1(t)$ and $X^2(t)$ are linearly independent a.a. (t, ω). Calculating $\Sigma(t)$ by Ito's lemma, we have

$$\Sigma(t) = J(c_1(t))I_{\delta(t)}\lambda \tag{3.125}$$

where $J(c_1(t))$ denotes the Jacobian of $c_1(t) = \left(c_1^1(t), c_1^2(t)\right)$ and is given by

$$J(c_1(t)) = \begin{bmatrix} \frac{\partial c_1^1(t)}{\partial \delta_1} & \frac{\partial c_1^1(t)}{\partial \delta_2} \\ \frac{\partial c_1^2(t)}{\partial \delta_1} & \frac{\partial c_1^2(t)}{\partial \delta_2} \end{bmatrix}. \tag{3.126}$$

Note that $\det(\Sigma(T)) = \det(J(c_1(T))) \det(I_{\delta(T)}) \det(\lambda)$. By definition $\det(I_{\delta(T)}) \neq 0$ and $\det(\lambda) \neq 0$, implying that $\det(\Sigma(T)) \neq 0$ if and only if $\det(J(c_1(T))) \neq 0$. From the clearing of the commodity market, we obtain

$$c_1^1(t) + c_1^2(t) = \delta_1(t). \tag{3.127}$$

Taking the derivative with respect to δ_2, we get

$$\frac{\partial c_1^1(t)}{\partial \delta_2} = -\frac{\partial c_1^2(t)}{\partial \delta_2}. \tag{3.128}$$

We need to show that (3.128) is non zero. To this end, note that by Proposition 21 we have that the sharing rule is non-linear. Moreover, by Proposition 20 we know that the utility function of the representative agent is non-separable in δ_1 and δ_2. By Proposition 19 we know that separable sharing rules and non-separable utility functions are only consistent with linear sharing rules. As we do not have linear sharing rules, this implies that the sharing rule must be non-separable. Non-separable sharing rules guarantees that (3.128) is non-zero and that

$$\det(J(c_1(T))) = J_{11}J_{22} - J_{12}J_{21} \neq 0 \tag{3.129}$$

since $J_{11}J_{22} < 0$ and $J_{12}J_{21} > 0$ or $J_{11}J_{22} > 0$ and $J_{12}J_{21} < 0$ where J_{ij} denotes element (ij) of $J(c_1(T))$. This proves 1.

To prove (2): Combining Equation (3.120) with Equation(3.121) yields

$$B_1 X^1(t) + B_2 X^2(t) = S_1(t) \tag{3.130}$$

with $B_j = 1/A_j$. In equilibrium, total wealth must be equal to the value of the stock market, i.e.,

$$X^1(t) + X^2(t) = S_1(t) + S_2(t). \tag{3.131}$$

Using Equation (3.130) and Equation (3.131), we obtain

$$\begin{bmatrix} B_1 & B_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X^1(t) \\ X^2(t) \end{bmatrix} = \begin{bmatrix} S_1(t) \\ S_1(t) + S_2(t) \end{bmatrix}. \tag{3.132}$$

This has a unique solution if $B_1 \neq B_2$, which holds whenever $\alpha_1 \neq \alpha_2$. This proves 2.

To prove (3): Subtracting Equation (3.120) from Equation (3.121) leads to

$$(1 - B_1) X^1(t) + (1 - B_2) X^2(t) = S_2(t). \tag{3.133}$$

We have now expressed stock prices as linear combinations of the wealth of agent 1 and agent 2 (see Equation (3.120) and Equation (3.133)). Let wealth of the agents evolve accordingly to

$$dX^1(t) = \mu_{X^1}(t)dt + \sigma_{X^1}(t)^T dW(t) \tag{3.134}$$

$$dX^2(t) = \mu_{X^2}(t)dt + \sigma_{X^2}(t)^T dW(t) \tag{3.135}$$

which implies

$$\begin{bmatrix} B_1 & B_2 \\ 1 - B_1 & 1 - B_2 \end{bmatrix} \begin{bmatrix} \sigma_{X_1^1}(t) & \sigma_{X_2^1}(t) \\ \sigma_{X_1^2}(t) & \sigma_{X_2^2}(t) \end{bmatrix} = \begin{bmatrix} S_1(t)\sigma_{11}(t) & S_1(t)\sigma_{12}(t) \\ S_2(t)\sigma_{21}(t) & S_2(t)\sigma_{22}(t) \end{bmatrix}. \tag{3.136}$$

As $\begin{bmatrix} \sigma_{X_1^1}(t) & \sigma_{X_2^1}(t) \\ \sigma_{X_1^2}(t) & \sigma_{X_2^2}(t) \end{bmatrix}$ is invertible, $\begin{bmatrix} S_1(t)\sigma_{11}(t) & S_1(t)\sigma_{12}(t) \\ S_2(t)\sigma_{21}(t) & S_2(t)\sigma_{22}(t) \end{bmatrix}$ is invertible if $\begin{bmatrix} B_1 & B_2 \\ 1 - B_1 & 1 - B_2 \end{bmatrix}$ is invertible. Finally, $\begin{bmatrix} B_1 & B_2 \\ 1 - B_1 & 1 - B_2 \end{bmatrix}$ is invertible if $\alpha_1 \neq \alpha_2$. This ends the proof.

$\square$

Remark 12. *Note that the only requirement for the proof above to work is less than perfect correlation between the wealth of agent one and agent two. In fact, we could solve the optimal portfolio even if the market was intrinsically incomplete, i.e., with more than two Brownian motions.*

Remark 13. *The above proof also relies on the number of goods and agents. We have two goods and two agents. This allows us to write down a system with two equations in two unknowns. If there where more goods than agents, we would not be able to identity all of the stock price processes from the above method.*

3.5.1.2   Taste

We construct an aggregate agent from the two agents populating the economy, which results in the following utility function

$$u(\delta_1, \delta_2) = \max_{\substack{c_1^1 + c_1^2 = \delta_1 \\ c_2^1 + c_2^2 = \delta_2}} a\varphi_1 \left(c_1^1 \left(c_2^1 \right)^{\alpha_1} \right) + (1 - a)\varphi_2 \left(c_1^2 \left(c_2^2 \right)^{\alpha_2} \right). \tag{3.137}$$

Proposition 23. *The market is incomplete if $\alpha_1 = \alpha_2$.*

Proof. Assume that $\alpha_1 = \alpha_2 = \alpha$. From the first order conditions, we obtain the following relation $\frac{c_1^1}{c_2^1} = \frac{c_1^2}{c_2^2} = \frac{y_2}{\alpha y_1}$ where y_1 and y_2 are the Lagrange multipliers from the optimization in (3.137). The relation, in turn, suggests that the sharing rule must take the following form

$$
\begin{aligned}
\left(c_1^1\right)^* &= A(\delta_1, \delta_2)\delta_1 \\
\left(c_2^1\right)^* &= A(\delta_1, \delta_2)\delta_2 \\
\left(c_1^2\right)^* &= (1 - A(\delta_1, \delta_2))\delta_1 \\
\left(c_2^2\right)^* &= (1 - A(\delta_1, \delta_2))\delta_2
\end{aligned}
\tag{3.138}
$$

where $(.)^*$ denotes optimality. The marginal utility of the representative agent must be proportional to the marginal utility of agent 1 (and agent 2) evaluated at the optimal consumption choice. Using the marginal utility of agent 1, we see that

$$
\frac{\partial u_1\left(\left(c_1^1\right)^*, \left(c_2^1\right)^*\right)}{\partial c_1^1}\delta_1 = \alpha \frac{\partial u_1\left(\left(c_1^1\right)^*, \left(c_2^1\right)^*\right)}{\partial c_2^1}\delta_2
\tag{3.139}
$$

which then, by using Proposition 13, implies that the market is incomplete. $\square$

The proposition states that when agents have identical taste and unit elasticity of substitution, then the market is incomplete. Note that the above easily extends to the case of $n - agent$ economies. If all the agents have same taste, the market will be incomplete. This follows from the fact that each agent consumes the same fraction of total output of each good. The approach discussed above already allows for heterogeneity in risk aversion, see Equation (3.94). This, then, implies that only heterogeneity in taste, but not heterogeneity in risk aversion, might help to restore the completeness of the financial market.

3.5.1.3 Heterogeneity in Beliefs

Now, we consider an economy with incomplete information.[11] Here, agents do not know the expected growth of output and thus need to infer it from the data. The question we are interested in is the following: Will heterogeneity in beliefs resolve the incompleteness problem?

We assume that the subjective probability measure of agent $j = 1, 2$ is equivalent to the objective probability measure.

The program which agents will need to solve takes the following form

$$
\max_{c_1, c_2} E_j \left[\int_0^T e^{-\rho s} u_j(c_1^j(s), c_2^j(s))ds + e^{-\rho T} u_j(c_1^j(T), c_2^j(T)) \right]
\tag{3.140}
$$

---

[11]The setting and notation below borrows from [Detemple & Murthy 1994].

$$s.t.$$

$$E_j \left[\int_0^T \xi_j(s) P(s)^\top C(s)^j ds + \xi_j(T) P(T)^\top C(T)^j \right] \le X_j(0).$$

Here, expectations are defined on agent j's probability measure. The process ξ_j denotes agent j's state price density.

Proposition 24. *If $\alpha_1 = \alpha_2$, $\sigma(t)$ is non-invertible.*

Proof. From the first order condition of the maximization problem, we know that the sharing rule must be of the form

$$\left(c_1^1(t) \right)^* = A(\delta_1(t), \delta_2(t), \eta(t)) \delta_1(t) \tag{3.141}$$

$$\left(c_2^1(t) \right)^* = A(\delta_1(t), \delta_2(t), \eta(t)) \delta_2(t) \tag{3.142}$$

where

$$\eta(t) = \frac{\xi_2(t)}{\xi_1(t)}. \tag{3.143}$$

We already know from the case with heterogeneous taste (and risk aversion) that such a sharing rule implies non-invertibility of σ. □

3.6 Conclusion

In this paper, we investigated the determinants of financial market completeness in potentially complete economies with multiple goods. Our main theorem shows that completeness depends on invertibility of a matrix containing non-linear functions of marginal rates of substitutions at one point in time.

Financial market completeness also depends on the choice of numeraire. Even if our theorem and our proposition are not satisfied under two numeraires, one of the numeraires may imply complete financial markets while the other numeraire may imply incomplete financial markets. Numeraire invariance holds with certainty only if the financial market is incomplete because the partial differential equation in our proposition holds.

Importantly, many popular utility functions such as Cobb-Douglas and logarithmic utility cannot be employed in multi-good economies when studying complete financial markets because they may imply colinear prices. Colinear prices cause, then, incompleteness of the financial market side of the economy. Although this may have no consequences for Pareto efficiency, as in [Cass & Pavlova 2004], it curtails the ability to study optimal portfolio choice, whether in a national setting or in an international setting such as in [Serrat 2001].

We do provide good news: Heterogeneity in taste can restore completeness of financial markets even if individual preferences exhibit unit elasticity of substitution. Taste heterogeneity leads to efficient allocations and to complete financial markets except with log-linear preferences.

Correlations

Abstract

We link stock market correlations in an equilibrium model to the level of risk aversion, time variation in aggregate risk aversion, and other fundamentals. Preference heterogeneity induces endogenous variations in equilibrium quantities that is reflected in aggregate risk aversion and in portfolio trade. Correlations increase in the level of risk aversion as well as in the difference in risk aversions. The model implies countercyclical stock market correlations, expected returns, and standard deviations and countercyclical quadratic variations of portfolio policies. Calibrations of the model match average industry correlations and changes of average industry correlations from business cycles peaks to troughs. Finally, we examine changes in industry stock market correlations, returns, and standard deviations and in quadratic variations of industry turnover and find, as the model predicts, positive relations.

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Keywords: Equity Return Correlations; Quadratic Variation of Portfolios; Heterogeneous Investors;

JEL Classification: G10; G11

4.1 Introduction

We study the economic mechanism underlying the variations of stock market correlations in an equilibrium model. We find that correlations rise with the level of risk aversion. The central ingredient of the model is *endogenous* variation in equilibrium quantities, which we obtain by introducing investors with heterogeneous risk aversion. Hence aggregate marginal utility varies due to changes in aggregate consumption and due to changes in consumption shares, or wealth shares. When consumption declines in a business cycle trough, aggregate risk aversion increases, which then increases correlations. The *endogenous* variation in the stochastic discount factor further increases correlations. The model generates countercyclical stock market correlations, expected returns, and standard deviations and countercyclical quadratic variations of portfolio trade among investors. Calibrations of the model match average industry correlations and changes of average industry correlations from business cycles peaks to troughs.

Why do stock market correlations increase in risk aversion? Consider a setting with uncorrelated dividends. When one asset receives a positive dividend shock, the net effect of this shock on the price and thus on the return is strictly positive. The positive net return has two components, the positive dividend effect and a negative discounting effect, i.e., the discount factor increases. The discounting effect is negative because of an increase in the market price of risk related to the dividend shock of the asset. The market price of risk increases as a consequence of the increased dividend risk, i.e., consumption risk, relative to other dividend streams. For another asset the discounting effect on the return is strictly positive as its market price of risk falls. Hence all assets experience positive returns after a positive dividend shock to one of the assets. [Cochrane *et al.* 2008] study this market clearing mechanism with a logarithmic representative investor. Importantly, we show that the impact of the discounting effect on all assets due to one dividend shock to one of the assets increases in risk aversion. Implying that the larger the risk aversion the smaller is the net return on the asset that experiences the positive dividend shock and that the larger is the return on all other assets. Therefore, the higher is the risk aversion the higher is the correlation between assets returns.

Why do stock market correlations increase in the difference of risk aversion among investors? When consumption declines, investors with high risk aversion hold a larger fraction of the economy, i.e., investors with low risk aversion delevarage.[3] Hence aggregate risk aversion increases, which then increases correlations due to

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[3][Cochrane 2008] argues for a risk aversion story, as evidenced by the amount of deleveraging and forced selling taking place within capital markets, behind the daily volatility of 75 percent in October 2008.

the risk aversion level effect. Although aggregate risk aversion is bounded from above by the largest risk aversion in the economy, the correlations in an economy with heterogeneous investors is significantly larger than the correlation obtained in a homogeneous investor economy populated by the investor with the largest risk aversion. This effect on correlations, and other endogenous quantities, arises due to the non-linearity of the consumption, or wealth, sharing rule among investors, see [Dumas 1989]. The non-linearity is related to the slope of the sharing rule. Increasing the steepness of the sharing rule by increasing the difference in risk aversion among investors also increases endogenous variations in the stochastic discount factor which then increases correlations and other endogenous quantities. When the sharing rule is steep, then the risk aversion of the representative agent is volatile. High volatility of the risk aversion leads to high volatility of the discount rates and consequently high stock retun volatility. The correlation between stock returns increase due to the fact that risk aversion is a common component in all assets' discount rates, and when the sharing rule is steep the volatility is mainly driven by the changes in risk aversion.

We want to emphasize the magnitude of the sharing rule effect on correlations and standard deviations. Even the basic power utility function allows for a match of the level of correlations and standard deviations, although we cannot replicate the level of the risk free rate and the aggregate mean excess return. In our model, unlike in many habit models as for example in [Campbell & Cochrane 1999], time variation in risk aversion is endogenous. Further, risk sharing is implemented by trade in the stock market and a riskless security. Importantly, trade in securities constitutes an observable economic channel that provides information about investor heterogeneity. Models with a representative investor can, at best, capture time variation due to investor heterogeneity in reduced form. Admittedly, analyzing trading activity in models, but also in the data, is challenging. Even so, we solve for the quadratic variation, or volatility, of portfolio policies as measure of trading intensity and put the business cycle implications on trade to the test. The endogenous variation in risk aversion and the trading implications set apart our work from studies such as [Chue 2005] and [Aydemir 2008], who employ habit models with a representative investor and thus exogenously specify an increase in risk aversion and in correlations in business cycle troughs.[4]

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[4][Aydemir 2008], [Ribeiro & Veronesi 2002], and [Pavlova & Rigobon 2008] model comovements in international stock market returns. [Aydemir 2008] extends the model in [Chue 2005] to an international setting and studies correlations with perfect and imperfect risk sharing across countries. [Ribeiro & Veronesi 2002] analyze fundamental country processes that are jointly affected by an unobservable global business cycle factor. Time variation in correlations of asset returns arise from the learning activity of the representative investor. [Pavlova & Rigobon 2008] study stock prices, exchange rates and the correlations of stock prices with multi-

We employ preference heterogeneity and "catching up with the Joneses" preferences, or ratio habit preferences, see [Chan & Kogan 2002].[5] As with standard power utility, countercyclical variations of endogenous quantities with the ratio habit preferences result from preference heterogeneity. One useful byproduct of ratio habits comes in the form of providing a good replication of the level of the risk free rate, the aggregate stock market mean excess return, and the aggregate stock market return standard deviation and this, in particular, relative to [Campbell & Cochrane 1999].[6] There are, however, two other more important reasons why we use ratio habit preferences. First, [Chan & Kogan 2002] preferences allow us to utilize relative consumption, consumption in excess of habit, as a stationary state variable. Second, since relative consumption is procyclical it may be interpreted as a business cycle indicator that defines booms and recessions.

The model captures only a portion of all potential differences among investors. For example, investors trade only in the market portfolio but do not trade in individual securities or in industry portfolios. Two implications arise from this. Firstly, stock market correlations in the model can differ only due to exogenous differences in dividend correlations since *endogenous* variation is a pure market effect. We, therefore, focus in our calibrations on average industry stock market correlations. Secondly, because our model has no cross-sectional implications, such as the influence of heterogeneous beliefs or of heterogeneous overconfidence in stock picking on correlations, this should bias against model calibrations as well as against empirical tests even at business cycle frequencies.

Parametrized examples of our model with ten industries explain conditional stock market return correlations.[7] Correlations are asymmetric, increasing during bad states of the economy but remaining low and stable during good states of nature. This feature of the model is consistent with [Longin & Solnik 2001], who argue that correlations increase in bear markets, but not in bull markets. Further, conditional stock market return correlations and return standard deviations move together, along with aggregate risk aversion, over the business cycle. This is an important matter

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goods. In their model, spill-over effects arise because of binding portfolio constraints.

[5]Notable papers addressing asset pricing with preference heterogeneity include [Dumas 1989] and [Wang 1996].

[6]The micro evidence on asset allocations suggests that investors have constant relative risk aversion preferences, see [Brunnermeier & Nagel 2008]. The ratio habit preferences in [Abel 1990], [Chan & Kogan 2002], and in our model, unlike other habit models including [Campbell & Cochrane 1999], imply constant relative risk aversion preferences.

[7]See [Bollerslev *et al.* 1988] and [Moskowitz 2003] for evidence on time variation in return correlations and especially for correlations moving over the business cycle. Other recent work on correlations include [Ang & Chen 2001] and [Dumas *et al.* 2003].

since the empirical literature on correlations argues for correlations and standard deviations moving together, [Longin & Solnik 1995] and [Karolyi & Stulz 1996], but also produces evidence consistent with the argument that these results in the literature can be spurious, [Longin & Solnik 2001]. Our calibrations match the change in correlations from boom to recession. They also match the variations in the standard deviation of the market return and in average industry standard deviations. Importantly, our calibration results are almost unchanged whether one uses volatile industry dividends and repurchases as aggregate consumption or industry dividends including repurchases and another large and smooth claim, which helps to replicate the time-series properties of aggregate consumption. Further, our calibration results are also almost unchanged whether one identifies booms and recessions via realized consumption growth or via relative consumption.

Figure 4.1 shows plots of average pairwise industry correlations, annualized 3-year ahead continuously compounded average market excess returns, average industry standard deviations and average quadratic variation of HP-filtered industry turnover with gray shaded areas denoting NBER recessions. We can see from the plots significant increases of the time series in recessions or sometimes around recession periods. Industry stock market correlations, market excess returns (industry excess returns), and standard deviations and quadratic variations of turnover show positive relations, correlation coefficients, with the NBER recession indicator.

4.2 The Economy

This section introduces a continuous-time exchange economy defined on the finite time span $[0, T]$, in which N risky securities and one riskless security are traded. Preferences are described by heterogeneous constant curvature of the utility function and linear process for the standard of living, "catching up with the Joneses" preferences, as in [Chan & Kogan 2002].

The dividend processes are assumed to have the following dynamics

$$d\delta_i(t) = \delta_i(t) \left(\mu_{\delta_i} dt + \sigma_{\delta_i}^\top dW(t) \right) \tag{4.1}$$

where $\delta_i(0) > 0$ and where $\top$ denotes the transpose with $i = 1, ..., N$. To simplify matters, we assume that the drift rates of the dividends μ_{δ_i} and the diffusion coefficients $\sigma_{\delta_i} \in R^N$ are constants with σ_δ positive definite. The N-dimensional Brownian motion, W, is defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$.[8]

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[8]The probability space is defined over the finite time horizon $[0, T]$, where Ω is the state space, $\mathcal{F}$ denotes the σ-algebra, P represents the probability measure, and the information structure $\mathcal{F}_{(.)}$ is generated by observations of the dividend processes with $\mathcal{F}_T = \mathcal{F}$.

Figure 4.1: **Correlations, Returns, Standard Deviations, and Quadratic Variation of Turnover.** The figure shows plots of average pairwise industry correlations, annualized 3-year ahead continuously compounded market excess returns, average industry standard deviations and average quadratic variation of HP-filtered industry turnover for the sample period 1927 to 2008. Industry return data is from Kenneth French's webpage. Turnover is the sum of industry turnover (using Kenneth French's industry classification) for firms appearing jointly in Compustat and CRSP.
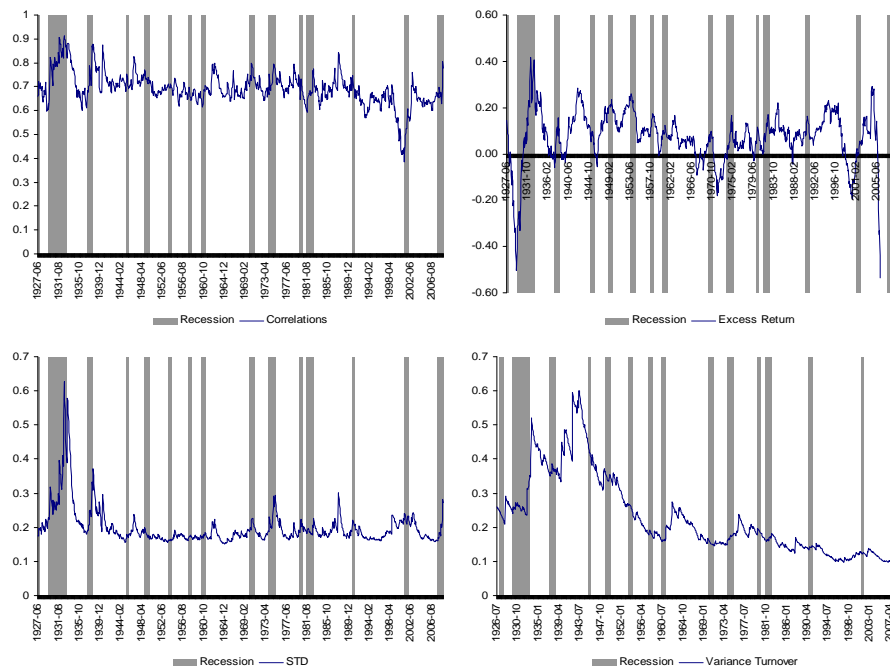
Aggregate consumption is the sum of the N dividend streams, i.e.,

$$C(t) = \sum_{i=1}^{N} \delta_i(t). \tag{4.2}$$

The dynamics of aggregate consumption are given by

$$dC(t) = C(t) \left(\mu_C(t)dt + \sigma_C(t)^\top dW(t) \right) \tag{4.3}$$

where

$$\begin{aligned} \mu_C(t) &= s_\delta(t)^\top \mu_\delta \\ \sigma_C(t) &= s_\delta(t)^\top \sigma_\delta, \end{aligned} \tag{4.4}$$

by definition dividend shares are given by

$$s_\delta(t) = (s_{\delta_1}(t), s_{\delta_2}(t), ..., s_{\delta_N}(t))^\top = \left(\frac{\delta_1(t)}{C(t)}, \frac{\delta_2(t)}{C(t)}, ..., \frac{\delta_N(t)}{C(t)} \right)^\top \tag{4.5}$$

where

$$\begin{aligned} \mu_\delta &= (\mu_{\delta_1}, \mu_{\delta_2}, ..., \mu_{\delta_N})^\top \\ \sigma_\delta &= (\sigma_{\delta_1}, \sigma_{\delta_2}, ..., \sigma_{\delta_N})^\top. \end{aligned} \tag{4.6}$$

Dividend share processes follow

$$ds_{\delta_i}(t) = s_{\delta_i}(t) \left(\left(\mu_{s_i}(t) - \sigma_{s_i}(t)^T \sigma_C(t) \right) dt + \sigma_{s_i}(t)^T dW(t) \right) \tag{4.7}$$

where

$$\begin{aligned} \mu_{s_i}(t) &= \mu_{\delta_i} - s_\delta(t)^T \mu_\delta \\ \sigma_{s_i}(t) &= \sigma_{\delta_i} - \sigma_\delta^T s_\delta(t). \end{aligned} \tag{4.8}$$

4.2.1 Financial Markets

There are N stocks, with stock i representing the claim to dividend i. Stock price processes show the following dynamics

$$dS_i(t) + \delta_i(t)dt = S_i(t) \left(\mu_i(t)dt + \sigma_i(t)^\top dW(t) \right) \tag{4.9}$$

with boundary conditions $S_i(T) = 0$, where $\mu_i(t)$ and $\sigma_i(t)$ as well as $S_i(0)$ represent equilibrium quantities. The locally risk free asset, $B(t)$, generates an instantaneous rate of return equal to $r(t)$. Price dynamics are

$$dB(t) = r(t)B(t)dt \tag{4.10}$$

with $B(0) = 1$. The equilibrium quantity $r(t)$ clears the bond market if the bond is in zero net supply.

4.2.2 Investors

We consider two classes of investors consuming continuously over time. The functional form of investors' utility, see [Chan & Kogan 2002], is given by

$$U_j(C, X) = E_0 \left[\int_0^T e^{-\rho t} u_j(C_j(t), X(t)) dt \right] \tag{4.11}$$

where $\rho > 0$, u represents the instantaneous utility function, C stands for the consumption rate, and X is the external economy wide living standard with $j = 1, 2$. Above $E_t[.]$ denotes the $\mathcal{F}_t$-conditional expectation with respect to the probability measure P.

The instantaneous utility function is given by

$$u_j(C_j(t), X(t)) = \frac{1}{1 - \gamma_j} \left(\frac{C_j(t)}{X(t)} \right)^{1 - \gamma_j} \tag{4.12}$$

where γ measures the local curvature of the utility function. We follow [Chan & Kogan 2002] in interpreting γ as the relative risk aversion parameter. The average historical standard of living evolves accordingly to

$$x(t) = x(0)e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-u)} c(u) du \tag{4.13}$$

where $x(t) = \log(X(t))$ and $c(t) = \log(C(t))$. The parameter λ governs the dependency of the living standard, $x(t)$, on past aggregate consumption experiences.

Relative consumption, $w(t) = c(t) - x(t)$, instead of aggregate consumption serves as state variable and follows a mean-reverting process

$$dw(t) = \lambda \left(\overline{w}(t) - w(t) \right) dt + \sigma_C(t)^\top dW(t) \tag{4.14}$$

with long-run mean

$$\overline{w}(t) = \frac{\mu_C(t) - \frac{1}{2}\sigma_C(t)^\top \sigma_C(t)}{\lambda}. \tag{4.15}$$

When consumption in the economy is high then $w(t)$ is also high. Hence, we interpret $w(t)$ as a business cycle indicator.

4.2.3 Equilibrium

Conditional on endowments and preferences, equilibrium is a collection of allocations and prices such that individuals' consumption profiles are optimal, also requiring the clearing of the commodity market, stock markets and bond market. Equilibrium is also concerned with portfolio policies that finance optimal consumption. We assume that equilibrium exists.[9] Construction of equilibrium (when it exists) and propositions appear in the Appendix C.1.

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[9]See [Anderson & Raimondo 2008b] for results on existence of equilibrium. Their results, however, can be applied to our model (with minor changes) only if we turn off the external standard of living.

4.2.4 Correlations, Returns, Volatility, and the Quadratic Variation of Portfolio Policies

The instantaneous equity returns, $dR(t)$, evolve accordingly to

$$dR_i(t) = \mu_i(t)dt + \sigma_i(t)^\top dW(t). \qquad (4.16)$$

The conditional variance, with respect to the probability measure P, of stock i's instantaneous return is defined as follows

$$Var_t(dR_i(t)) = \|\sigma_i(t)\|^2 dt \qquad (4.17)$$

where

$$\|\sigma_i(t)\| = \sqrt{\sigma_i(t)^\top \sigma_i(t)}. \qquad (4.18)$$

The conditional covariance between asset i and k, with $k = 1, .., N$, is given by

$$Cov_t\left(dR_i(t), dR_k(t)\right) = \sigma_i(t)^\top \sigma_k(t)dt. \qquad (4.19)$$

Then, we calculate the conditional correlation[10] between asset i and asset k as

$$Corr_t\left(dR_i(t), dR_k(t)\right) = \frac{\sigma_i(t)^\top \sigma_k(t)}{\|\sigma_i(t)\| \|\sigma_k(t)\|}. \qquad (4.20)$$

The trading activity of investors separates our economy from other economies that model stock market return correlations with a representative investor. As trading volume or turnover in continuous-time economies is not well defined, i.e., it is infinite, we instead employ the quadratic variation, or volatility, of portfolio policies as measure of trading intensity.[11] The quadratic variation, QV, of optimal portfolio policies is by definition

$$d\left[\pi_j(t)\right] = QV_j(t)dt \qquad (4.21)$$

where π denotes investors portfolio policies, Equations C.10-C.11, with $j = 1, 2$. Since Equation 4.21 is tedious we refer the reader to the Appendix C.3.

4.3 Correlations with Two Stocks

In this section, we provide numerical examples of the effects of risk aversion and preference heterogeneity (in risk aversion) on correlations in two-stock economies. Further, we elaborate briefly on the influence of other fundamentals on correlations.

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[10]In the remainder of the paper, we use the terms 'conditional correlation' and 'correlation' interchangeably.

[11][?] and [Longstaff & Wang 2008], among others, also employ the quadratic variation of portfolio policies to measure trading intensity.

We solve the economies through Monte Carlo simulations. Technical details of the simulations are in Appendix C.4. Our plots show correlations as a function of the consumption share of the most risk averse investor and the share of the first dividend process with turned off habit process. The investor with high risk aversion consumes a larger share of aggregate consumption when consumption is low and vice versa, see [Dumas 1989] and [Wang 1996] among others. With [Chan & Kogan 2002] preferences the plots show correlations as function of relative consumption, w, and the share of the first dividend process. Relative consumption is path dependent and not identical to the consumption share of the most risk averse investor, albeit correlated with it, with turned on habit process. Overall, the shape of the correlation functions are not sensitive to the identification of the state of the economy, via habit or via aggregate consumption. Figures with correlations as a function of the consumption share of the most risk averse investor and the share of the first dividend process with [Chan & Kogan 2002] preferences are available from the authors.

The economy contains two independent and identically distributed dividend (i.i.d.) processes. The dividend drift coefficients μ_δ are set at 0.02, and the dividend diffusion coefficients σ_δ are set at 0.05. We set the starting values for the dividends to 1 and the maturity of the economy to 50 years. The utility weight in the social planner problem is 0.5. The discount rate for time preference of both investors is set to 1% and while the habit persistence, λ, is set at 0.1. The parameters are motivated by model calibrations except the i.i.d. feature of dividends, the starting values for dividends, the maturity of the economy and the endowments.

4.3.1   Homogeneous Preferences

The economy is populated by (i) two homogeneous investors with logarithmic utility, (ii) two homogeneous investors with power utility with a risk aversion coefficient of 3, or (iii) two homogeneous investors with power utility with a risk aversion coefficient of 10. For each of these economies we consider two cases: habit process turned off and habit process, as in Equation 4.13, turned on.

The three top plots in Figure 4.2 show the correlation function with homogeneous investors and turned off habit process. In the top left plot, the representative investor is of a logarithmic type. Although the dividend processes are uncorrelated, stock returns are positively correlated with correlations up to 0.029. Apparently, the correlation function reaches its maximum when the dividend processes contribute 50% to aggregate consumption.

The top middle and right plots highlight the effect of risk aversion on correlations with risk aversion coefficients of 3 and 10, respectively. Holding everything else constant, the correlation functions are increasing in risk aversion. Correlations reach 0.05 in the economy with a risk aversion coefficient of 3, and 0.086 in the economy with a risk aversion of 10. The top right plot in Figure 4.2 suggests that the

correlation function peaks twice, once before the 50% share of the dividends, or the diagonal of the two Brownian motions, and once after the 50% share. This feature of the correlation function is not due to the heightened risk aversion. The two other versions of the economy also exhibit this feature, although we cannot detect this in the plots because the dip in correlations is too small.[12]

The bottom plots of Figure 4.2 show correlation functions when the habit process is turned on. As the Malliavin derivative in Equation C.15 for the logarithmic case is independent of the external standard of living the diffusion coefficient do not depend on the Malliavin derivative. Therefore, the correlation function is identical to the correlation function with logarithmic investor and turned off habit process, bottom left plot versus top left plot. The top middle and right plots highlight the effect of the habit process with homogeneous investors. The procyclical variation in the correlations, low correlations when the habit level is low and vice versa, contradicts [Bollerslev *et al.* 1988], [Moskowitz 2003], and [Longin & Solnik 2001]. This result is related to the procyclical variation in expected stock returns and volatility, also inconsistent with the data, with homogeneous risk aversion in [Chan & Kogan 2002]. We are therefore led to conclude that the ratio habit preferences in our model economy cannot match countercyclical variations in asset pricing moments. Nevertheless, as without ratio habits, correlations increase in the level of risk aversion. Importantly, habit persistence significantly increases correlations, bottom middle and left plot. One last observation we make is that correlations reach a local minimum when the dividend share of dividend one equals 0.5.

Why does the correlation function peak twice? The shape of the correlation function in top plots of Figure 4.2 are entirely determined by stock price diffusion coefficients. The diffusion terms, for the economies (i) to (iii), are depicted in top plots and bottom left plot of Figure 4.3 and document two important features of the model diffusion terms. First, own dividend shocks produce larger diffusion terms, σ_{11} and σ_{22}, than dividend shocks of other dividend streams, σ_{12} and σ_{21}. Second, diffusion terms of one stock are partially offsetting, that is when σ_{11} is high (dividend one is high) then σ_{12} is low and vice versa. Note that this relation holds for both stocks. The diffusion term σ_{11} peaks when the share of dividend one is between 0.5 and 1 and reaches a minimum when the share of dividend one is between 0 and 0.5. Consumption risk reaches a minimum when the share of dividend one equals 0.5 and

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[12][Cochrane *et al.* 2008] consider a two stock economy with a logarithmic investor. Their Figure 7 is analogous to the three plots in the top row of Figure 4.2, and identical to the top left plot in Figure 4.2 of this paper. Note that the correlation function in the top left plot is smaller, 0.025 instead of 0.3, than that found in Figure 7 in [Cochrane *et al.* 2008]. This is because we use a dividend volatility of 0.05 while [Cochrane *et al.* 2008] use a dividend volatility of 0.2.

Figure 4.2: **Correlations with Homogeneous Preferences.** The figure shows the conditional return correlation between stock 1 and stock 2 as a function of dividend share, $\delta_1(0)/(\delta_1(0) + \delta_2(0))$, or as a function of dividend share and relative consumption, Equation 4.13. The figure contains six plots with the following risk aversion, γ, for the first and second investor, respectively: Log - Log, 3 - 3, 10 - 10 (top row plots) with standard power preferences and Log - Log, 3 - 3, 10 - 10 with [Chan & Kogan 2002] preferences (bottom row plots). Investors time preference, ρ, is set at 0.01. Habit persistence, λ, is set at 0.1. Both stocks have identical dividend drift and diffusion coefficients. The drift is set at 0.02 while the diffusion coefficient is 0.05. Dividends are uncorrelated. The horizon of the economy, T, is set at 50 years.
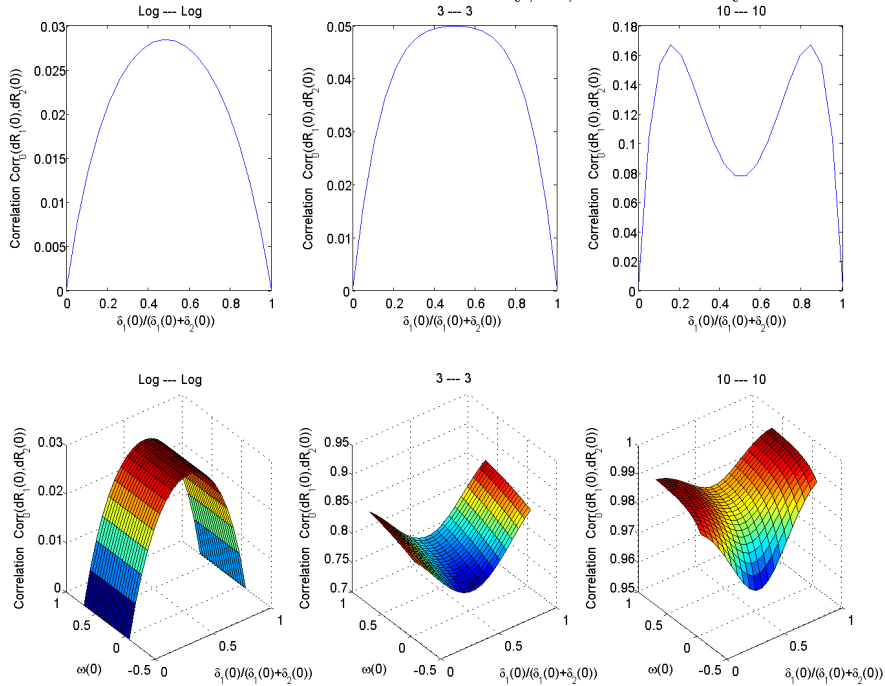
Figure 4.3: **Correlations and Diffusion Terms.** The figure shows equity diffusion terms of three economies, Log - Log, 3 - 3, 10 - 10, as a function of dividend share, $\delta_1(0)/(\delta_1(0) + \delta_2(0))$, in a stylized economy with two stocks. The sum of the diffusion term products in the bottom middle and left plots corresponds to the covariance between stock 1 and stock 2. Investors time preference, ρ, is set at 0.01. Both stocks have identical dividend drift and diffusion coefficients. The drift is set at 0.02 while the diffusion coefficient is 0.05. Dividends are uncorrelated. The horizon of the economy, T, is set at 50 years.
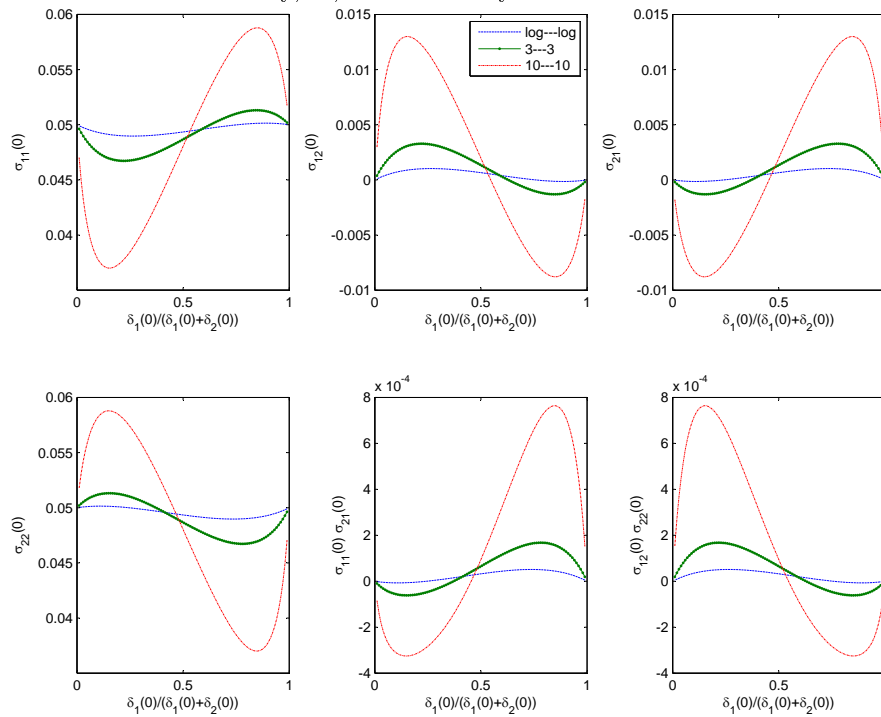
reaches maximum at dividend one shares of 0 and 1. The bottom middle and bottom right plots show the products of $\sigma_{11} \times \sigma_{21}$ and $\sigma_{12} \times \sigma_{22}$ and these functions resemble the shapes of σ_{11} and σ_{22}. The sum of the two products yields the covariance between stock one and stock two. The overall conclusion that we draw from bottom middle and bottom right plots of Figure 4.3 is that the covariance between stock one and stock two peaks twice simply because of the peak in σ_{11} and in σ_{22}. As correlations are nothing but a normalization of covariances it is also the case that correlations peak twice due to the peak in σ_{11} and in σ_{22}.

4.3.2 Risk Aversion Level Effect

The three top plots in Figure 4.2 show that correlations increase in risk aversion. We now develop the intuition for this result. Consider a setting with two or many uncorrelated dividends. When one dividend experiences a positive Brownian shock, the net effect of this shock on the price and the return of the stock associated with the dividend stream must be positive. The positive net return has two components. A positive dividend effect and a negative discounting effect. The negative discounting effect is caused by an increase in the market price of risk related to the positive dividend shock of the asset. The market price of risk increases as a consequence of the increased dividend risk, i.e., consumption risk, relative to second dividend stream or other dividend streams. For the second asset or other assets the discounting effect on the return is strictly positive as its or their market price of risk falls. Hence all assets experience positive returns after a positive dividend shock to one of the assets. Importantly, Figure 4.4 shows that the impact of the discounting effect on all assets due to one dividend shock to one of the assets increases in risk aversion. Hence, the larger the risk aversion, the larger is the negative discounting effect and the smaller is the net return on the asset that experiences the positive dividend shock. Further, the larger the risk aversion, the larger is the positive discounting effect and the larger is the return on the second asset or all other assets. Therefore, the higher is the risk aversion the higher is the correlation between assets returns. We conjecture, as highlighted in Figure 4.4, that correlations converge to one when risk aversion approaches infinity. Numerical examples support this conjecture. However, proving the conjecture goes beyond the goals of this paper.

4.3.3 Heterogeneous Preferences

We now introduce heterogeneity in risk aversion. We consider the following risk aversion pairs: $3 - \log$ and $10 - \log$. All other parameters are kept constant. The two top plots in Figure 4.5 show the correlation function with turned off habit process while the bottom plots show the correlation function when the habit process is turned on.

Figure 4.4:   **Correlations and Risk Aversion.** The figure shows equity returns due to a positive dividend shock to dividend one in a stylized economy with two stocks. The return of stock one —gray shaded area— increases in (the own) positive dividend shock, dividend effect, but decreases in the discounting effect —white area— due to a two trees effect, see [Cochrane *et al.* 2008]. R_1 denotes the net effect. The discounting effect —R_2— on the second stock, discounting cross-effect, is positive. The dividend effect and the discounting cross-effect cause positive correlation. The correlation, via the discounting effects, increases in risk aversion.
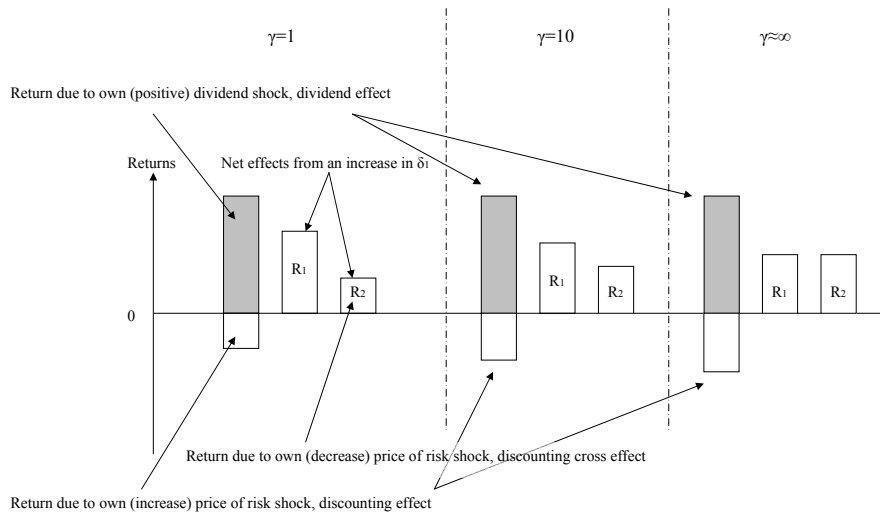
Introducing heterogeneity in risk aversion, top plots of Figure 4.5, has a strong and interesting impact on the correlation function. Correlations are increasing in the bad state of nature, that is, correlations are the larger the lower is the realization of aggregate dividends. Importantly, from the top left plot we learn that the correlation function increases in the difference in risk aversion. This sharing rule effect on correlations and other quantities is large as correlations in the $3 - \log$ economy range from 0.05 to slightly above 0.2 while in the $10 - \log$ economy correlations reach almost 0.8.

With [Chan & Kogan 2002] preferences, bottom plots of Figure 4.5, correlations range from slightly below 0.4 to approximately 0.6 in the $3 - \log$ economy and range from 0.5 to almost 1.0 in the $10 - \log$ economy. Hence, we conclude again that habit persistence significantly increases correlations. As without habit persistence correlations increase in the bad state of nature, low ω. Because with habit persistence and homogeneous preferences correlations decrease in ω it must be that heterogeneity in risk aversion drives this result.

4.3.4 Sharing Rule Effect

Investors with high risk aversion hold a larger fraction of the economy after a drop in aggregate consumption. Because aggregate risk aversion is wealth weighted it increases after a drop in aggregate consumption. The increase in risk aversion leads to an increase in correlations due to the risk aversion level effect. This, however, does not explain the entire increase in correlations. For instance, correlations in top middle plot of Figure 4.2 with aggregate risk aversion set at 3 reach 0.05 while correlations in top left plot of Figure 4.5 with risk aversion pair set at $3 - \log$ reach 0.2. Because aggregate risk aversion is bounded from above at 3 the level of risk aversion cannot explain the equilibrium correlations in Figure 4.5.

Why are correlations significantly larger in an economy with heterogeneous investors than the correlations obtained in homogeneous investor economies populated by the investor with the largest risk aversion from the heterogeneous investor economy? In the model with heterogeneous investors the marginal utility of the aggregate investor varies due to changes in aggregate consumption and due to changes in consumption shares, or wealth shares, of individual investors. When small changes in consumption imply large changes in relative wealth then endogenous variations, including correlations, are high.4.6 plots the consumption share of the most risk averse, the slope of the consumption share, the quadratic variation, the correlation and the volatility of the market. It is evident that in the model, all these quantities are highly correlated. When the slope of the sharing rule is steep, then the risk aversion of the representative agent is highly volatile. The volatility of the risk aversion implies volatile market price of risk. Since all risk factors depend on the risk aversion of the

Figure 4.5: **Correlations with Heterogeneous Preferences.** The figure shows the conditional return correlation between stock 1 and stock 2 as a function of dividend share, $\delta_1(0)/\left(\delta_1(0) + \delta_2(0)\right)$, and consumption share of the most risk averse investor, Equation C.2, or as a function of dividend share and relative consumption, Equation 4.13. The figure contains four plots with the following risk aversion, γ, for the first and second investor, respectively: 3 - Log and 10 - Log (top row plots) with standard power preferences and 3 - Log and 10 - Log with [Chan & Kogan 2002] preferences (bottom row plots). Investors time preference, ρ, is set at 0.01. Habit persistence, λ, is set at 0.1. The weight on investors is 0.5. Both stocks have identical dividend drift and diffusion coefficients. The drift is set at 0.02 while the diffusion coefficient is 0.05. Dividends are uncorrelated. The horizon of the economy, T, is set at 50 years.
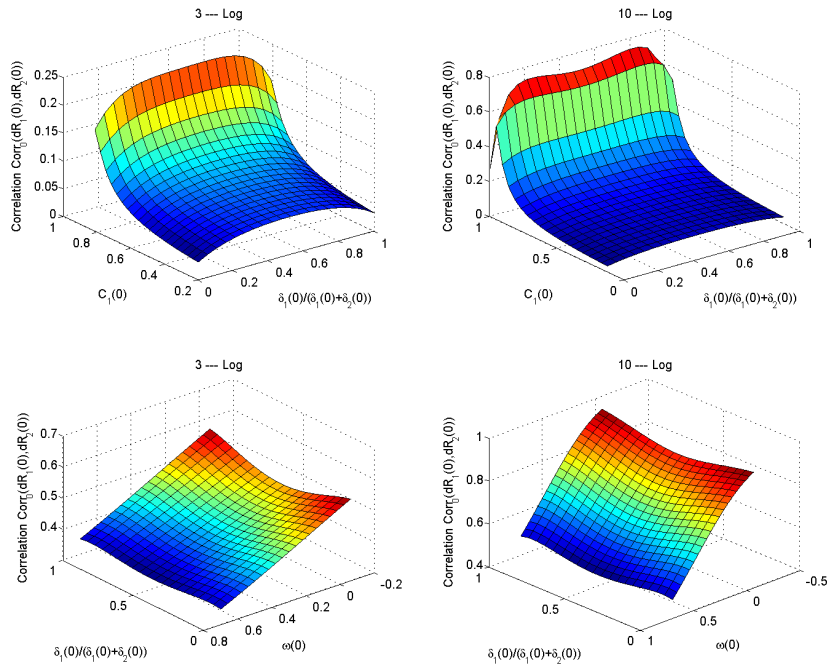
representative agent, this effect has an economy wide impact on all discount rates. The highly volatile market price of risk translates into highly volatile discount rates, which in turn leads to increased stock return volatility. The correlations increase because of the fact that risk aversion is common for all risk factors, and when the sharing rule is steep most of the volatility is induced via the volatility of the risk aversion. The quadratic variation is high because the least risk averse agent must deleverage. However, in order to deleverage his position he must sell to the more risk averse agent. In oder for the more risk averse agent to take a larger position in the stock market, the price of risk must increase, and thus stock prices fall. The top left plot of Figure 4.7 shows the conditional return correlations between stock 1 and stock 2 obtained in an economy with risk aversion pairs $3-Log$. The plot also shows the conditional return correlations between stock 1 and stock 2 obtained in an economy with homogeneous risk aversion in which aggregate risk aversion corresponds to aggregate risk aversion in the heterogeneous investor economy. The difference in correlations between the homogeneous economy and the heterogeneous economy is entirely due to endogenous variations. The top right plot shows the slope of the consumption or wealth sharing rule between the agents. We can see that the shape of the slope of the sharing rule and the difference between homogeneous and heterogeneous correlations are identical. Endogenous variation in equilibrium quantities, which one can obtain by introducing heterogeneous investors, lead to the increase in correlations above the correlation in a corresponding homogeneous investor economy.

The bottom left plot of Figure 4.7 shows the conditional return correlations between stock 1 and stock 2 obtained in an economy with risk aversion pairs $30-Log$ as well as the conditional return correlations between stock 1 and stock 2 obtained in a corresponding economy with homogeneous risk aversion. We learn from the plots that correlations increase in the difference in risk aversion between investors.

Lastly, note that when aggregate consumption declines to very low levels then and only then correlations drop in the bad state of nature. It is, therefore, an important calibration question whether a fall in correlations in the bad state of nature may be observed for realizations of the model economy.

4.3.5 Empirical Predictions

Empirical literature on correlations, on the one hand, argues for correlations and standard deviations moving hand in hand over the business cycle.[13] On the other

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[13][Longin & Solnik 1995] find that correlations rise in periods of high volatility. [Hamilton & Lin 1996] argue that stock market volatility and the business cycle comove.

Figure 4.6: **Sharing Rule.** The figure shows the fraction of consumption consumed by the most risk avers agent (sharing rule), the slope of the sharing rule, the quadratic variation, return correlations between stock 1 and stock 2 and the stock market volatility as a function of aggregate consumption. The risk aversion pair is $3 - Log$ where the weight on investors is 0.5. Both stocks have identical dividend drift and diffusion coefficients. The drift is set at 0.02 while the diffusion coefficient is 0.05. Dividends are uncorrelated. The horizon of the economy, T, is set at 50 years.
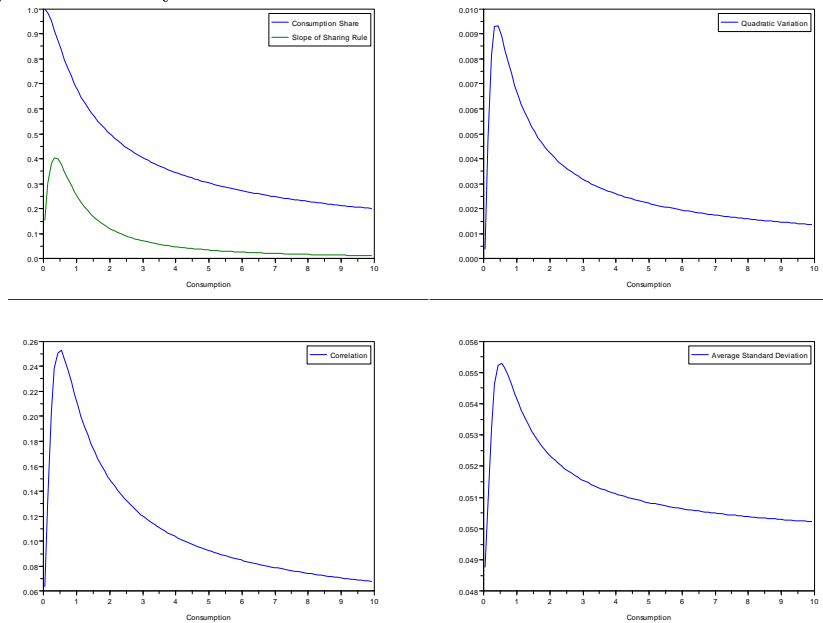
Figure 4.7: **Sharing Rule Effect.** The figure shows the conditional return correlations between stock 1 and stock 2 as a function of aggregate dividends, $C(0)$, and slopes of the sharing rule as a function of aggregate dividends. Plots show correlations of economies with heterogeneous risk aversion, γ, for the first and second investor, respectively: 3 - Log and 30 - Log as well as homogeneous risk aversion in which aggregate risk aversion corresponds to aggregate risk aversion in the heterogeneous investor economy. Investors time preference, ρ, is set at 0.01. Habit persistence, λ, is set at 0.1. The weight on investors is 0.5. Both stocks have identical dividend drift and diffusion coefficients. The drift is set at 0.02 while the diffusion coefficient is 0.05. Dividends are uncorrelated. The horizon of the economy, T, is set at 50 years.
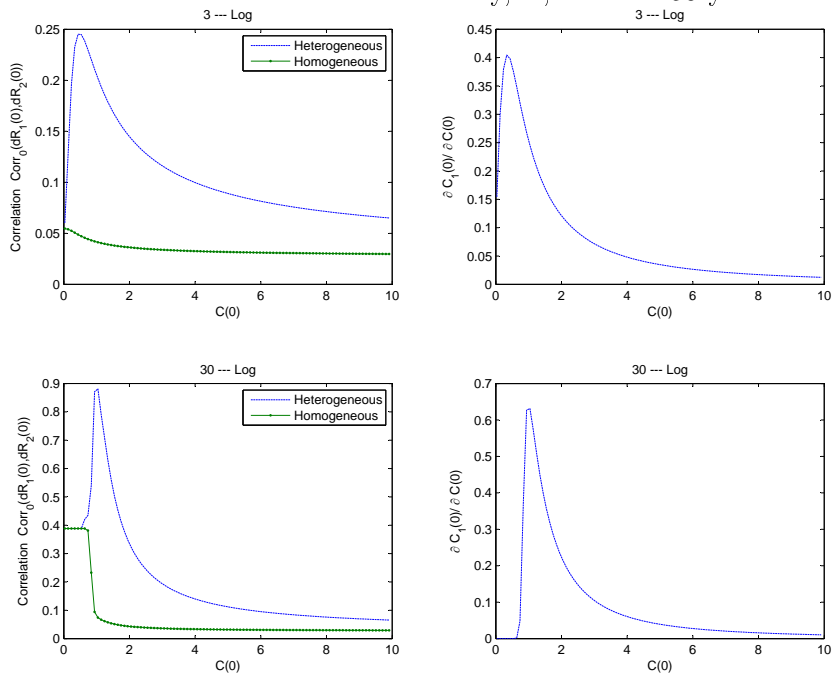
hand, [Longin & Solnik 2001] argue that these results in the literature can be spurious. Therefore, it is natural to ask for relevant empirical implications of our model for correlations and standard deviations.

The plots in Figure 4.8 show the fraction of total wealth held by the most risk averse investor in the two stocks, the quadratic variation of the most risk averse investors' portfolio, the correlation between stock one and stock two and the standard deviation of the market. We see from the plots that all quantities increase in the bad state of nature. Therefore, our model suggests that the findings in the empirical literature regarding heightened correlations and heightened standard deviations in bad times may not be spurious. The plots in Figure 4.9 show the same relations with [Chan & Kogan 2002] preferences and the same conclusion applies.

One way to test the countercyclical relation between correlations, returns, standard deviations and variance of portfolio policies is to employ a business indicator. Further, this opens the door for a principle component analysis. To our best knowledge such a relation between these four time series has not been studied in the empirical literature.[14]

4.3.6 The Influence of Fundamentals other than Preferences on Correlations

For brevity, we do not report additional figures for the model correlations, but the figures are available from the authors. First, return correlations are increasing in the dividend growth. Second, studying the influence of the dividend volatility on correlations we find that return correlations are increasing in dividend volatility. Third, we introduce correlation at the dividend level. We gain the following insights from this exercise: Return correlations increase compared to the baseline case, and endogenous correlation still play an important role since correlations are always above the dividend correlation, except when dividend correlation approaches one, and endogenously generated correlation is smaller when dividends are already correlated. Forth, we analyze economies with different time horizons: 20 years and 100 years and learn that the shape of the correlation function is slightly increasing in the horizon.[15]

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[14][Lamoureux & Lastrapes 1990] argue that volume explains standard deviations. [Gallant *et al.* 1992] show that trading volume has positive relation with absolute volatility changes. [Longin & Solnik 1995] argue that correlations and standard deviations move together.

[15]In an earlier version of this paper, we also considered economies with heterogeneous time preferences and heterogeneous beliefs (with and without learning). Calibrations and plots of these economies are available upon request.

Figure 4.8:   **Wealth Share, Quadratic Variation of Portfolio,
Correlations and Standard Deviations with Heterogeneous
Preferences.**   The Figure in the top left corner shows total wealth
of the most risk averse investor to total wealth. The figure in the top
right corner shows the quadratic variation of equity of the most risk
averse investor. The bottom left corner shows the correlation between
stock one and stock two. The bottom right corner shows the standard
deviation of the market portfolio. Risk aversion coefficients, γ, are: 3
- Log. Investors time preference, ρ, is set at 0.1. Habit persistence, λ,
is set at 0.1. The weight on investors is 0.5. Dividend specifications
are as in Figure 4.2. The horizon of the economy, T, is set at 50 years.
Consumption share is consumption of investor 1, Equation (C.2).
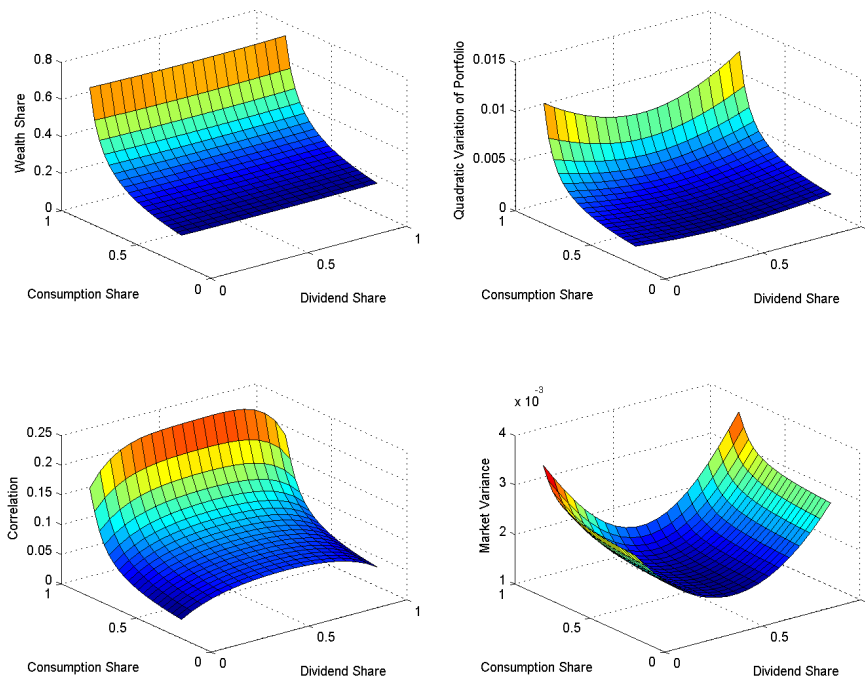
Figure 4.9: **Wealth Share, Quadratic Variation of Portfolio, Correlations and Standard Deviations with [Chan & Kogan 2002] Preferences.** The Figure in the top left corner shows the fraction of the total wealth of the most risk averse investor in the market portfolio. The figure in the top right corner shows the quadratic variation of equity of the most risk averse investor. The bottom left corner shows the correlation between stock one and stock two. The bottom right corner shows the standard deviation of the market portfolio. Risk aversion coefficients, γ, are: 3 - Log. Investors time preference, ρ, is set at 0.1. Habit persistence, λ, is set at 0.1. The weight on investors is 0.5. Dividend specifications are as in Figure 4.2. The horizon of the economy, T, is set at 50 years. Consumption share is consumption of investor 1, Equation C.2, while habit level is defined in Equation 4.13.
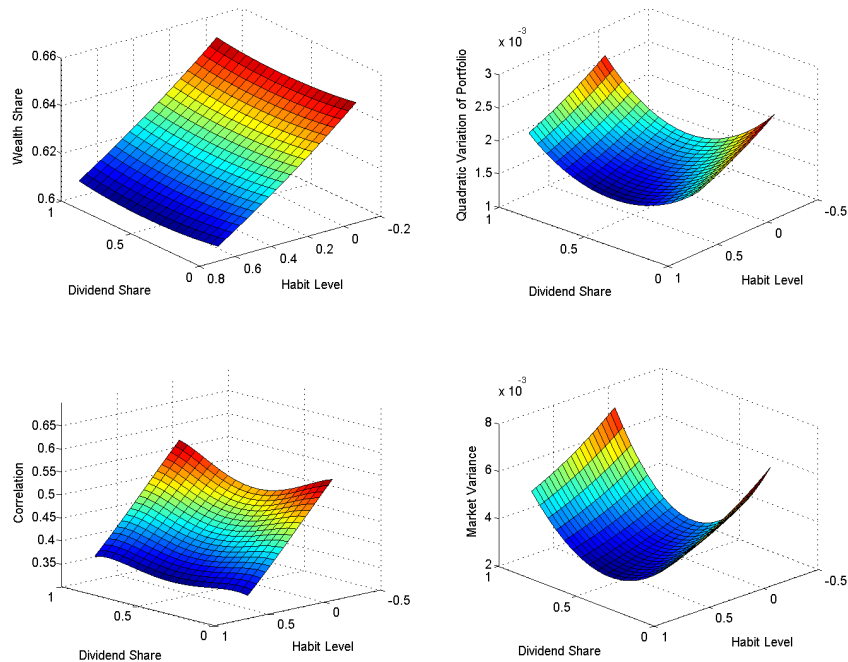
We also considered economies with up to one hundred investors with heterogeneous risk aversion. We found that the shape and level of the correlation function depends largely on the difference in risk aversion between two consecutive investors. With Normally distributed risk aversion and with uniformly distributed risk aversion the correlation function decreases in aggregate consumption except for very low realizations. Overall, two heterogeneous investors capture the impact of heterogeneity in risk aversion on correlations well, although the distribution of investors may be important for correlations.

4.4   Calibration

This section extends the model to ten stocks —indexes— to address empirically three questions: Can our model match the average level of correlations? And, more importantly,[16] can our model match the change in correlations as well as the change in other equilibrium quantities over the business cycle?

We gather monthly dividends, stock market returns, repurchases and market capitalization from CRSP, for the sample period, 1951 to 2005. From the monthly data, we aggregate dividends to yearly data. The data is grouped into 10 industries using Kenneth French's industry classification. We then split the sample into periods of boom and recession using the NBER business cycle indicator. The summary statistics for dividends with and without repurchases as well as stock market returns can be found in Table 4.1 while correlations statistics are in Table 4.2.

Next, the dividend processes in our model are calibrated to the data using the average growth and the average variance. Since our theoretical model assumes that the dividend volatility and dividend correlation are constants, we do not make a separate calibration for the booms and recessions. Furthermore, prior research [see [Ribeiro & Veronesi 2002]] has shown that correlations of the fundamentals are more stable than the correlations of returns over the business cycle. However, note that the volatility of dividends is increasing slightly in recessions and this clearly works against our calibrated models. For a second calibration we employ an another large and smooth claim, which helps to replicate the time-series properties of aggregate consumption.

In the data we calculate the probability of being in recession by taking the average of the BCI. We then simulate the distribution of ω and find the ω that corresponds to the probability of being in a recession.

Table 4.3 reports moments from the data and from two calibrations. The first calibration is a $20 - \log$ economy in which aggregate consumption equals the div-

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[16][Dumas *et al.* 2003] study correlations in an international context and using a representative agent framework, find that the level of correlations can be matched, and so there is no excess correlation puzzle. However, they do not address the time-variation in correlations.

Table 4.1: **Summary Statistics — Dividends & Returns.** The table summarizes descriptive statistics (mean and standard deviation (STD)) of industry dividends, industry dividends & repurchases, industry returns (booms and recessions), and the market portfolio (Market) using Kenneth French's industry classification. Booms and recessions are identified using the NBER business cycle indicator. Industries are Consumer NonDurables (NoDur), Consumer Durables (Durbl), Manufacturing (Manuf), Energy (Enrgy), Business Equipment (HiTec), Telephone and Television Transmission (Telcm), Shops, Health (Hlth), Utilities (Utils), and Other. The sample consists of all firms appearing jointly in Compustat and CRSP, for the sample period 1950 to 2007. Returns are annualized from monthly observations.

	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Market
	Dividends (inflation and population growth adjusted)										
Mean	0.03374	0.00119	0.01438	0.03322	0.05172	0.04484	0.02661	0.06957	0.02365	0.06821	0.03992
STD	0.10380	0.17692	0.08184	0.07842	0.20302	0.09152	0.07070	0.05584	0.05022	0.08145	0.05507
	Dividends & Repurchases (inflation and population growth adjusted)										
Mean	0.04920	0.01950	0.03491	0.05487	0.11195	0.05227	0.06098	0.08514	0.02886	0.08625	0.05437
STD	0.17797	0.22857	0.15574	0.15930	0.26700	0.18397	0.20313	0.16393	0.09411	0.15619	0.09369
	Returns (inflation adjusted) - Full sample										
Mean	0.09036	0.09773	0.08635	0.10709	0.10254	0.07758	0.08910	0.10658	0.07529	0.09073	0.08640
STD	0.14375	0.18406	0.16346	0.17519	0.21902	0.15598	0.17177	0.17415	0.13224	0.16298	0.14546
	Returns (inflation adjusted) - Boom										
Mean	0.09105	0.12148	0.10354	0.12954	0.11771	0.08691	0.09236	0.11129	0.07741	0.11224	0.10078
STD	0.13432	0.17395	0.15258	0.16364	0.20774	0.15409	0.15730	0.16498	0.12338	0.14971	0.13396
	Returns (inflation adjusted) - Recession										
Mean	0.08634	-0.03989	-0.01326	-0.02301	0.01466	0.02354	0.07021	0.07933	0.06299	-0.03389	0.00309
STD	0.19023	0.23130	0.21467	0.22846	0.27528	0.16643	0.23996	0.22072	0.17571	0.22258	0.19867

Table 4.2: **Summary Statistics — Dividend & Repurchase Correlations.** The table summarizes pair-wise correlations of industry portfolio dividends and of industry portfolio dividends & repurchases using Kenneth French's industry classification. Industries are Consumer NonDurables (NoDur), Consumer Durables (Durbl), Manufacturing (Manuf), Energy (Enrgy), Business Equipment (HiTec), Telephone and Television Transmission (Telcm), Shops, Health (Hlth), Utilities (Utils), and Other. The sample consists of all firms appearing jointly in Compustat and CRSP, for the sample period 1950 to 2007.

	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
				Dividend Correlations						
NoDur	1.00000									
Durbl	0.18342	1.00000								
Manuf	0.06677	0.43621	1.00000							
Enrgy	0.03797	0.21397	0.38493	1.00000						
HiTec	0.03439	0.16496	0.00611	0.09691	1.00000					
Telcm	0.23522	0.14905	0.30288	0.17856	-0.10590	1.00000				
Shops	0.02765	0.42752	0.42482	0.43905	0.03053	0.30916	1.00000			
Hlth	0.04576	0.22559	0.05440	0.28761	0.10486	-0.14368	0.32285	1.00000		
Utils	0.16513	0.26534	0.25284	0.18794	0.10587	0.36858	0.41965	0.02327	1.00000	
Other	0.21226	0.34789	0.36654	0.37266	0.14497	0.19315	0.42656	0.04092	0.08717	1.00000
Average	0.19827									
				Dividend & Repurchase Correlations						
NoDur	1.00000									
Durbl	0.43813	1.00000								
Manuf	0.32118	0.41952	1.00000							
Enrgy	0.24487	0.36414	0.40745	1.00000						
HiTec	0.02949	0.19177	0.47249	0.19020	1.00000					
Telcm	-0.10497	0.20925	0.07696	0.20349	0.03350	1.00000				
Shops	0.25547	0.27865	0.38651	-0.01269	0.18353	0.07082	1.00000			
Hlth	0.16818	0.08646	0.06828	-0.12243	-0.26574	-0.04993	0.16271	1.00000		
Utils	0.13811	0.11247	0.04789	-0.12235	0.10224	0.28798	0.10467	-0.04603	1.00000	
Other	0.28184	0.34657	0.38163	0.30779	0.32310	0.19839	0.36734	0.04747	0.29393	1.00000
Average	0.17512									

idends and repurchases from the 10 industries. The second calibration is $30 - \log$ economy in which aggregate consumption equals the dividends and repurchases from the 10 industries plus another large and smooth claim. The table also reports the parameters used to calibrate the economy.

Table 4.3: **Summary Statistics — Calibrations.** The table summarizes stock market moments (Mean, Boom, Recession) and corresponding moments from two calibrations as well as model parameters employed in the calibrations. In the model with risk aversion coefficients 20 - Log industry dividends represent aggregate dividends while in the model with risk aversion coefficients 30 - Log an unpriced industry is introduced such that aggregate consumption in the model matches the data.

	Data	Model - 20-Log	Model - 30-Log
Excess Return Market	6.3 %	5.0 %	5.1 %
Stdev Market	15.0 %	20.5 %	17.9 %
Risk-free Rate	0.9 %	1.6 %	1.3 %
Average Correlation	0.63	0.66	0.60
Average Stdev	17.0 %	24.2 %	22.4 %
Correlation - Recession	0.72	0.77	0.71
Stdev Market - Recession	19.7 %	27.0 %	22.7 %
Risk Aversion - High	-	20	30
Risk Aversion - Low	-	1	1
Utility Weight on Most Risk Averse	-	0.75	0.99982
Time Discount Factor	-	0.01	0.01
Delta	-	0.11	0.1

Two insights emerge from Table 4.3: First, both models match the mean correlation from the data and the correlation in the recession fairly well. Second, the models also match the other moments. Because the second calibration matches moments of aggregate consumption the model requires a higher risk aversion for the first agent to match moments. Overall, our match is comparable to [Chan & Kogan 2002]

To see the impact of the business cycle on model correlations and other quantities, Figures 4.10-4.10 show the distribution of ω, expected market return, average correlations and market standard deviation as a function of ω. The results make clear that our model explains correlations the other matched quantities over the business cycle.

The most important implication from Figures 4.10-4.10 is that neither correlations nor other equilibrium quantities are expected to fall in recessions because these

Figure 4.10:   **Calibration I.** The Figure shows ω, stock market expected returns, correlations and standard deviations. Risk aversion coefficients, γ, are: 20 - Log. All other parameters are as in Table 4.3.
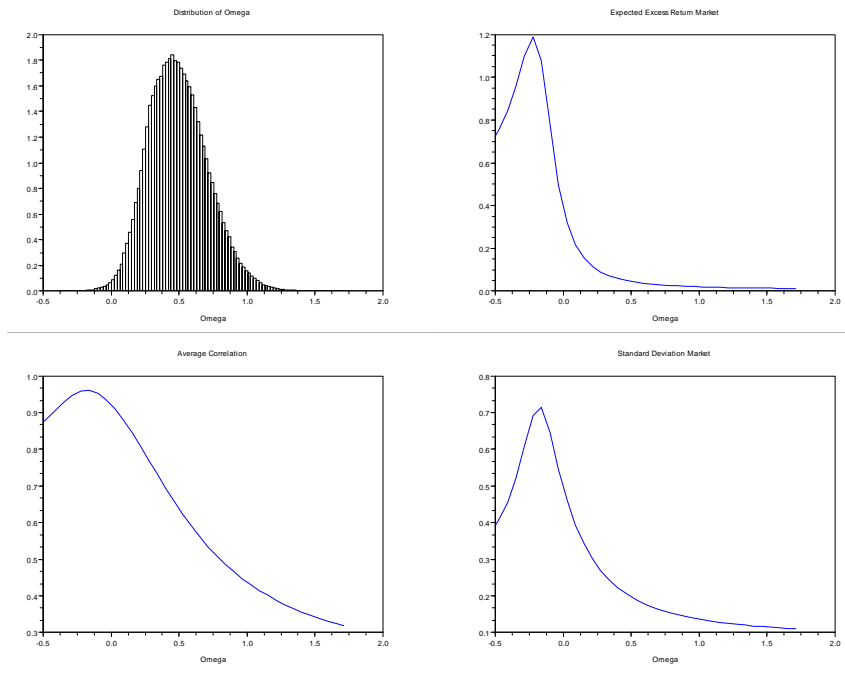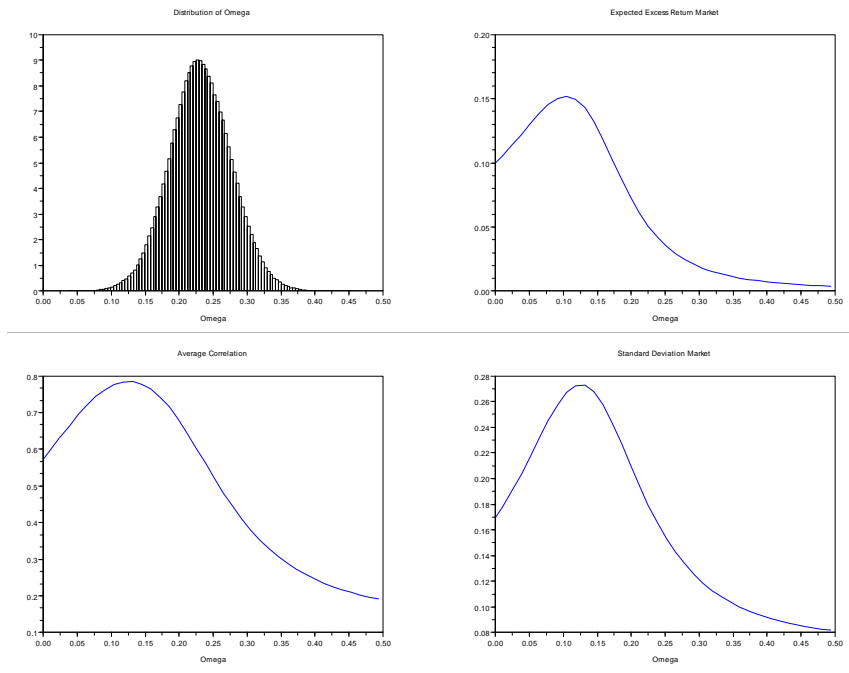
Figure 4.11: **Calibration II.** The Figure shows ω, stock market expected returns, correlations and standard deviations. Risk aversion coefficients, γ, are: 30 - Log. All other parameters are as in Table 4.3.

states have zero measure.

4.5   Conclusions

We study equity return correlations in multi-stock economies with heterogeneous agents. We identify two endogenous effects on correlations: First, correlations are increasing in risk aversion. Second, heterogeneity in risk aversion produces excess correlation via increased variability in the state price density.

Calibrations of the model match average industry correlations and changes of average industry correlations from business cycles peaks to troughs. To further support our point that heterogeneity in risk aversion is a major driver of correlations, standard deviations, returns and quadratic variations of portfolios we examine changes in industry stock market correlations, returns, and standard deviations and in quadratic variations of industry turnover and find positive relations. We also conduct a principle component analysis and find evidence that supports our model.

Asset Prices and Real Exchange Rates with Deep Habits

A.1 Derivation of Equilibrium

In this section I derive equilibrium by using standard Martingale methods (see [Cox & Huang 1989], [Karatzas *et al.* 1990]). The first order conditions for the maximization problem are given by

$$\frac{\partial u\left(C_H(t), C_F(t), X_H(t), X_F(t), t\right)}{\partial C_H} = y\xi(t) \tag{A.1}$$

$$\frac{\partial u\left(C_H(t), C_F(t), X_H(t), X_F(t), t\right)}{\partial C_F} = ye(t)\xi(t). \tag{A.2}$$

In the following it will be convenient to define the quantities

$$Q_i(t) = C_i(t)s_i(t)$$

$$s_\beta(t) = \frac{Q_H(t)^\beta}{Q_H(t)^\beta + Q_F(t)^\beta}$$

To calculate the real exchange rate and the state price density we need the partial derivatives

$$\frac{\partial Z}{\partial Q_H} = Zs_\beta Q_H^{-1} \tag{A.3}$$

$$\frac{\partial Z}{\partial Q_F} = Z\left(1 - s_\beta\right) Q_F^{-1} \tag{A.4}$$

$$\frac{\partial^2 Z}{\partial Q_H^2} = -\left(1 - \beta\right) Zs_\beta\left(1 - s_\beta\right) Q_H^{-2} \tag{A.5}$$

$$\frac{\partial^2 Z}{\partial Q_F^2} = -\left(1 - \beta\right) Zs_\beta\left(1 - s_\beta\right) Q_F^{-2} \tag{A.6}$$

$$\frac{\partial^2 Z}{\partial Q_F \partial Q_H} = \left(1 - \beta\right) Zs_\beta\left(1 - s_\beta\right) Q_H^{-1}Q_F^{-1}. \tag{A.7}$$

Using the partial derivatives we get that the equilibrium state price density, ξ, and the real exchange rate e are given by

$$\xi(t) = \frac{1}{y}e^{-\rho t}Z(t)^{1-\beta-\gamma}Q_H^{\beta-1} \tag{A.8}$$

$$e(t) = \left(\frac{Q_H}{Q_F}\right)^{1-\beta}. \tag{A.9}$$

To derive the risk-free rate, the market price of risk and the real exchange rate dynamics we need the processes for Q_i and Z. By Ito's lemma we have that

$$dQ_i(t) = Q_i(t)\left(\mu_{Q_i}(t)dt + \sigma_{Q_i}(t)^\top dB(t)\right) \tag{A.10}$$

where

$$\mu_{Q_i}(t) = \mu_{C_i}(t) + \phi_i\left(\frac{\overline{s_i}}{s_i(t)} - 1\right) + \lambda_i\left(s_i(t)\right)\sigma_{C_i}^\top \sigma_{C_i} \tag{A.11}$$

$$\sigma_{Q_i}(t) = \left(1 + \lambda_i\left(s_i(t)\right)\right)\sigma_{C_i}. \tag{A.12}$$

Applying Ito's lemma to Z we get

$$dZ(t) = Z(t)\left(\mu_z(t)dt + \sigma_Z(t)^\top dB(t)\right) \tag{A.13}$$

where

$$\mu_z(t) = s_\beta(t)\mu_{Q_i}(t) + (1 - s_\beta(t))\mu_{Q_i}(t)$$
$$-\frac{1}{2}s_\beta(t)\left(1 - s_\beta(t)\right)(1 - \beta) \tag{A.14}$$

$$\left(\sigma_{Q_H}(t)^\top \sigma_{Q_H}(t) + \sigma_{Q_F}(t)^\top \sigma_{Q_F}(t) - 2\sigma_{Q_H}(t)^\top \sigma_{Q_F}(t)\right) \tag{A.15}$$

and

$$\sigma_Z(t) = s_\beta(t)\sigma_{Q_H}(t) + (1 - s_\beta(t))\sigma_{Q_F}(t). \tag{A.16}$$

We have that

$$d\left(Z(t)^{1-\beta-\gamma}\right) = Z(t)^{1-\beta-\gamma}\left(\begin{pmatrix}(1-\beta-\gamma)\mu_Z(t)+\\\frac{1}{2}(\beta+\gamma)(\beta+\gamma-1)\sigma_Z(t)^\top\sigma_Z(t)\\+(1-\beta-\gamma)\sigma_Z(t)^\top dB(t)\end{pmatrix}dt\right) \tag{A.17}$$

and

$$d\left(Q_H(t)^{\beta-1}\right) = Q_H(t)^{\beta-1}\left(\begin{pmatrix}(\beta-1)\mu_{Q_H}(t)+\\\frac{1}{2}(\beta-1)(\beta-2)\sigma_{Q_H}(t)^\top\sigma_{Q_H}(t)\\+(\beta-1)\sigma_{Q_H}(t)^\top dB(t)\end{pmatrix}dt\right). \tag{A.18}$$

Applying Ito's lemma to $\frac{1}{y}e^{-\rho t}Z(t)^{1-\beta-\gamma}Q_H^{\beta-1}$ we get:

$$\frac{d\left(\frac{1}{y}e^{-\rho t}Z(t)^{1-\beta-\gamma}Q_H^{\beta-1}\right)}{\frac{1}{y}e^{-\rho t}Z(t)^{1-\beta-\gamma}Q_H^{\beta-1}} = \left(-\left(\begin{array}{c}\rho+(\gamma+\beta-1)\,\mu_Z(t)\\+(1-\beta)\,\mu_{Q_H}(t)\\-\frac{1}{2}\,(\beta+\gamma)\,(\gamma+\beta-1)\,\sigma_Z(t)^\top\sigma_Z(t)\\-\frac{1}{2}\,(\beta-1)\,(\beta-2)\,\sigma_{Q_H}(t)^\top\sigma_{Q_H}(t)\\+(\gamma+\beta-1)\,(\beta-1)\,\sigma_Z(t)^\top\sigma_{Q_H}(t)\end{array}\right)dt\\-\left((\gamma+\beta-1)\,\sigma_Z(t)+(1-\beta)\,\sigma_{Q_H}(t)\right)^\top dB(t)\right). \tag{A.19}$$

Comparing with the process for ξ we see that

$$\begin{aligned}r_H(t) &= \rho+(\gamma+\beta-1)\,\mu_Z(t)+(1-\beta)\,\mu_{Q_H}(t)\\&\quad-\frac{1}{2}\,(\beta+\gamma)\,(\gamma+\beta-1)\,\sigma_Z(t)^\top\sigma_Z(t)\\&\quad-\frac{1}{2}\,(\beta-1)\,(\beta-2)\,\sigma_{Q_H}(t)^\top\sigma_{Q_H}(t)\\&\quad+(\gamma+\beta-1)\,(\beta-1)\,\sigma_Z(t)^\top\sigma_{Q_H}(t)\end{aligned} \tag{A.20}$$

and

$$\theta(t)=(\gamma+\beta-1)\,\sigma_Z(t)+(1-\beta)\,\sigma_{Q_H}(t). \tag{A.21}$$

The dynamics of the real exchange rate is given by

$$\begin{aligned}de(t) &= d\left(\left(\frac{Q_H}{Q_F}\right)^{1-\beta}\right)\\&= e(t)\left(\mu_e(t)dt+\sigma_e(t)^\top dB(t)\right)\end{aligned} \tag{A.22}$$

where

$$\begin{aligned}\mu_e(t) &= (1-\beta)\left(\mu_{Q_H}(t)-\mu_{Q_F}(t)+\sigma_{Q_F}(t)^\top\left(\sigma_{Q_F}(t)-\sigma_{Q_H}(t)\right)\right)\\&\quad+\frac{1}{2}\beta\,(\beta-1)\,(\sigma_{Q_H}(t)-\sigma_{Q_F}(t))^T\,(\sigma_{Q_H}(t)-\sigma_{Q_F}(t))\end{aligned} \tag{A.23}$$

and

$$\sigma_e(t)=(1-\beta)\,(\sigma_{Q_H}(t)-\sigma_{Q_F}(t)) \tag{A.24}$$

To derive the stock price diffusion matrix note that we have

$$\xi(t)S_H(t) = E_t\left[\int_t^T \xi(s)C_H(s)ds\right] \tag{A.25}$$

$$\xi(t)S_F(t) = E_t\left[\int_t^T \xi(s)e(s)C_F(s)ds\right]. \tag{A.26}$$

By applying Ito's lemma to the left hand side and Clark-Ocone's Theorem to the right hand side, and then matching the diffusion terms we get

$$
\begin{aligned}
\sigma_H(t) &= \theta(t) + \frac{E_t\left[\int_t^T \xi(s)C_H(s)\left(D_t \ln \xi(s) + D_t \ln C_H(s)\right)ds\right]}{E_t\left[\int_t^T \xi(s)C_H(s)ds\right]} \quad\text{(A.27)} \\
\sigma_F(t) &= \theta(t) + \frac{E_t\left[\int_t^T \xi(s)e(s)C_F(s)\left(D_t \ln \xi(s) + D_t \ln e(s)\right)ds\right]}{E_t\left[\int_t^T \xi(s)C_F(s)ds\right]} \\
&\quad + \frac{E_t\left[\int_t^T \xi(s)e(s)C_F(s)D_{tF}(s)ds\right]}{E_t\left[\int_t^T \xi(s)C_F(s)ds\right]}
\end{aligned}
$$

$$\text{(A.29)}$$

A.2 Derivation of Equilibrium - Heterogeneous Agent Economy

In this section I derive the equilibrium for the model with heterogeneous preferences. I use the martingale method (see [Cox & Huang 1989], [Karatzas *et al.* 1990]). From the solution to central planner problem we have that the marginal utility of the representative agent is proportional to the home agent's marginal utility evaluated at his optimal consumption, i.e.,

$$
\nabla u(C_H(t), C_F(t), X_H(t), X_H(t)) = a \nabla u_H(C_H^H(t), C_F^H(t), X_H^H(t), X_F^H(t)). \quad \text{(A.30)}
$$

As a consequence we can use the marginal utility of the home agent as the pricing kernel. Define the following quantity

$$
s_\beta^\lambda(t) = \frac{\lambda_H^{1-\beta} Q_H^H(t)^\beta}{\lambda_H^{1-\beta} Q_H^H(t)^\beta + (1-\lambda_H)^{1-\beta} Q_F^H(t)^\beta}. \quad \text{(A.31)}
$$

The following partial derivatives will be useful

$$\frac{\partial Z_H}{\partial Q_H^H} = Z_H s_\beta^\lambda \left(Q_H^H\right)^{-1} \tag{A.32}$$

$$\frac{\partial Z_H}{\partial Q_F^H} = Z_H \left(1 - s_\beta^\lambda\right) \left(Q_F^H\right)^{-1} \tag{A.33}$$

$$\frac{\partial^2 Z_H}{\partial \left(Q_H^H\right)^2} = -Z_H \left(1 - \beta\right) s_\beta^\lambda \left(1 - s_\beta^\lambda\right) \left(Q_H^H\right)^{-2} \tag{A.34}$$

$$\frac{\partial^2 Z_H}{\partial \left(Q_F^H\right)^2} = -Z_H \left(1 - \beta\right) s_\beta^\lambda \left(1 - s_\beta^\lambda\right) \left(Q_F^H\right)^{-2} \tag{A.35}$$

$$\frac{\partial^2 Z_H}{\partial Q_H^H \partial Q_F^H} = Z_H \left(1 - \beta\right) s_\beta^\lambda \left(1 - s_\beta^\lambda\right) \left(Q_H^H\right)^{-1} \left(Q_F^H\right)^{-1}. \tag{A.36}$$

By Ito's lemma we have the following

$$dQ_H^H(t) = Q_H^H(t) \left(\mu_{Q_H^H}(t)dt + \sigma_{Q_H^H}(t)^\top dB(t)\right) \tag{A.37}$$

$$dQ_F^H(t) = Q_F^H(t) \left(\mu_{Q_F^H}(t)dt + \sigma_{Q_F^H}(t)^\top dB(t)\right) \tag{A.38}$$

where

$$\begin{aligned}
Q_H^H(t)\mu_{Q_H^H}(t) &= \frac{\partial g_H^H}{\partial Q_H} Q_H(t)\mu_{Q_H}(t) + \frac{\partial g_H^H}{\partial Q_F} Q_F(t)\mu_{Q_F}(t) \\
&\quad + \frac{1}{2} \frac{\partial^2 g_H^H}{\partial \left(Q_H\right)^2} Q_H(t)^2 \sigma_{Q_H}(t)^\top \sigma_{Q_H}(t) \\
&\quad + \frac{1}{2} \frac{\partial^2 g_H^H}{\partial \left(Q_F\right)^2} Q_F(t)^2 \sigma_{Q_F}(t)^\top \sigma_{Q_F}(t) \\
&\quad + \frac{\partial^2 g_H^H}{\partial Q_H \partial Q_F} Q_H(t)Q_F(t)\sigma_{Q_H}(t)^\top \sigma_{Q_F}(t)
\end{aligned} \tag{A.39}$$

$$\begin{aligned}
Q_F^H(t)\mu_{Q_F^H}(t) &= \frac{\partial g_F^H}{\partial Q_H} Q_H(t)\mu_{Q_H}(t) + \frac{\partial g_F^H}{\partial Q_F} Q_F(t)\mu_{Q_F}(t) \\
&\quad + \frac{1}{2} \frac{\partial^2 g_F^H}{\partial \left(Q_H\right)^2} Q_H(t)^2 \sigma_{Q_H}(t)^\top \sigma_{Q_H}(t) \\
&\quad + \frac{1}{2} \frac{\partial^2 g_F^H}{\partial \left(Q_F\right)^2} Q_F(t)^2 \sigma_{Q_F}(t)^\top \sigma_{Q_F}(t) \\
&\quad + \frac{\partial^2 g_F^H}{\partial Q_H \partial Q_F} Q_H(t)Q_F(t)\sigma_{Q_H}(t)^\top \sigma_{Q_F}(t)
\end{aligned} \tag{A.40}$$

and

$$Q_H^H(t)\sigma_{Q_H^H}(t) = \frac{\partial g_H^H}{\partial Q_H} Q_H(t)\sigma_{Q_H}(t) + \frac{\partial g_H^H}{\partial Q_F} Q_F(t)\sigma_{Q_F}(t) \tag{A.41}$$

$$Q_F^H(t)\sigma_{Q_F^H}(t) = \frac{\partial g_F^H}{\partial Q_H} Q_H(t)\sigma_{Q_H}(t) + \frac{\partial g_F^H}{\partial Q_F} Q_F(t)\sigma_{Q_F}(t). \tag{A.42}$$

Moreover, Z_H follows

$$dZ_H(t) = Z_H(t)\left(\mu_{Z_H}(t)dt + \sigma_{Z_H}(t)^\top dB(t)\right) \tag{A.43}$$

where

$$\begin{aligned}
\mu_{Z_H}(t) &= s_\beta^\lambda(t)\mu_{Q_H^H}(t) + \left(1 - s_\beta^\lambda(t)\right)\mu_{Q_F^H}(t) \\
&\quad - \frac{1}{2}(1-\beta)s_\beta^\lambda(t)\left(1 - s_\beta^\lambda(t)\right)\sigma_{Q_H^H}(t)^\top \sigma_{Q_H^H}(t) \\
&\quad - \frac{1}{2}(1-\beta)s_\beta^\lambda(t)\left(1 - s_\beta^\lambda(t)\right)\sigma_{Q_F^H}(t)^\top \sigma_{Q_F^H}(t) \\
&\quad + (1-\beta)s_\beta^\lambda(t)\left(1 - s_\beta^\lambda(t)\right)\sigma_{Q_H^H}(t)^\top \sigma_{Q_F^H}(t)
\end{aligned} \tag{A.44}$$

and

$$\sigma_{Z_H}(t) = s_\beta^\lambda(t)\sigma_{Q_H^H}(t) + \left(1 - s_\beta^\lambda(t)\right)\sigma_{Q_F^H}(t). \tag{A.45}$$

Using the partial derivatives we have that

$$\xi(t) = \frac{1}{y}e^{-\rho t}Z_H(t)^{1-\beta-\gamma}Q_H^H(t)^{\beta-1}. \tag{A.46}$$

By Ito's lemma we have

$$\frac{d\left(\frac{1}{y}e^{-\rho t}Z_H(t)^{1-\beta-\gamma}Q_H^H(t)^{\beta-1}\right)}{\frac{1}{y}e^{-\rho t}Z_H(t)^{1-\beta-\gamma}Q_H^H(t)^{\beta-1}} = \left(\begin{array}{c} -\left(\begin{array}{c} \rho + (1-\beta-\gamma)\mu_{Z_H}(t) \\ +(1-\beta)\mu_{Q_H^H}(t) \\ -\frac{1}{2}(\beta+\gamma)(\beta+\gamma-1) \\ \sigma_{Z_H}(t)^\top \sigma_{Z_H}(t) \\ -\frac{1}{2}(\beta-1)(\beta-2) \\ \sigma_{Q_H^H}(t)^\top \sigma_{Q_H^H}(t) \\ +(\beta+\gamma-1)(\beta-1) \\ \sigma_{Z_H}(t)^\top \\ \sigma_{Q_H^H}(t) \end{array}\right)dt \\ -\left(\begin{array}{c} (\beta+\gamma-1)\sigma_{Z_H}(t) \\ +(1-\beta)\sigma_{Q_H^H}(t) \end{array}\right)^\top dB(t) \end{array}\right). \tag{A.47}$$

Comparing with the process for ξ we have the following

$$\begin{aligned}
r_H(t) &= \rho + (1-\beta-\gamma)\mu_{Z_H}(t) + (1-\beta)\mu_{Q_H^H}(t) \tag{A.48} \\
&\quad - \frac{1}{2}(\beta+\gamma)(\beta+\gamma-1)\sigma_{Z_H}(t)^\top \sigma_{Z_H}(t) \\
&\quad - \frac{1}{2}(\beta-1)(\beta-2)\sigma_{Q_H^H}(t)^\top \sigma_{Q_H^H}(t) \\
&\quad + (\beta+\gamma-1)(\beta-1)\sigma_{Z_H}(t)^\top \sigma_{Q_H^H}(t)
\end{aligned}$$

and

$$\theta_H(t) = (\beta + \gamma - 1)\,\sigma_{Z_H}(t) + (1 - \beta)\sigma_{Q_H^H}(t). \tag{A.49}$$

In equilibrium the relative price of the foreign good in terms of the home good is given by equation (2.68). By Ito's lemma we have

$$\frac{dP_F(t)}{P_F(t)} = \mu_{P_F}(t)dt + \sigma_{P_F}(t)^\top dB(t) \tag{A.50}$$

where

$$\begin{aligned}
\mu_{P_F}(t) &= (1 - \beta)\left(\mu_{Q_H^H}(t) - \mu_{Q_F^H}(t) + \sigma_{Q_F^H}(t)^\top\left(\sigma_{Q_F^H}(t) - \sigma_{Q_H^H}(t)\right)\right) \\
&\quad + \frac{1}{2}\beta(1 - \beta)\left(\sigma_{Q_H^H}(t) - \sigma_{Q_F^H}(t)\right)^\top\left(\sigma_{Q_H^H}(t) - \sigma_{Q_F^H}(t)\right)
\end{aligned} \tag{A.51}$$

and

$$\sigma_{P_F}(t) = (1 - \beta)\left(\sigma_{Q_H^H}(t) - \sigma_{Q_F^H}(t)\right). \tag{A.52}$$

The price index in the home and the foreign country is given by (2.70) and (2.71) respectively. Applying Ito's lemma yields

$$\begin{aligned}
\frac{dP^H(t)}{P^H(t)} &= \mu_{P^H}(t)dt + \sigma_{P^H}(t)^\top dB(t) \tag{A.53} \\
\frac{dP^F(t)}{P^F(t)} &= \mu_{P^F}(t)dt + \sigma_{P^F}(t)^\top dB(t) \tag{A.54}
\end{aligned}$$

where

$$\begin{aligned}
\mu_{P^H}(t) &= (1 - \lambda_H)\left(\frac{P_F(t)}{P^H(t)}\right)\mu_{P_F}(t) \tag{A.55} \\
&\quad + \frac{1}{2}(1 - \beta)(1 - \lambda_H)\left(\frac{P_F(t)}{P^H(t)}\right)\left(\left(\frac{P_F(t)}{P^H(t)}\right)(1 - \lambda_H) - 1\right) \\
&\quad \sigma_{P_F}(t)^\top\sigma_{P_F}(t) \\
\mu_{P^F}(t) &= (1 - \lambda_F)\left(\frac{P_F(t)}{P^F(t)}\right)\mu_{P_F}(t) \tag{A.56} \\
&\quad + \frac{1}{2}(1 - \beta)(1 - \lambda_F)\left(\frac{P_F(t)}{P^F(t)}\right)\left(\left(\frac{P_F(t)}{P^F(t)}\right)(1 - \lambda_F) - 1\right)\sigma_{P_F}(t)^\top \\
&\quad \sigma_{P_F}(t)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{P^H}(t) &= (1 - \lambda_H)\left(\frac{P_F(t)}{P^H(t)}\right)\sigma_{P_F}(t) \tag{A.57} \\
\sigma_{P^F}(t) &= (1 - \lambda_H)\left(\frac{P_F(t)}{P^F(t)}\right)\sigma_{P_F}(t). \tag{A.58}
\end{aligned}$$

The real exchange rate is given by the ratio of the foreign price index to the home price index. By Ito's lemma we have

$$de(t) = e(t) \left(\mu_e(t)dt + \sigma_e(t)^\top dW(t) \right) \tag{A.59}$$

where

$$\mu_e(t) = \mu_{PF}(t) - \mu_{PH}(t) - \sigma_{PH}(t)^\top \sigma_e(t) \tag{A.60}$$

$$\sigma_e(t) = \sigma_{PF}(t) - \sigma_{PH}(t). \tag{A.61}$$

The stock price diffusion coefficients can be derived as for the case with a representative agent with appropriate change of the Malliavin derivatives. The optimal portfolios are found by the approach of [Cox & Huang 1989], where the Malliavin derivatives are in Appendix A.3.

A.3 Derivation of the Malliavin Derivatives

In this section I derive the Malliavin derivatives. The first section considers the Malliavin derivatives in the homogeneous agent economy. The second section only deals with the Malliavin derivatives that are special for the heterogeneous agent economy.

A.3.1 Malliavin Derivatives - Homogeneous Agent Economy

In this section I derive the expressions for the Malliavin derivatives used in the equilibrium expression for the diffusion coefficients. The Malliavin derivatives of interest are $D_t\mu_{C_i}(u), D_tC_i(u), D_ts_i(u), D_tQ_i(u), D_tZ(u), D_t\xi(u)$ and $D_te(u)$ for $u \geq t$ and $i \in \{H, F\}$. Note that each Malliavin derivative is a vector, where each element refers to the Malliavin derivative with respect to the first and second Brownian motion. The Malliavin derivative of the expected consumption growth, $\mu_{C_i}(t)$, is given by

$$\begin{aligned} D_t\mu_{C_i}(t) &= D_t \left(e^{-\alpha_i u}\mu_{C_i}(0) + \alpha_i\overline{\mu}_{C_i} \int_0^u e^{\alpha_i(s-u)}ds + \int_0^u e^{\alpha_i(s-u)}\nu_i^\top dB(s) \right) \\ &= \int_t^u D_t e^{\alpha_i(s-u)}\nu_i^\top dB(s) \\ &= e^{\alpha_i(t-u)}\nu_i^\top. \end{aligned} \tag{A.62}$$

The Malliavin derivative of the consumption processes are given by

$$
\begin{aligned}
D_t C_i(u) &= C_i(u) D_t \left(\int_0^u \left(\mu_{C_i}(s) - \frac{1}{2} \sigma_{c_i}^\top \sigma_{c_i} \right) ds + \sigma_{c_i}^\top B(u) \right) \\
&= C_i(u) \left(\int_t^u D_t \mu_{C_i}(s) ds + \sigma_{c_i}^\top \right) \\
&= C_i(u) \left(\int_t^u e^{\alpha_i(t-s)} \nu_i^\top ds + \sigma_{c_i}^\top \right) \\
&= C_i(u) \left(\nu_i^\top \left(\frac{1 - e^{\alpha_i(t-u)}}{\alpha_i} \right) + \sigma_{c_i}^\top \right) \quad\quad\quad \text{(A.63)}
\end{aligned}
$$

To compute the Malliavin derivatives of the surplus consumption ratios, first note that the surplus consumption ratios follows

$$
ds_i(t) = \phi_i \left(\overline{s}_i - s_i(t) \right) dt + s_i(t) \lambda \left(s_i(t) \right) \sigma_{c_i}^\top dB(t) \quad\quad \text{(A.64)}
$$

where $s_i(0) = s_i$ with $s_i \in (0, 1]$. The first variation process of the surplus consumption ratios are given by

$$
Y_i(t) = \exp \left(- \int_0^t \left(\phi_i + \frac{1}{2} \sigma_{Y_i}^\top(u) \sigma_{Y_i}(u) \right) du + \int_0^t \sigma_{Y_i}^\top(u) dB(u) \right) \quad\quad \text{(A.65)}
$$

where

$$
\sigma_{Y_i}(t) = \frac{\lambda \left(s_i(t) \right) \left(1 - 2\lambda \left(s_i(t) \right) \right)}{2(1 - s_i(t))} \sigma_{c_i}. \quad\quad \text{(A.66)}
$$

Using the relation between the first variation process and the Malliavin derivatives we get (see appendix A.4)

$$
D_t s_i(u) = s_i(t) \lambda \left(s_i(t) \right) \sigma_{c_i} Y_i(u) Y_i(t)^{-1}. \quad\quad \text{(A.67)}
$$

Next I derive the Malliavin derivatives of Q_i

$$
\begin{aligned}
D_t Q_i(u) &= D_t \left(C_i(u) s_i(u) \right) \\
&= C_i(u) D_t s_i(u) + s_i(u) D_t C_i(u) \\
&= Q_i(u) \left(\nu_i^\top \left(\frac{1 - e^{\alpha_i(t-u)}}{\alpha_i} \right) + \sigma_{c_i}^\top \right) \\
&\quad + C_i(u) s_i(t) \lambda \left(s_i(t) \right) \sigma_{c_i} Y_i(u) Y_i(t)^{-1}. \quad\quad \text{(A.68)}
\end{aligned}
$$

The Malliavin derivative of Z follows from the chain-rule

$$
\begin{aligned}
D_t Z(u) &= \frac{\partial Z(u)}{\partial Q_i} D_t Q_i(u) \\
&= Z(u) s_\beta(u) Q_H^{-1}(u) D_t Q_i(u). \quad\quad \text{(A.69)}
\end{aligned}
$$

The Malliavin derivative of the state price density is

$$
\begin{aligned}
D_t \xi(u) &= D_t \left(\frac{1}{y} e^{-\rho u} Z(u)^{1-\beta-\gamma} Q_H^{\beta-1}(u) \right) \\
&= \frac{1}{y} e^{-\rho t} Z(u)^{1-\beta-\gamma} D_t Q_H(u)^{\beta-1} \\
&\quad + \frac{1}{y} e^{-\rho t} \left(D_t Z(u)^{1-\beta-\gamma} \right) Q_H(u)^{\beta-1} \\
&= \xi(u) \left((1-\beta-\gamma) D_t \ln Z(u) + (\beta-1) D_t \ln Q_H(u) \right) \quad \text{(A.70)}
\end{aligned}
$$

where

$$
D_t \ln Z(u) = \frac{D_t Z(u)}{Z(u)} \tag{A.71}
$$

$$
D_t \ln Q_H(u) = \frac{D_t Q_H(u)}{Q_H(u)}. \tag{A.72}
$$

Finally, we have that the Malliavin derivative of $e(u)$ is

$$
\begin{aligned}
D_t e(u) &= D_t \left(\frac{Q_H(u)}{Q_F(u)} \right)^{1-\beta} \\
&= (1-\beta) \left(\frac{Q_H(u)}{Q_F(u)} \right)^{-\beta} \left(\frac{D_t Q_H(u)}{Q_F(u)} - \frac{Q_H(u) D_t Q_F(u)}{Q_F(u)^2} \right) \\
&= e(u) (1-\beta) \left(D_t \ln Q_H(u) - D_t \ln Q_F(u) \right). \tag{A.73}
\end{aligned}
$$

A.3.2 Malliavin Derivatives - Heterogeneous Agent Economy

In this section I derive the Malliavin derivatives in the heterogeneous agent economy. The Malliavin derivatives of interest are $D_t Q_j^i(u)$, $D_t X_j^i(u)$ and $D_t C_j^i(u)$ for $i, j = H, F$. The remaining Malliavin derivatives can be found by substituting the corresponding values from the heterogeneous agent economy into the formulas for the homogeneous agent economy. The Malliavin derivative of $Q_j^i(u)$ for $t < u$ is

$$
\begin{aligned}
D_t Q_j^i(u) &= D_t g_j^i \left(Q_H(u), Q_F(u) \right) \\
&= \frac{\partial g_j^i \left(Q_H(u), Q_F(u) \right)}{\partial Q_H} D_t Q_H(u) \\
&\quad + \frac{\partial g_j^i \left(Q_H(u), Q_F(u) \right)}{\partial Q_F} D_t Q_F(u). \tag{A.74}
\end{aligned}
$$

To calculate the Malliavin derivative of $X_j^i(u)$, note that

$$
\begin{aligned}
X_j^i(u) &= \omega_i X_j(u) \\
&= \omega_i C_j(u) \left(1 - s_j(u) \right). \tag{A.75}
\end{aligned}
$$

Using the expression in (A.75)

$$
\begin{aligned}
D_t X_j^i(u) &= D_t \left(\omega_i C_j(u) \left(1 - s_j(u) \right) \right) \\
&= \omega_i D_t \left(C_j(u) \left(1 - s_j(u) \right) \right) \\
&= \omega_i \left(D_t C_j(u) - s_j(u) D_t C_j(u) - C_j(u) D_t s_j(u) \right). \qquad \text{(A.76)}
\end{aligned}
$$

Finally, the Malliavin derivative of $C_j^i(u)$ is

$$
\begin{aligned}
D_t C_j^i(u) &= D_t \left(Q_j^i(u) + X_j^i(u) \right) \\
&= D_t Q_j^i(u) + D_t X_j^i(u). \qquad \text{(A.77)}
\end{aligned}
$$

A.4 The first variation process and Malliavin derivatives

In this section I will briefly discuss Malliavin calculus and the first variation process. Let $X^x(t)$ be an Ito process given by

$$
\begin{aligned}
dX^x(t) &= \mu(X^x(t))dt + \sigma(X^x(t))^\top dB(t) \qquad &\text{(A.78)} \\
X^x(0) &= x \qquad &\text{(A.79)}
\end{aligned}
$$

where it is assumed that μ and σ are C^1 and satisfy standard condition such that there exists a unique strong solution to the SDE. Define the first variation process, $Y(t) = \frac{\partial}{\partial x} X^x(t)$, as follows

$$
dY(t) = \mu'(X^x(t))Y(t)dt + \sigma'(X^x(t))^\top Y(t)dB(t) \qquad \text{(A.80)}
$$

i.e.

$$
Y(t) = \exp \left(\begin{array}{c} \int_0^t \left(\mu'(X^x(u)) - \frac{1}{2}\sigma'(X^x(u))^\top \sigma'(X^x(u)) \right) du \\ + \int_0^t \sigma'(X^x(u))^\top dB(u) \end{array} \right). \qquad \text{(A.81)}
$$

Now consider the Malliavin derivative of $X^x(t)$

$$
Z(t) := D_s X^x(t) = \int_s^t \mu'(X^x(u))D_s X^x(u)du + \int_s^t \sigma'(X^x(u))^\top D_s X^x(u)dB(u). \qquad \text{(A.82)}
$$

It then follows that

$$
\begin{aligned}
dZ(t) &= \mu'(X^x(t))Z(t)dt + \sigma'(X^x(t))^\top Z(t)dB(t) \qquad &\text{(A.83)} \\
Z(s) &= \sigma(X^x(t))
\end{aligned}
$$

with the following solution

$$Z(t) = \sigma(X^x(s)) \exp \left(\begin{array}{c} \int_0^t \left(\mu'(X^x(u)) - \frac{1}{2}\sigma'(X^x(u))^\top \sigma'(X^x(u)) \right) du \\ + \int_0^t \sigma'(X^x(u))^\top dB(u). \end{array} \right) \quad (A.84)$$

Comparing with $Y(t)$ we see that

$$D_s X(t) = \sigma(X^x(s))Y(t)Y(s)^{-1} \quad (A.85)$$

for $t \geq s$.

A.5   Computational Procedure

In this section I describe the numerical procedure to solve for the equilibrium quantities. The model is solved using Monte-Carlo (MC) simulations. The state variables are simulated forward using an Euler scheme with 20000 sample paths and 10000 time steps. I use antithetic sampling to reduce the variance. For each time step I calculate the optimal allocation of the habit adjusted consumption between the two agents by solving the system of first order condition from the central planner problem

$$f(Q_H^H, Q_F^H, Q_H^F, Q_F^F; Q_H, Q_F) = 0 \quad (A.86)$$

where

$$f(Q_H^H, Q_F^H, Q_H^F, Q_F^F; Q_H, Q_F) = \begin{bmatrix} a\frac{\partial u_H}{\partial Q_H^H} - y_1 \\ a\frac{\partial u_H}{\partial Q_F^H} - y_2 \\ (1-a)\frac{\partial u_H}{\partial Q_H^F} - y_1 \\ (1-a)\frac{\partial u_H}{\partial Q_F^F} - y_2 \\ Q_H^H + Q_H^F - Q_H \\ Q_F^H + Q_F^F - Q_F \end{bmatrix} \quad (A.87)$$

where y_1 and y_2 are the Lagrange multipliers. The system is solved by Newton's method. I iterate until $\left\| f(Q_H^H, Q_F^H, Q_H^F, Q_F^F; Q_H, Q_F) \right\| < 10^{-7}$. The derivatives of the optimal allocations Q_j^i for $i,j = H, F$ are calculated using finite differences. The time integrals are calculated by using the trapezoid rule with 10000 steps.

A.6   Change of Numeraire

In this section I discuss how changing the numeraire good impacts the equilibrium. In the following I will work with the generic numerairs A and B. The numeraire can be a single good or a basket of goods. The basket does not have to be a simple linear combination of the goods, but can take more complicated forms.

Proposition 25. *The relation between the equilibrium Sharp Ratios under two different numeraires (A and B) is given by*

$$\theta_B(t) = \theta_A(t) - \sigma_{p_A}(t).$$ (A.88)

Furthermore, the relation between the risk free rates is given by

$$r_B(t) = r_A(t) - \mu_{p_A}(t) + \theta_A^T(t)\sigma_{p_A}(t).$$ (A.89)

Stock price diffusion coefficients under two numeraires relate via

$$\sigma_B(t) = \sigma_A(t) - \sigma_{p_A}(t)$$ (A.90)

where the subscript A (B) denotes the equilibrium quantities in the economy with A (B) as numeraire.

Financial Market Completeness in Multi-Good Economies

B.1 Examples

In this appendix, we examine classes of utility functions that imply financial market incompleteness. For all examples that follow, we assume the parameters of the utility functions to satisfy the standard restrictions.

B.1.1 Log Utility

This case is also discussed in [Cass & Pavlova 2004]. The utility function of the representative agent takes the form of

$$u(c_1, c_2, ..., c_n) = \sum_{i=1}^{n} a_i \log c_i. \tag{B.1}$$

The above utility function can, alternatively, be expressed as

$$u(c_1, c_2, ..., c_n) = \varphi \left(\prod_{i=1}^{n} c_i^{a_i} \right) \tag{B.2}$$

where $\varphi(x) = \log x$. This implies that $u \in U_{IC}^1$ and, thus, the financial market is incomplete.

B.1.2 Non-Separable Cobb-Douglas

For this case, the utility function is given by

$$u(c_1, c_2, ..., c_n) = c_1^{a_1} c_2^{a_2} ... c_n^{an}. \tag{B.3}$$

Again, the above utility function can be expressed alternatively, as

$$u(c_1, c_2, ..., c_n) = \varphi \left(\prod_{i=1}^{n} c_i^{a_i} \right) \tag{B.4}$$

with $\varphi(x) = x$. Hence, the Cobb-Douglas utility function belongs to U_{IC}^1. Therefore, the financial market is again incomplete.

B.1.3   Generalized Non-Separable Cobb-Douglas

This utility function is given by

$$u(c_1, c_2, ..., c_n) = \frac{(c_1^{a_1} c_2^{a_2} ... c_n^{an})^{1-\rho}}{1-\rho}. \tag{B.5}$$

Assume $\varphi(x) = \frac{x^{1-\rho}}{1-\rho}$ from which we see that the generalized Cobb-Douglas will be in U_{IC}^1 and, thus, the financial market is again incomplete.

B.1.4   Separation   between   Non-Traded   and   Traded Goods as in [Serrat 2001]

[Serrat 2001] studies an economy with two countries. Each country has access to two output processes where one good is traded while the second good is non-traded. [Kollmann 2006], however, proves that the diffusion matrix in the [Serrat 2001] economy is non-invertible. We illustrate here that this result can be readily verified by applying Proposition 13 (or Proposition 14). The utility function of the representative agent employed in [Serrat 2001] can be expressed as follows (with a slight simplification relative to the original utility function)

$$u(c_1, c_2, c_3, c_n) = \frac{1}{q} \left(c_1^q + c_2^q \right) \left(ac_3^\alpha + b\delta_4^\beta \right). \tag{B.6}$$

This can also be expressed as follows

$$u(c_1, c_2, c_3, c_n) = \varphi \left((c_1^q + c_2^q) c_3^\alpha, (c_1^q + c_2^q) c_4^\alpha \right) \tag{B.7}$$

with $\varphi(x, y) = \frac{1}{q}(ax + by)$. This suggests that the utility function is in U_{IC}^2 and, thus, the financial market is incomplete.

B.2   Malliavin Derivatives

In this appendix, we derive explicit expressions for the Malliavin derivatives in Section 3.5. The Malliavin derivative of dividend process $i = 1, ..., N$ is

$$D_t \delta_i(s) = \delta_i(s) \lambda_i. \tag{B.8}$$

The Malliavin derivative of the optimal consumption for agent $j = 1, ..., J$ of good $i = 1, ..., N$ is

$$D_t c_i^j(s) = \sum_{k=1}^{N} \left(\frac{\partial c_i^j(s)}{\partial \delta_k} \delta_k(s) \lambda_k \right). \tag{B.9}$$

Next, we calculate the Malliavin derivative of the state price density

$$
\begin{aligned}
D_t \xi(s) &= D_t \left(\frac{\partial u(\delta(s); a)}{\partial \delta_1} \right) \\
&= \sum_{k=1}^{N} \left(\frac{\partial^2 u(\delta(s); a)}{\partial \delta_1 \partial \delta_k} \delta_k(s) \lambda_k \right).
\end{aligned}
\tag{B.10}
$$

The Malliavin derivative of commodity price $i = 1, ..., N$ is

$$
\begin{aligned}
D_t P_i(s) &= D_t \left(\frac{\frac{\partial u(\delta(s); a)}{\partial \delta_i}}{\frac{\partial u(\delta(s); a)}{\partial \delta_1}} \right) \\
&= \sum_{k=1}^{N} \left(\left(\frac{\frac{\partial^2 u(\delta(s); a)}{\partial \delta_i \partial \delta_k}}{\frac{\partial u(\delta(s); a)}{\partial \delta_1}} - \frac{\frac{\partial u(\delta(s); a)}{\partial \delta_i}}{\frac{\partial u(\delta(s); a)}{\partial \delta_1}} \frac{\frac{\partial^2 u(\delta(s); a)}{\partial \delta_1 \partial \delta_k}}{\frac{\partial u(\delta(s); a)}{\partial \delta_1}} \right) \delta_k(s) \lambda_k \right).
\end{aligned}
\tag{B.11}
$$

The Malliavin derivatives of the log consumption process, log state price density and the log commodity prices are

$$D_t \ln c_i^j(s) = \frac{D_t c_i^j(s)}{c_i^j(s)} \tag{B.12}$$

$$D_t \ln \xi(s) = \frac{D_t \xi(s)}{\xi(s)} \tag{B.13}$$

$$D_t \ln P_i(s) = \frac{D_t P_i(s)}{P_i(s)}. \tag{B.14}$$

Correlations

C.1   Propositions

The propositions for the economy in Section 4.2 are presented here. Before we state the propositions, a brief summary of the optimization problem of the investors is presented.

Investor j solves a static optimization problem (see [Cox & Huang 1989] and [Karatzas *et al.* 1987])

$$U_j\left(C, X\right) \ \ s.t. \ E_0\left[\int_0^T \eta(0,t)C_j(t)dt\right] \leq Y_j\left(0\right) \tag{C.1}$$

where $Y_j\left(0\right)$ is the wealth of investor j at $t = 0$, for $j = 1, 2$, and η denotes the stochastic discount factor (state price density).

The constrained maximization can then be reformulated as an unconstrained optimization problem by employing a Lagrange multiplier, y. The first order conditions and market clearing lead to the following optimal consumption sharing rule

$$\left(\frac{C_1(t)}{X(t)}\right) + \left(\frac{C_1(t)}{X(t)}\right)^{\frac{\gamma_1}{\gamma_2}}\left(\frac{y_1}{y_2}\right)^{\frac{1}{\gamma_2}} = \left(\frac{C(t)}{X(t)}\right). \tag{C.2}$$

Denote the solution of the above equation as

$$z_1(t) = z_1\left(C_1(t), X(t)\right) = \left(\frac{C_1(t)}{X(t)}\right)^* \tag{C.3}$$

parametrized with γ_1, γ_2, y_1, and y_2.

One can show (see [Cuoco & He 1994] and [Karatzas *et al.* 1990]) that the Lagrange multipliers in the sharing rule in Equation C.2 are equal to one over the weight on investors in the utility of a representative investor. With two investors this implies $\frac{y_1}{y_2} = \frac{1-a}{a}$ where a is the weight on investor 1. We employ this identity between Lagrange multipliers and weights on investors in our calibrations and examples in order to set the wealth (Lagrange multiplier) of the investors at date zero.

For a few risk aversion combinations the sharing rule in Equation C.2 can be solved for z_1 in closed form, see Wang (1996). Alternatively, a numerical solution is required. To compute equilibrium, we insert the (numerical) solution of the sharing rule into the marginal utility of the reference investor, e.g. investor 1, and employ the later to discount cash flows.

Below superscripts denote derivatives of the utility function with respect to one of its arguments. Using standard Martingale methods, it is easy to prove the following propositions.

Proposition 1
When the economy is in equilibrium, the state price density, the risk-free rate, and the Sharp Ratio, respectively, will be given by

$$\eta(0,t) = e^{-\rho t}\frac{u_1^C(C_1(t), X(t))}{u_1^C(C_1(0), X(0))}$$

$$r(t) = \rho + \gamma_1\mu_{z_1}(t) + \lambda w(t) - \frac{1}{2}\gamma_1(\gamma_1+1)\sigma_{z_1}(t)^\top \sigma_{z_1}(t)$$

$$\theta(t) = \gamma_1\sigma_{z_1}(t) \tag{C.4}$$

where

$$dz_1(t) = z_1(t)\left(\mu_{z_1}(t)dt + \sigma_{z_1}(t)^T dW(t)\right) \tag{C.5}$$

with

$$\mu_{z_1}(t) = \frac{z_1^C(t)}{z_1(t)}\mu_C(t) + \frac{1}{2}\frac{z_1^{CC}(t)}{z_1(t)}C(t)^2\sigma_C(t)^T\sigma_C(t) + \frac{z_1^X(t)}{z_1(t)}X(t)\lambda\omega(t) \tag{C.6}$$

and with

$$\sigma_{z_1}(t) = z_1^C(t)\left(\frac{C(t)}{z_1(t)}\right)\sigma_C(t). \tag{C.7}$$

Proposition 2
When the economy is in equilibrium, stock prices and the components of instantaneous stock price volatilities are given by

$$
\begin{aligned}
S_i(t) &= \frac{1}{\eta(0,t)}E_t\left[\int_t^T \eta(0,s)\delta_i(s)ds\right] \\
\sigma_i(t) &= \theta(t) + \frac{E_t\left[\int_t^T D_t\left(\eta(0,s)\delta_i(s)\right)ds\right]}{\eta(0,t)S_i(t)} \\
&= \theta(t) + \frac{E_t\left[\int_t^T\left(\eta(0,s)\delta_i(s)h(t,s)\right)ds\right]}{\eta(0,t)S_i(t)} + \sigma_{\delta_i} \tag{C.8}
\end{aligned}
$$

where D denotes the Malliavin derivative[1] and h is found at Appendix C.2.

Proposition 3
When the economy is in equilibrium, wealth allocations will be

$$Y_j(t) = \frac{1}{\eta(0,t)} E_t \left[\int_t^T \eta(0,s) C_j(s) ds \right], \tag{C.9}$$

portfolio policies are given by

$$\pi_j(t) = \left(\sigma(t)^\top \right)^{-1} \left[\theta(t) Y_j(t) + \frac{\psi_j(t)}{\eta(0,t)} \right] \tag{C.10}$$

where

$$\psi_j(t) = E_t \left[\int_t^T \left\{ \eta(0,s) C_j(s) h(t,s) + \eta(0,s) H_j(t,s) \right\} ds \right] \tag{C.11}$$

where H is found at Appendix C.2.

C.2   The Malliavin Derivatives

The Malliavin calculus is a generalization of the calculus of variations (see [Nualart 1995b]). One useful result from Malliavin calculus concerns the Clark-Ocone theorem (see [Detemple *et al.* 2003b], and the references therein), which allows for the explicit identification of the Ito integral in the martingale representation theorem (see [Cox & Huang 1989]).

All the Malliavin derivatives used in the paper are standard. The Malliavin derivatives are given by (for $u > t$ with $u, t \in [0,T]$)

$$D_{k,t} \delta_i(u) = \delta_i(u) \sigma_{\delta_{i,k}} \tag{C.12}$$

or in vector notation

$$D_t \delta_i(u) = \delta_i(u) \sigma_{\delta_i} \tag{C.13}$$

and

$$
\begin{aligned}
D_t X(u) &= X(u) D_t \log X(u) \\
&= X(u) \lambda \int_t^u e^{-\lambda(t-v)} D_t c(v) dv \\
&= X(u) \lambda \int_t^u e^{-\lambda(t-v)} s_\delta(v)^\top \sigma_\delta(v) dv
\end{aligned}
\tag{C.14}
$$

---

[1]D denotes a vector operation, and D_i denotes Malliavin derivatives with respect to the i'th Brownian component.

$$
\begin{aligned}
D_t\eta(0,u) &= D_t\left(e^{-\rho u}z_1\left(C_1(u),X(u)\right)^{-\gamma_1}\frac{1}{X(u)}\right)\\
&= \eta(0,u)\left[-\gamma_1\left(\frac{z_1^C}{z_1}\sum_i^N \delta_i(u)\sigma_{\delta_i}\right)\right]\\
&\quad -\eta(0,u)\left[\lambda\left(1+\gamma_1\frac{z_1^X}{z_1}X(u)\right)\int_t^u e^{-\lambda(t-v)}u_\delta(v)^\top\sigma_\delta(v)dv\right]\\
&= \eta(0,u)h(t,u)
\end{aligned}
\tag{C.15}
$$

$$
\begin{aligned}
D_tC_j(u) &= \frac{\partial C_j}{\partial C}\sum_i^N \delta_i(u)\sigma_{\delta_i} + \frac{\partial C_j}{\partial X}X(u)\lambda\int_t^u e^{-\lambda(t-v)}u_\delta(v)^\top\sigma_\delta(v)dv\\
&= H_j(t,u)
\end{aligned}
\tag{C.16}
$$

$$
\begin{aligned}
D_t s_{\delta_i}(u) &= D_t\left(\frac{\delta_i(u)}{C(u)}\right)\\
&= \frac{D_t\delta_i(u)}{C(u)} - \frac{\delta_i(u)D_tC(u)}{C(u)^2}\\
&= \frac{\delta_i(u)}{C(u)}\sigma_{\delta_i} - \frac{\delta_i(u)C(u)\sigma_C(u)}{C(u)^2}\\
&= \frac{\delta_i(u)}{C(u)}\left(\sigma_{\delta_i} - \sigma_C(u)\right)\\
&= s_{\delta_i}(u)\sigma_{s_i}(u)
\end{aligned}
\tag{C.17}
$$

$$
\begin{aligned}
D_tC(u) &= D_t\sum_{i=1}^N \delta_i(u)\\
&= \sum_{i=1}^N D_t\delta_i(u)\\
&= \sum_{i=1}^N \delta_i(u)\sigma_{\delta_i}\\
&= C(u)\sum_{i=1}^N s_{\delta_i}(u)\sigma_{\delta_i}\\
&= C(u)\sigma_C(u)
\end{aligned}
\tag{C.18}
$$

$$
\begin{aligned}
D_t \sigma_C(u)^T &= D_t \left(\left(\sigma_\delta^T s_\delta(u) \right)^T \right) \\
&= D_t \left(s_\delta(u)^T \sigma_\delta \right) \\
&= D_t \left(\sum_{i=1}^N s_{\delta_i}(t) \sigma_{\delta_i}^T \right) \\
&= \sum_{i=1}^N \left(D_t s_{\delta_i}(t) \right) \sigma_{\delta_i}^T \\
&= \sum_{i=1}^N s_{\delta_i}(u) \sigma_{s_i}(u) \sigma_{\delta_i}^T
\end{aligned}
\tag{C.19}
$$

$$
\begin{aligned}
D_t \left(C(u) \sigma_C(u)^T \right) &= D_t \sum_{i=1}^N \left(\delta_i(u) \sigma_{\delta_i}^T \right) \\
&= \sum_{i=1}^N \left(D_t \delta_i(u) \right) \sigma_{\delta_i}^T \\
&= \sum_{i=1}^N \delta_i(u) \sigma_{\delta_i} \sigma_{\delta_i}^T \\
&= C(u) \sum_{i=1}^N s_{\delta_i}(u) \sigma_{\delta_i} \sigma_{\delta_i}^T
\end{aligned}
\tag{C.20}
$$

$$
\begin{aligned}
D_t H_j(t,u)^T &= D_t\left(\frac{\partial C_j(u)}{\partial C}C(u)\sigma_C(u)^T\right) \\
&\quad + D_t\left(\frac{\partial C_j(u)}{\partial X}X(u)\lambda\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv\right) \\
&= \left(\frac{\partial^2 C_j(u)}{\partial C^2}C(u)\sigma_C(u) + \frac{\partial^2 C_j(u)}{\partial C\partial X}X(u)\lambda\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv\right) \\
&\quad C(s)\sigma_C(u)^T + \frac{\partial C_j(u)}{\partial C}C(u)\sum_{i=1}^N s_{\delta_i}(u)\sigma_{\delta_i}(u)\sigma_{\delta_i}^T \\
&\quad + \left(\frac{\partial^2 C_j(u)}{\partial X\partial C}C(u)\sigma_C(u)\right)X(u)\lambda\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv \\
&\quad + \frac{\partial^2 C_j(u)}{\partial X^2}X(u)^2\lambda^2\left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv\right) \\
&\quad \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv\right) \\
&\quad + \frac{\partial C_j(u)}{\partial X}X(u)\lambda^2\left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv\right) \\
&\quad \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv\right) \\
&\quad + \frac{\partial C_j(u)}{\partial X}X(u)\lambda\int_t^u e^{-\lambda(t-v)}\sum_{i=1}^N s_{\delta_i}(v)\sigma_{s_i}(v)\sigma_{\delta_i}^T dv \\
&= G_j(t,u).
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
D_t h(t,u)^T &= -\gamma_1 \left(D_t \left(\frac{z_1^C(u)}{z_1(u)} \right) C(u)\sigma_C(u)^T + \frac{z_1^C(u)}{z_1(u)} D_t \left(C(u)\sigma_C(u)^T \right) \right) \\
&\quad -\lambda \left(1 + \gamma_1 \frac{z_1^X(u)}{z_1(u)} X(u) \right) \int_t^u e^{-\lambda(t-v)} D_t \sigma_C(v)^T dv \\
&\quad -\gamma_1 \lambda \left(D_t \left(\frac{z_1^X(u)}{z_1(u)} \right) X(u) + \left(\frac{z_1^X(u)}{z_1(u)} \right) D_t X(u) \right) \\
&\quad \int_t^u e^{-\lambda(t-v)} \sigma_C(v)^T dv \\
&= -\gamma_1 \left(\frac{z_1^{CC}(u)C(u)\sigma_C(u)}{z_1(u)} \right) C(u)\sigma_C(u)^T \\
&\quad -\gamma_1 \lambda \left(\frac{z_1^{CX}(u)X(u)\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv}{z_1(u)} \right) C(u)\sigma_C(u)^T \\
&\quad +\gamma_1 \left(\frac{z_1^C(u)}{z_1(u)^2} \right) \left(z_1^C(u)C(u)\sigma_C(u) \right) C(u)\sigma_C(u)^T \\
&\quad +\gamma_1 \lambda \left(\left(\frac{z_1^X(u)}{z_1(u)^2} \right) \left(z_1^C(u)X(u)\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv \right) \right) \\
&\quad C(u)\sigma_C(u)^T - \gamma_1 \frac{z_1^C(u)}{z_1(u)} C(u) \sum_{i=1}^N s_{\delta_i}(u)\sigma_{\delta_i}\sigma_{\delta_i}^T \\
&\quad -\lambda \left(1 + \gamma_1 \frac{z_1^X(u)}{z_1(u)} X(u) \right) \int_t^u e^{-\lambda(t-v)} \sum_{i=1}^N s_{\delta_i}(v)\sigma_{s_i}(v)\sigma_{\delta_i}^T dv \\
&\quad -\gamma_1 \lambda \frac{z_1^{XC}(u)C(u)\sigma_C(u)}{z_1(u)} X(u) \int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv \\
&\quad -\gamma_1 \lambda^2 \frac{z_1^{XX}(u)}{z_1(u)} X(u)^2 \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv \right) \\
&\quad \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv \right) \\
&\quad +\gamma_1 \lambda \left(\frac{z_1^X(u)}{z_1(u)^2} \right) \left(z_1^C(u)C(u)\sigma_C(u) \right) X(u) \int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv \\
&\quad +\gamma_1 \lambda^2 \left(\frac{z_1^X(u)}{z_1(u)^2} \right) z_1^X(u)X(u)^2 \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv \right) \\
&\quad \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv \right) \\
&\quad -\gamma_1 \lambda^2 \left(\frac{z_1^X(u)}{z_1(u)} \right) X(u) \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)dv \right) \\
&\quad \left(\int_t^u e^{-\lambda(t-v)}\sigma_C(v)^T dv \right) \\
&= g(t,u). \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (C.22)
\end{aligned}
$$

C.3 Quadratic Variation of Portfolio Policies

To calculate the diffusion coefficient of $\pi_j(t)$ apply Ito's lemma on C.10. As we are only interested in diffusion terms, that is $\sigma_\sigma(t)$, $\sigma_\theta(t)$, $\sigma_{Y_j}(t)$, $\sigma_{\psi_j}(t)$, and $\sigma_\eta(t)$, below we drop all drift terms.

The diffusion of the Sharpe ratio, $\sigma_\theta(t)$, is given by

$$\sigma_\theta(t) = \gamma(t)\sigma_{\sigma_C}(t) + \sigma_C(t)\sigma_\gamma(t) \tag{C.23}$$

where

$$d\theta(t) = \gamma(t)d\sigma_C(t) + \sigma_C(t)d\gamma(t) + d\gamma(t)d\sigma_C(t). \tag{C.24}$$

To obtain $d\sigma_\gamma$ apply Ito's lemma on z_1

$$
\begin{aligned}
d\left(\frac{\partial z_1(t)}{\partial C}\right) &= \frac{\partial^2 z_1(t)}{\partial C^2}C(t)\left(\mu_C(t)dt + \sigma_C(t)^T dW(t)\right) \\
&\quad + \frac{\partial^2 z_1(t)}{\partial C \partial X}\lambda\omega(t)X(t)dt + \frac{\partial^3 z_1(t)}{\partial C^3}C(t)^2\sigma_C(t)^T\sigma_C(t) \\
&= \left(\frac{\partial^2 z_1(t)}{\partial C^2}C(t)\mu_C(t) + \frac{\partial^2 z_1(t)}{\partial C \partial X}\lambda\omega(t)X(t)\right)dt \tag{C.25} \\
&\quad + \left(\frac{\partial^3 z_1(t)}{\partial C^3}C(t)^2\sigma_C(t)^T\sigma_C(t)\right)dt \\
&\quad + \frac{\partial^2 z_1(t)}{\partial C^2}C(t)\sigma_C(t)^T dW(t) \tag{C.26}
\end{aligned}
$$

we also have that

$$
\begin{aligned}
d\left(\frac{C(t)}{z_1(t)}\right) &= \left(\frac{C(t)}{z_1(t)}\right)\left(\mu_C(t) - \mu_{z_1}(t) + \sigma_{z_1}(t)^T\left(\sigma_{z_1}(t) - \sigma_C(t)\right)\right)dt \\
&\quad + \left(\frac{C(t)}{z_1(t)}\right)\left(\sigma_{z_1}(t) - \sigma_C(t)\right)^T dW(t) \tag{C.27}
\end{aligned}
$$

next we obtain

$$
\begin{aligned}
d\sigma_C(t) &= d\left(\sigma_\delta^T s_\delta(t)\right) \\
&= \sigma_\delta^T ds_\delta(t) \\
&= \sigma_\delta^T I_{s_\delta}(t)\left(\nu_s(t)dt + \sigma_s(t)dW(t)\right) \\
&= \mu_{\sigma_C}(t)dt + \sigma_{\sigma_C}(t)dW(t) \tag{C.28}
\end{aligned}
$$

where

$$\sigma_{\sigma_C}(t) = \sigma_\delta^T I_{s_\delta}(t)\sigma_s(t) \tag{C.29}$$

where

$$\mu_{\sigma_C}(t) = \sigma_\delta^T I_{s_\delta}(t)\nu_s(t) \tag{C.30}$$

and where

$$ds_\delta(t) = I_{s_\delta}(t)\left(\nu_s(t)dt + \sigma_s(t)dW(t)\right) \tag{C.31}$$

with

$$\nu_s(t) = \begin{bmatrix} \mu_{s_1}(t) - \sigma_{s_1}(t)^T \sigma_C(t) \\ . \\ . \\ \mu_{s_N}(t) - \sigma_{s_N}(t)^T \sigma_C(t) \end{bmatrix} \tag{C.32}$$

which is the compact form of Equations 4.7-4.8. Define an aggregate agent as

$$U(C, X) = a \frac{z_1^{1-\gamma_1}}{1 - \gamma_1} + (1 - a) \frac{z_1^{1-\gamma_2}}{1 - \gamma_2} \tag{C.33}$$

and to obtain $d\gamma_C$ note that

$$\gamma(t) = \gamma_1 \frac{\partial z_1(t)}{\partial C} \left(\frac{C(t)}{z_1(t)} \right) \tag{C.34}$$

represents the coefficient of relative risk aversion (induced by market clearing in equilibrium) with dynamics

$$
\begin{aligned}
d\gamma(t) &= [.] \, dt + \gamma_1 \frac{\partial^2 z_1(t)}{\partial C^2} \left(\frac{C(t)}{z_1(t)} \right) C(t) \sigma_C(t)^T dW(t) \\
&\quad + \gamma_1 \frac{\partial z_1(t)}{\partial C} \left(\frac{1}{z_1(t)} \right) C(t) \sigma_C(t)^T dW(t) - \gamma_1 \left(\frac{\partial z_1(t)}{\partial C} \right)^2 \left(\frac{C(t)}{z_1(t)^2} \right) \\
&\quad C(t) \sigma_C(t)^T dW(t) \\
&= [.] \, dt + \gamma_1 \left(\frac{\partial^2 z_1(t)}{\partial C^2} \left(\frac{C(t)}{z_1(t)} \right) + \frac{\partial z_1(t)}{\partial C} \left(\frac{1}{z_1(t)} \right) - \left(\frac{\partial z_1(t)}{\partial C} \right)^2 \left(\frac{C(t)}{z_1(t)^2} \right) \right) \\
&\quad C(t) \sigma_C(t)^T dW(t) \\
&= [.] \, dt + \sigma_\gamma(t)^T dW(t).
\end{aligned}
\tag{C.35}
$$

The stock price diffusion coefficients, $\sigma_{\sigma_i}(t)$, are derived below. First recall Equation C.8 and denote

$$Q_{i1}(t) = E_t \left[\int_t^T \eta(0, s) \, \delta_i(s) h(t, s) ds \right] \tag{C.36}$$

which leads to

$$
\begin{aligned}
Q_{i1}(t) &= E_t \left[\int_0^T \eta(0, s) \, \delta_i(s) h(t, s) ds \right] - \int_0^t \eta(0, s) \, \delta_i(s) h(t, s) ds \\
&= Q_{i1}^M(t) - \int_0^t \eta(0, s) \, \delta_i(s) h(t, s) ds
\end{aligned}
\tag{C.37}
$$

where Q_{i1}^M stands for the martingale part of Q_{i1}. By Clark-Ocone's theorem

$$dQ_{i1}^M(t) = \sigma_{Q_{i1}}(t) dW(t) \tag{C.38}$$

where

$$\sigma_{Q_{i1}}(t)^T = E_t \left[\int_0^T D_t \left(\eta(0,s) \, \delta_i(s) h(t,s)^T \right) ds \right] \tag{C.39}$$

$\sigma_{Q_{i1}}(t)$ is a $N \times N$ matrix. Calculating the Malliavin derivative leads to

$$D_t \left(\eta(0,s) \, \delta_i(s) h(t,s)^T \right)$$
$$= \delta_i(s) h(t,s)^T D_t \eta(0,s) + \eta(0,s) \, h(t,s)^T D_t \delta_i(s) + \eta(0,s) \, \delta_i(s) D_t h(t,s)^T$$
$$= \eta(0,s) \, \delta_i(s) h(t,s) h(t,s)^T + \eta(0,s) \, \delta_i(s) \sigma_{\delta_i} h(t,s)^T + \eta(0,s) \, \delta_i(s) g(t,s). \tag{C.40}$$

Using the above expressions and the following relation

$$d \left(\eta(0,t) \, S_i(t) \right) = \eta(0,t) \, S_i(t) \left(\sigma_i(t) - \theta(t) \right)^T dW(t) \tag{C.41}$$

leads to

$$
\begin{aligned}
d\sigma_i(t) &= [.] \, dt + \gamma(t) d\sigma_C(t) + \sigma_C(t) d\gamma(t) \\
&\quad + \left(\frac{1}{\eta(0,t) \, S_i(t)} \right) \sigma_{Q_{i1}}(t) dW(t) + \frac{Q_{i1}(t)}{\eta(0,t) \, S_i(t)} \left(\theta(t) - \sigma_i(t) \right)^T dW(t) \\
&= [.] \, dt + \gamma(t) \sigma_{\sigma_C}(t) + \sigma_C(t) \sigma_\gamma(t)^T dW(t) \\
&\quad + \left(\frac{1}{\eta(0,t) \, S_i(t)} \right) \left(\sigma_{Q_{i1}}(t) + (\theta(t) - \sigma_i(t)) (\theta(t) - \sigma_i(t))^T \right) dW(t) \\
&= [.] \, dt + \left(\gamma(t) \sigma_{\sigma_C}(t) + \sigma_C(t) \sigma_\gamma(t)^T + \sigma_{Q_{i1}}(t) \right) dW(t) \\
&\quad + \left((\theta(t) - \sigma_i(t)) (\theta(t) - \sigma_i(t))^T \right) dW(t) \\
&= [.] \, dt + \sigma_{\sigma_i}(t) dW(t). \tag{C.42}
\end{aligned}
$$

Hedging term diffusion coefficients are derived next. Recall Equation C.11 and apply Clark-Ocone's theorem to obtain

$$d\psi_j(t) = [.] \, dt + \sigma_{\psi_j}(t) dW(t) \tag{C.43}$$

where $\sigma_{\psi_j}(t)$ represents a $N \times N$ matrix given by

$$
\begin{aligned}
\sigma_{\psi_j}(t)^T &= E_t \left[\int_t^T D_t \left\{ \eta(0,s) C_j(s) h(t,s)^T + \eta(0,s) H_j(t,s)^T \right\} ds \right] \\
&= E_t \left[\int_t^T \left\{ \eta(0,s) C_j(s) h(t,s) h(t,s)^T + \eta(0,s) C_j(s) g_j(t,s) \right\} ds \right] \\
&\quad + E_t \left[\int_t^T \left\{ \eta(0,s) h(t,s) H_j(t,s)^T + \eta(0,s) G_j(t,s) \right\} ds \right]. \tag{C.44}
\end{aligned}
$$

Finally, $\sigma_{Y_j}(t)$ and $\sigma_\eta(t)$ are easily obtained from Equations C.9 and C.4.

C.3.1   Example

Define

$$A(t) = \left(\sigma(t)^T \right)^{-1} \tag{C.45}$$

and

$$v(t) = \left(\theta(t) Y_j(t) + \frac{\psi_j(t)}{\eta(0,t)} \right) \tag{C.46}$$

and rewrite the optimal portfolio, Equation C.10, as follows

$$\pi_j(t) = A(t)v(t) = \left[\begin{array}{c} A_{11}(t)v_1(t) + A_{12}(t)v_2(t) \\ A_{21}(t)v_1(t) + A_{22}(t)v_2(t) \end{array} \right]. \tag{C.47}$$

A direct application of Ito's lemma (drift terms are disregarded) leads to

$$
\begin{aligned}
d\pi_j(t) &= \left[\begin{array}{c} d\pi_{j1}(t) \\ d\pi_{j2}(t) \end{array} \right] \\
&= d \left[\begin{array}{c} A_{11}(t)v_1(t) + A_{12}(t)v_2(t) \\ A_{21}(t)v_1(t) + A_{22}(t)v_2(t) \end{array} \right] \\
&= \left[\begin{array}{c} A_{11}(t)dv_1(t) + dA_{11}(t)v_1(t) + A_{12}(t)dv_2(t) + dA_{12}(t)v_2(t) \\ A_{21}(t)dv_1(t) + dA_{21}(t)v_1(t) + A_{22}(t)dv_2(t) + dA_{22}(t)v_2(t) \end{array} \right] \\
&= \left[\begin{array}{c} A_{11}(t)\sigma_{v_1}(t) + v_1(t)\sigma_{A_{11}}(t) + A_{12}(t)\sigma_{v_2}(t) + v_2(t)\sigma_{A_{12}}(t) \\ A_{21}(t)\sigma_{v_1}(t) + v_1(t)\sigma_{A_{21}}(t) + A_{22}(t)\sigma_{v_2}(t) + v_2(t)\sigma_{A_{22}}(t) \end{array} \right] \left[\begin{array}{c} dW_1(t) \\ dW_2(t) \end{array} \right]
\end{aligned} \tag{C.48}
$$

We have that

$$\left(\sigma(t)^T \right)^{-1} = \left(\left[\begin{array}{cc} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{array} \right]^T \right)^{-1} = \frac{1}{|\sigma(t)|} \left[\begin{array}{cc} \sigma_{22}(t) & -\sigma_{21}(t) \\ -\sigma_{12}(t) & \sigma_{11}(t) \end{array} \right] \tag{C.49}$$

with

$$|\sigma(t)| = \sigma_{22}(t)\sigma_{11}(t) - \sigma_{12}(t)\sigma_{21}(t). \tag{C.50}$$

Using the fact that

$$d\sigma_{ij}(t) = [.]\, dt + \sigma_{\sigma_{ij}}(t)^T dW(t) \tag{C.51}$$

enables to obtain the following result

$$
\begin{aligned}
d\,|\sigma(t)| &= [.]\, dt + \left(\sigma_{22}(t)\sigma_{\sigma_{11}}(t)^T + \sigma_{11}(t)\sigma_{\sigma_{22}}(t)^T \right) dW(t) \\
&\quad - \left(\sigma_{12}(t)\sigma_{\sigma_{21}}(t)^T + \sigma_{21}(t)\sigma_{\sigma_{12}}(t)^T \right) dW(t) \\
&= [.]\, dt + \sigma_{|\sigma(t)|}(t)^T dW(t).
\end{aligned} \tag{C.52}
$$

Combining the above leads to

$$
\begin{aligned}
dA_{11}(t) &= [.]\, dt + \left(\sigma_{\sigma_{22}}(t) - \sigma_{|\sigma(t)|}(t) \right)^T dW(t) \\
dA_{12}(t) &= [.]\, dt + \left(-\sigma_{\sigma_{21}}(t) - \sigma_{|\sigma(t)|}(t) \right)^T dW(t) \\
dA_{21}(t) &= [.]\, dt + \left(-\sigma_{\sigma_{12}}(t) - \sigma_{|\sigma(t)|}(t) \right)^T dW(t) \\
dA_{22}(t) &= [.]\, dt + \left(\sigma_{\sigma_{11}}(t) - \sigma_{|\sigma(t)|}(t) \right)^T dW(t).
\end{aligned} \tag{C.53}
$$

Turning to the dynamics of $v(t)$, we have

$$
\begin{aligned}
dv(t) &= d\left(\theta(t)Y_j(t) + \frac{\psi_j(t)}{\eta(0,t)}\right) \\
&= \theta(t)dY_j(t) + Y_j(t)d\theta(t) + \frac{1}{\eta(0,t)}d\psi_j(t) + \psi_j(t)d\left(\frac{1}{\eta(0,t)}\right) \\
&= [.]\,dt + \theta(t)Y_j(t)\sigma_{Y_j}(t)^T dW(t) + Y_j(t)\sigma_\theta(t)dW(t) \\
&\quad + \left(\frac{1}{\eta(0,t)}\sigma_{\psi_j}(t)^T - \frac{\psi_j(t)}{\eta(0,t)}\theta(t)^T\right)dW(t) \\
&= [.]\,dt + \sigma_v(t)dW(t).
\end{aligned} \tag{C.54}
$$

Finally, the quadratic variation of the portfolio policy, Equation 4.21, is thus given by

$$
QV_j(t) = \sigma_{\pi_j}(t)^T \sigma_{\pi_j}(t)dt \tag{C.55}
$$

C.4 Technical Details of Monte Carlo Simulations

In this section I describe the numerical procedure to solve for the equilibrium quantities. The model is solved using Monte-Carlo (MC) simulations. The state variables are simulated forward using an Euler scheme with 10000 sample paths and 1000 time steps. I use antithetic sampling to reduce the variance. For each time step I calculate the optimal allocation of the habit adjusted consumption between the two agents by solving the central planner problem. The sharing rule is solved by Newton's method.The derivatives of of the optimal allocations are calculated using finite differences. The time integrals are calculated by using the trapezoid rule with 1000 steps.

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