BI NORWEGIAN SCHOOL OF MANAGEMENT

Discussion Paper
2/2004

ISSN: 0807-3406

The Correspondence Principle and Structural Stability in Non-Maximum Systems

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June 22, 2004

Abstract

The correspondence principle suggests a link between asymptotic stability properties of equilibria of economic models and the equilibrium response to data that describe the model or the model environment. However, this link has been impaired by a logical-mathematical deficiency. This paper, by introducing a conceptual requirement of (local) structural stability as part of the principle hypotheses, rectifies the relation between qualitative properties of equilibria and the analysis of variations. Two related examples are given. The first completes Dierkers' proof of a unique equilibrium in regular Arrow–Debreu economies, where all price systems are locally stable relative to a tâtonnement process. The second validates linear approximation analysis of deterministic continuous time rational expectation models. The paper's focus on local analysis makes it possible to handle potentially difficult problems in a straightforward manner.

JEL Classification: C60, C62

Keywords: correspondence principle, structural stability, regular economies, rational expectations

* I wish to thank Alexandra C. Katz, Espen R. Moen and in particular Atle Seierstad for helpful discussions and valuable comments.

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> [...] James Clerk Maxwell applied stability theory to the study of the rings of planet Saturn. He decided that they must be composed by many small separate bodies, rather than being solid or fluid, for only in the former case are there *stable* solutions of the equations of motions. He discovered that while solid or fluid rings were mathematically possible, the slightest perturbation would destroy their configuration. (Hirsch & Smale [1974], p. 191.)

Economic theory has developed tools for the study of equilibria in various environments. While arguably the core results of these studies have to do with what could be described as *intrinsic* qualities of equilibria (existence, determinacy, efficiency etc.), a systematic analysis of *variations*, that is, how the set of equilibria responds to changes in parts of the environment, is also important. Samuelson [1941,1942,1947] envisioned a link between the two fields of study in what he termed the correspondence principle (CP). In non-maximum systems, the principle roughly amounts to saying that if one studies a phenomenon that can be assumed to be stable, then this information can be used in analyzing comparative statical and (possibly) a limiting form of comparative dynamical properties of equilibria.[1]

However, the fascinating CP has been impaired by a mathematical-logical deficiency: conditions imposed from dynamical considerations are only sufficient, not necessary, for the assumed stability property. This incongruity mirrors the inference problems regarding the dynamics of a non-linear system from its linearization around a stationary point.

The CP only uses information about the initial equilibrium when analyzing changes in parameters or exogenous variables. I argue that the application of the CP presupposes that the perturbed equilibrium is of the same "kind" as the initial one (for instance that a locally asymptotically stable equilibrium remains so). A precise formulation of this idea requires that

[1] Velupillai [1973] attributes the first (implicit) formulation of the correspondence principle in economics to Hicks (Hicks [1939] p. 62) and traces its origin to fundamental work in theoretical physics, in particular to Niels Bohr.

I will not comment on the CP with respect to the other branch of application, notably where the equilibrium conditions are (or can be reduced to) first order conditions for a parameterized optimization problem. In this class of situations, the interplay with second order conditions takes the place of stability conditions. Malliaris & Broch [1989] gives a comprehensive treatment in an optimal control framework.

In systems where the dynamics can be represented in a one-dimensional phase space, a particularily powerful global approach can be taken to the CP, see Bhagwati, Brecher & Hatta [1987] (see also the reference to Echenrique in note 5 below).

a notion of qualitative invariance and an allowable class of perturbations are defined. They lead to the concept of structural stability (SS), and in particular to local structural stability (LSS). I include LSS as a part of the CP hypotheses and thereby reexamine the relation between variations and the qualitative structure of equilibria.[2]

The application of the fundamental conceptual requirement of LSS resurrects the CP and yields results relevant to the analysis of models in general. I give two examples that are related to the CP. The first completes Dierker's proof of uniqueness of Walrasian equilibrium in regular Arrow–Debreu-economies that have asymptotically stable equilibria with respect to the tâtonnement process (Dierker [1972], Theorem 2). The second validates linear approximation analysis of deterministic continuous time rational expectations models, where a conditional asymptotic stability notion is part of the solution concept.

There are some technicalities involved in the definition of SS. However, by focusing on local analysis around fixed points, rather than working in an equilibrium correspondence framework and global properties of flows, this paper deals with potentially difficult material in a straightforward way.

The paper is organized as follows: Section 1 sets the scene for the CP and explains the various situations where it has been applied. Section 2 motivates the (L)SS requirement and provides the necessary mathematical terminology and definitions. Section 3 is on the resurrection of the CP. Section 4 is on Walrasian equilibria, and Section 5 on the analysis of forward looking models. Section 6 concludes.

1. The Correspondence Principle

I assume initially existence of a finite-dimensional vector of parameters or exogenous variables μ in some (open) subset M of $\mathbf{R}^l$, and that the model world is described by a system of equations $f(x;\mu)=0$. Here, $f: X \times M \to \mathbf{R}^n$, and X – the set of conceivable states – is some (open) subset of $\mathbf{R}^n$. The equilibrium set E,

[2] Fuchs [1975] is, to the best of my knowledge, the first to formally introduce the notion of SS to economic theory. He calls SS a "consistency requirement" for dynamical models and provides a link between certain continuity assumptions and sufficiency theorems for structural stability, including typicality results. The present paper focuses on continuous time evolution and assumes a simple necessary condition for SS that is necessary and sufficient for LSS (a "global" sufficiency result is reported in Sec. 4).

(1) $E(\mu) = \left\{ x \in X \,\middle|\, f(x;\mu) = 0 \right\}$,

is assumed to be non-empty on a sufficiently large part of *M*. The variation question can be posed as: how does the constellation of equilibrium variables, the set *E*, vary in *X* as the parameter system varies in *M*? In full generality, this amounts to a study of the equilibrium correspondence $E: M \to X$.[3]

A standard approach is to do a local analysis of an equilibrium $\bar{x} \in E(\mu)$, given a smoothness assumption $f \in C^k\left(X \times M\right)$, $1 \le k \le \infty$, and *assuming* a regularity property of full rank of the Jacobian matrix $D_x f(\bar{x}, \mu)$ at the point. Then, by the implicit function theorem (IFT), there are open sets $X' \subset X$ and $M' \subset M$ around $\bar{x}$ and μ, and a C^k function $g: M' \to X'$ such that $f(x;\mu) = 0$ hold for $(x, \mu) \in X' \times M'$ if and only if $x = g(\mu)$.

Hence, given full rank, the IFT ensures that comparative statics is well-defined locally (isolated equilibria depend in a continuous way on the data of the economy), and[4]

(2) $Dg(\mu) = -D_x f(g(\mu);\mu)^{-1} D_\mu f(g(\mu);\mu)$.

However, due to the system-determinedness of the endogenous variables, it is hard to derive what Samuelson called "fruitful theorems in comparative statics". To further this aim, he suggested studying $g(\mu)$ with reference to an out-of-equilibrium dynamical system ("fast dynamics")[5]

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[3] Balasko [1988] and Mas-Colell [1985] give an exhaustive account of this approach in general equilibrium theory (see also the references in Sec. 4 below). See Van Damme [1991] for applications to game theory.

[4] The determinacy and robustness properties are called "persistence". Note that the assumptions of the IFT are sufficient, not necessary, for these qualities (but necessary for g to be C^k), see Mas-Colell [1985], Theorem C.3.2 p. 20. The requirement of LSS will be seen to imply the full rank assumption (called regularity in Sect. 4). The IFT does not provide existence of the "initial" equilibrium (Sec. 4 refers to a theorem that does).

[5] "Where the interactions between individuals are concerned, the scope of fruitful comparative statics may be greatly extended by […] the apprehension of the correspondence principle, whereby the comparative statical behavior of a system is seen to be closely related to its dynamical stability properties" (Samuelson [1947] p. 351).

In strategic settings, deriving comparative statics results is especially difficult due to a high degree of simultaneity created by the equilibrium notions. A recent approach applies the concept of so-called supermodular games to compare equilibria. Echenrique [2002] uses out of equilibrium dynamics and the CP to refine equilibria in this context.

(3) $\frac{dx}{dt} \stackrel{def}{=} \dot{x} = f(x;\mu)$.

In this context, f is a vector field on X, and its vanishing point, $g(\mu) \in E$ a member of the class of solutions to the differential equations (I discuss the legimaticy of calling (3) a dynamical system and its interpretation in Sec. 2.1). Following standard mathematical terminology, I call $g(\mu)$ a fixed point. Samuelson argued that for the calculation of Dg to make sense, the fixed point has to be *asymptotically stable*.[6] The assumption that the observed equilibrium is stable would then give information about the structure of the Jacobian $D_x f(g(\mu);\mu)$ that could aid in signing the elements of (2).

However, the suggested duality between comparative statics and stability is problematic due to the fact that the restrictions imposed on $D_x f(g(\mu);\mu)$ by the condition of asymptotic stability of the fixed point are *not necessary* for the stability of (3): Let $\Lambda(D_x f(\bar{x};\mu))$ be the collection of eigenvalues of $D_x f$ at the fixed point, and let $\mathrm{Re}(\lambda_i)$ denote the real part of the eigenvalue λ_i. The condition $\mathrm{Re}(\lambda_i(\mu)) < 0$, $i = 1,...,n$ (allowing for possible multiplicities) is necessary and sufficient for the asymptotic stability of the linear approximation system $\dot{x} = D_x f(g(\mu);\mu)(x - g(\mu))$, but only sufficient, not necessary, for the asymptotic stability of the non-linear system (3).[7] Hence Arrow & Hahn's [1971] statement (albeit on the assumption of gross substitutes in general equilibrium contexts, p. 321) is accurate:

[6] Let φ^f denote the flow of (3) and $B_r(a)$ an open ball around a in $\mathbf{R}^n$ with radius r. A fixed point $\bar{x} \in E(\mu)$ is *stable* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all "initial conditions" $x \in B_\delta(\bar{x})$ and $t \geq 0$, $\varphi^f(t,x) \in N_\varepsilon(\bar{x})$. The fixed point is *asymptotically stable* if it is stable and there exists a $\delta > 0$ such that for all $x \in B_\delta(\bar{x})$, $\lim_{t\to\infty} \varphi^f(t,x) = \bar{x}$. If there are regions in $\mathbf{R}^n$ relative to which $\bar{x}$ is (asymptotically) stable, then $\bar{x}$ is called *conditionally stable* (Coddington & Levinson [1955] Ch. 13.4), see Section 5 below.

[7] Mistakes recur in the literature on this point. Consider for example the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \mu x_1^2 x_2\end{aligned}.$$

At the fixed point $\bar{x} = (0,0)$, $\Lambda(D_x f(\bar{x};\mu)) = \{-i, i\}$, hence $\bar{x}$ is a *center* of the linear flow $\exp(tD_x f(\bar{x};\mu))x$; solutions are periodic with the same period (Hirsch & Smale [1974] p. 95). However, since $\bar{x}$ is an example of a *non-hyperbolic* fixed point (see Sec. 2.2 below), conclusions about the local behaviour of the nonlinear system cannot be drawn from the linearization. In fact, $\bar{x}$ is globally asymptotically stable for $\mu > 0$ and unstable for $\mu < 0$ (Guckenheimer & Holmes [1983], p. 13). The qualitative change in the structure of solutions at $\mu = 0$ is called a (local) *bifurcation*, and $\mu = 0$ a bifurcation value.

All these restrictions share the characteristic that they are not necessary for the task for which they were invented; they are only sufficient and this explains why the correspondence principle "isn't".

The CP has also been used in settings where the dynamics is considered a real time process ("slow dynamics") and where the (economic) interpretation of the equilibrium concept is richer. The distinction between adjustment and real time dynamics may be explained by a digression into the mathematically equivalent formulation[8]

$$
(3') \qquad \begin{array}{l} F^1(y_1, \dot{y}_1, ..., y_1^{(q_1)}, ..., y_p, \dot{y}_p, ..., y_p^{(q_p)}; \mu) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ F^p(y_1, \dot{y}_1, ..., y_1^{(q_1)}, ..., y_p, \dot{y}_p, ..., y_p^{(q_p)}; \mu) = 0 \end{array} ,
$$

where $y_i^{(q_i)}$ denotes the highest derivative of the function y_i that enters the system, whose order is $q = \sum q_i$. As first pointed out by Frisch [1929[9],1936], equilibrium in a model of an economy extending over time has two meanings: i) an *instantaneous equilibrium*; the $p+q$ variables $y_1, \dot{y}_1, ..., y_1^{(q_1)}, ..., y_p, \dot{y}_p, ..., y_p^{(q_p)}$ ii) a *total dynamic equilibrium*; the *p*-dimensional solution trajectory $y(t;\cdot) = (y_1(t;\cdot), ..., y_p(t;\cdot))$ of (3'), see Section 2.2 for details.

In the comparative static analysis of (3), a permanent change in the parameter system from one constant level to another is envisioned (see Sec. 2.1), and the dynamics is *not* taken to describe the transition path between the fixed points; under fast adjustment dynamics, only the asymptotic states are observable. Under the slow dynamics envisioned in (3'), data can vary in a much richer way (an infinity of parameter system paths can be considered in comparative dynamic analysis, see Sec. 5).

In the context of real time dynamics, the CP has been applied in two ways. First, to point-in-time analysis of the instantaneous equilibrium. Under standard continuity assumptions on

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The necessary conditions for asymptotic stability of the non-linear system (3) are (allowing for multiplicities) $\mathrm{Re}(\lambda_i(\mu)) \leq 0$, $i = 1, ..., n$, Hirsch & Smale [1974] p. 187.

[8] By solving (3') with respect to the highest derivatives and by introducing all the other derivatives that appear in the system as new variables, the so-called normal form (3) can be obtained, see Pontryagin [1962] Ch. 1.4. The terminology of fast and slow dynamics is used e.g. in Balasko [1988].

$y(t;\cdot)$, all variables, except the highest derivative of each function $y_i(t;\cdot)$, are tied down by q initial conditions[10] that describe the historically given state of the system and that render these variables invariant (momentarily "exogenous") under variations in the parameter system. The p highest derivatives $y_1^{(q_1)},...,y_p^{(q_p)}$ are determined by the p functional equations in (3'). An adjustment process refers to these variables, the assumed stability of which is used to infer their momentary reactions to parameter shifts and to place restrictions on the functional equations. This analysis of (3') is very similar to the fixed point analysis in (3).

The other application of the CP, a limiting form of comparative dynamic analysis, refers to (eventual) stationary states $\bar{y} = (\bar{y}_1,...,\bar{y}_p)$ of (3'), defined by the algebraic equations

$$\begin{aligned} &F^1(\bar{y}_1,0,...0,\bar{y}_p,0,...,0;\mu) = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ &F^p(\bar{y}_1,0,...0,\bar{y}_p,0,...,0;\mu) = 0 \end{aligned}$$

Although $\bar{y}$ technically can be analysed as a fixed point in a (q-dimensional) system in normal form, the question of existence and (asymptotic) stability of stationary states in real time models raises different problems than that of the fixed point in (3).[11]

Here I touch upon unresolved "ontological" problems surrounding the CP, that in the case of adjustment dynamics reflect the lack of a necessary or axiomatic disequilibrium process and the relativity of the comparative static implications to the dynamical system.[12] Except for the motivation for (L)SS in Section 2.1, these problems will not be addressed. Note, however, that in rational expectations models – real time models where a conditional asymptotic stability notion is part of the *solution concept* – the question of the legitimacy of the asymptotic stability requirement has a different status. (The choice of a stable solution path has been characterized as a variant of the CP, see e.g. Sargent [1987] p. 39.) Finally, note that in this class of

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[9] This article has only recently partly been made available in English (see the reference list).

[10] $y_1(0) = {}_0y_1, \dot{y}_1(0) = {}_0\dot{y}_1,..., y_1^{(q_1-1)}(0) = {}_0y_1^{(q_1-1)},..., y_p(0) = {}_0y_p, \dot{y}_p(0) = {}_0\dot{y}_p,..., y_p^{(q_p-1)}(0) = {}_0y_p^{(q_p-1)}$.

[11] Samuelson [1941,1942,1947] applied the CP in the two first-mentioned situations, that is, with reference to adjustment dynamics. However, he clearly saw the possibility of using the CP in the last mentioned setting ([1947], Ch. 12), and discussed the apparatus introduced by Frisch in detail.

[12] See Kemp [1986] for an overview.

models the instantaneous equilibrium cannot be subject to comparative static analysis with respect to any instant of time.[13]

2. Structural Stability: Motivation and Definitions

2.1 Informal discussion

Figure 1 illustrates the situation analyzed in (2) for a small change in the parameter system from μ^0 to μ^1, with corresponding fixed points $\bar{x}^0 = g(\mu^0)$ and $\bar{x}^1 = g(\mu^1)$.

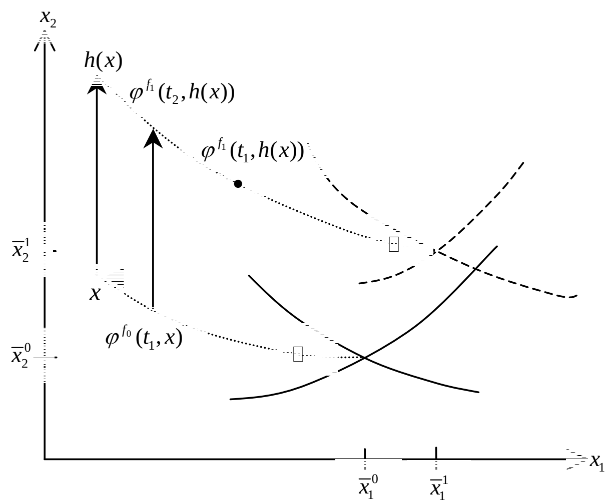

Figure 1. Adjustment dynamics and qualitative invariance around the fixed points $\bar{x}^0$ and $\bar{x}^1$ in the phase space. The solid and dashed curves (isoclines) are the graphs of $f(x;\mu^0) = 0$ and $f(x;\mu^1) = 0$, respectively. The homeomorphism $h(\cdot)$ connecting trajectories is defined in Section 2.2. (In a real time interpretation, a trajectory might have a kink at moments when the parameter system shifts or possibly a discontinuity in forward-looking models, see Sec. 2.2 and 5.)

Note that (2) only uses information about the initial equilibrium, including the structure assigned to $D_x f(g(\mu);\mu)$ from an assumption of asymptotic stability (or conditional asymptotic stability). The focus on the independence of initial conditions that the asymptotic stability notion embodies, is sufficient only when the reason for the postulated stable economy to

[13] The continuity assumption is dispensed with for some components of $y(t;\cdot)$ at certain points in time, determined by when new "information" becomes available to the economy and by the analysts' partitioning of the variables into predetermined ("backward looking") and non-predetermined ("forward looking"). This means that some of the lower derivatives or variables on level form are not pinned down by the state given by the historical development of the system; the instantaneous equilibrium is underdetermined as a function of the parameter system at any one moment of time (some of the initial conditions are replaced with an assumption of convergence of the total dynamic equilibrium as time goes to infinity). As pointed out by Sargent [1987], to do comparative static analyses at given time points requires that there is no dependence of variables at one moment of time on equilibrium values of those at later moments (see Ch. 1.9 and Sec. 5 below).

be out of equilibrium is a perturbation of the impulse kind, and not the persistent parameter shift envisioned in (2). Applying the asymptotic stability notion to the family of solution curves to (3), it is important to note that the vector field *f* itself has undergone permanent change, see Figure 1. Hence, for "fruitful" analysis the CP must (implicitly) assume that the resulting fixed point $\overline{x}^1$ is of the same "kind" as $\overline{x}^0$: that an (eventually conditionally) asymptotically stable fixed point remains so. I consider the effect of making this notion explicit, and require for *f* that the solutions to (3) remain qualitatively invariant around the fixed point in face of (possibly small) variations in *M*.[14]

I am, in fact, going to make a somewhat tougher demand on *f*. The set of parameters or exogenous variables singled out for analysis is somewhat arbitrary, and it would also be arbitrary to limit the invariance property to the subset of parameters explicitly chosen by the analyst to be studied by (2). Furthermore, even though the most typical "economic variations" are studied as variations in finite-dimensional spaces of parameters like *M*, plenty of situations are not (one could, for instance, think of economies defined by demand functions or more general spaces of agents characteristics). In the context of adjustment dynamics, it is of particular importance that the system (3) is typically not a derived one, but rather ad hoc in the sense that one would be willing to look at any transformed system

(4) $\dot{x} = F \circ f(x), \ \ F \in \mathfrak{I}^1,$

where $\mathfrak{I}^1$ is the set of C^1 functions such that $F(0)=0$, $F^i = F^i(f^i(x))$, and $\nabla F \gg 0$, $i = 1,...,n$ (see F. Hahn [1982]).[15] These observations motivate demanding qualitative invariance with respect to more general variations in the vector field. Section 2.2 refines these ideas.

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[14] Frisch and Samuelson, in the context of real time dynamics, clearly (implicitly) assumed LSS. Frisch defined a system originally in a stationary equilibrium as stable if it after an exposure to a small "rupture in the initial conditions, or in *the influencing coefficients* [---] tends back to the original equilibrium situation, or *to another equilibrium situation* close to the original one" (Frisch [1936] p. 102; my emphasis). "In the case of stable systems the response to a *permanent alteration* gives us a description of the actual path followed by a system in *going from one 'comparative static level' to another*" (Samuelson [1947] p. 252; my emphasis).

[15] A possible response to the degrees of freedom is a requirement of stability independent of adjustment speed (D-stability). This gives additional information as to as to the structure of $D_x f(g(\mu);\mu)$, see Arrow [1974].

2.2 C^r, ε-perturbations, (Local) Structural Stability and Hyperbolicity

The system (3) defines a dynamical system on the state space X by the fundamental theory of differential equations (Hirsch & Smale [1974], Ch. 8). Hence, let $\phi^f : \Omega \to X$ be the *flow* generated by the vector field *f*, that is $\Omega \subset \mathbf{R} \times X$ an open set, and for each $x \in X$ the map $t \to \varnothing^f(t,x) = \varnothing_t^f(x)$ the solution passing through *x* for $t = 0$. This solution is defined and is unique for some (possibly small) interval in $\mathbf{R}$ under the stated assumptions. With the exception of some minor comments in Section 5 on forward looking models, I do not discuss existence, but take for granted the solution over a sufficiently large interval. In terms of the flow, fixed points satisfy $\varphi^f(t,\overline{x}) = \overline{x}$ for all *t*. Note that (possibly) except for forward variables at a finite number of time points, solution curves (trajectories) $\phi^f(\cdot; x, \mu) : \mathbf{R} \to X$ are C^1 in initial conditions $x \in X$ and parameters $\mu \in M$ (Perko [2001], Theorem 2 p. 84).

As argued in Section 2.1, under the CP it must be assumed that the dynamics defined by (3) is not radically altered by small variations in the vector field. This requires that a notion of qualitative invariance and an allowable class of perturbations are defined. They lead to the idea of structural stability and the requirement that the mathematical model should belong to the set of dynamical systems on *X* which satisfy what I call local structural stability (LSS).[16] I first consider the question of when two flows have the same qualitative behavior:

DEFINITION (FLOW/TOPOLOGICAL EQUIVALENCE OF VECTOR FIELDS). Let $f_0 \in C^1(X_0)$ and $f_1 \in C^1(X_1)$, where X_0 and X_1 are open subsets of $\mathbf{R}^n$. The systems $\dot{x} = f_0(x)$ and $\dot{x} = f_1(x)$ are flow or topologically equivalent if there is a homeomorphism[17] $h : X_0 \to X_1$ which maps trajectories $\varnothing_t^{f_0}(x)$ of f_0 onto trajectories $\varnothing_t^{f_1}(x)$ of f_1 and preserves their time orientation. If $X_0 = X_1 = X$, the systems (or fields) are flow or topologically equivalent on *X*: $f_0 \sim f_1$.

[16] One could argue independently, that is, not only in the sphere of the CP, that (most) models of economic theory should be drawn from the set of models which are (L)SS. Since model construction is based on approximation and simplification, parameters are determined inaccurately etc., it seems reasonable to endow systems with properties that are not too sensitive to small changes if insights from the model analysis are going to be transferred to the real process under consideration (Arnold [1983] Ch. 3, see also Fuchs [1975]). These questions, however, are beyond the scope of the paper.

Note that the terminology of LSS is not used uniformly. It is adopted by Fuchs [1974]. Devaney [1986] and Hubbard & West [1995] write on structurally stable fixed points.

A direct consequence of the definition is that for any x and t_1 there exists t_2 such that $h(\varphi^{f_0}(t_1, x)) = \varphi^{f_1}(t_2, h(x))$. As illustrated in Figure 1, drawing the vector fields f_0 and f_1 on rubber sheets, they are topologically equivalent "if one rubber sheet can be stretched and put onto the other so that the trajectories of one equation coincide with the trajectories of the other, as oriented curves" (Hubbard & West [1995], p. 204). This continuous deformation of the phase portrait (also called a C^0-equivalence) sets up a one-to-one correspondence between the fixed points of the two fields and distinguishes between sinks, saddles and sources. The fact that the parameterization of time is not necessarily preserved reflects that one is mainly interested in the geometry of solutions (Hubbard and West give a more detailed motivation for the definition).

Turning to the perturbations under which topological equivalence is required, consider first the discussion of the IFT in Section 1. The basic requirement (in addition to full rank) is that the derivatives of *f* and of the finite-dimensionally perturbed *f* are close, through the assumption that *f* be $C^k(M)$. I make a similar restriction on the variation in *f*, but open for an infinite-dimensional perturbation as motivated in Section 2.1. Letting $\|\cdot\|$ be the Eucledian norm, I use the following definition of nearness of two vector fields f_0 and f_1:

DEFINITION (C^r, ε-PERTURBATION). If $f_0 \in C^k(\mathbf{R}^n)$, $k, r \in \mathbf{Z}^+$, $r \le k$, and $\varepsilon > 0$, then f_1 is a C^r-perturbation of size ε if there is a compact set $K \subset \mathbf{R}^n$ such that $f_0 = f_1$ on the set $\mathbf{R}^n - K$ and for all $(i_1, ..., i_n)$ with $i_1 + ... + i_n = i \le r$ we have $\left\|(\partial^i / \partial x_1^{i_1} ... \partial x_n^{i_n})(f_0 - f_1)\right\| < \varepsilon$.[18]

I can now state the following definition of SS (based on the requirement that the initial and perturbed vector field have components and first derivatives which are close throughout *K*):

[17] A function $h: X_0 \to X_1$ is a *homeomorphism* if it is continuous, one-to-one, onto, and has a continuous inverse h^{-1}.
[18] The definition is from Guckenheimer & Holmes [1983], p. 38, and is a special case of the ones in e.g. Hirsch & Smale [1974] and Perko [2001] which use properties of function spaces. Devaney [1986], Ch. 1.9 gives a geometric interpretation of C^r, ε-perturbations (restricts the possible variations in the isoclines in Fig. 1 above and 2a below). Note that the variations (via μ and *F*) in Section 2.1 are examples of C^1, ε-perturbations.

DEFINITION (C^1 STRUCTURAL STABILITY, SS). A vector field $f \in C^1(\mathbf{R}^n)$ is C^1 structurally stable (SS) if there exists an $\varepsilon > 0$ such that all C^1, ε-perturbations of *f* are topologically equivalent to *f*.

The concept of SS involves difficult global questions that inter alia have to do with existence and preservation of closed orbits, lack of saddle connections, and more generally properties of the so-called non-wandering set. However, in this paper SS is introduced as a maintained hypothesis of the CP, and as a property to hold *locally* around fixed points. Hence, it will be sufficient to insist on the following weaker requirement:

DEFINITION (C^1 LOCAL STRUCTURAL STABILITY, LSS). A vector field $f \in C^1(\mathbf{R}^n)$ is C^1 locally structurally stable (LSS) at the fixed point $\bar{x}$ if there is a neighborhood *N* of $\bar{x}$ and an $\varepsilon > 0$ such that all vector fields f' that are C^1, ε-close to *f* on *N* are topologically equivalent to *f*.

The condition of LSS is replaceable by a simple condition of *hyperbolicity*, that is, that all the eigenvalues of $D_x f$ evaluated at $\bar{x}$ have real parts that are different from zero. This prerequisite, which excludes zero and purely imaginary eigenvalues at equilibrium, is the key to the powerful mathematical theorems that will be used below.[19]

PROPOSITION. The vector field $f \in C^1(\mathbf{R}^n)$ is LSS if and only if the fixed point $\bar{x}$ is hyperbolic.

PROOF OUTLINE.[20] On necessity, see Guckenheimer & Holmes [1983] p. 40—41. Sufficiency: By theorems 1 and 2 in Hirsch & Smale [1974] p. 305, there is a neighborhood of $\bar{x}$ and a set

[19] For systems of higher dimentions ($n \geq 3$), a requirement of SS is non-trivial. (For two-dimensional flows, there are theorems of the set of SS-fields being open and dense in the set of C^1-fields that are traverse on the boundary of a suitably defined compact set (SS is a *generic* property): every such system that is not SS can be approximated arbitrarily closely by SS-systems. However, it is known that no such result can exist in higher dimensions, see Hirsch & Smale [1974] Ch. 16.3. The non-triviality of the requirement of LSS (hyperbolicity) can be seen from the discussion of degeneracy or codimension of (local) bifurcations at fixed points in Guckenheimer & Holmes [1983].

The substitution of the hyperbolicity for explicit equations for homeomorphisms to determine whether a given function is LSS is useful, as finding $h(\cdot)$ is demanding (see Devaney [1986], Example 9.4, for the simplest possible case). (A global sufficiency result is stated in Sec. 4 below.)

[20] Hubbbard & West [1995] Ch. 8*.6 gives an alternative proof under slightly more restrictive assumptions.

of C^1, ε-close fields such that for any such perturbation f' there is a unique $\bar{x}': f'(\bar{x}') = 0$ (in general $\bar{x}' \neq \bar{x}$). Furthermore, for any $\delta > 0$, $\varepsilon > 0$ can be chosen such that $\|\bar{x} - \bar{x}'\| < \delta$, where $\bar{x}'$ has the same number of eigenvalues with positive and negative real part as $\bar{x}$. By the Hartman–Grubman theorem in Section 3 below, there is a neighborhood U (U') of $\bar{x}$ ($\bar{x}'$) on which f (f') is topologically equivalent (in fact conjugate) to $D_x f\big|_{x=\bar{x}}$ ($D_x f'\big|_{x=\bar{x}'}$). For ε sufficiently small, $D_x f\big|_{x=\bar{x}}$ and $D_x f'\big|_{x=\bar{x}'}$ are topologically equivalent by the result on linear isomorphisms in Hirsch & Smale [1974] p. 309. Hence, for sufficiently small δ the following holds in a neighborhood N of $\bar{x}$: $f \sim D_x f\big|_{x=\bar{x}} \sim D_x f'\big|_{x=\bar{x}'} \sim f'$.

3. Resurrection of the Correspondence Principle

In Section 1, I commented on the persistence of equilibria ensured by the IFT assumptions. It is interesting to note that the property of LSS implies full rank of the Jacobian $D_x f(\bar{x};\mu)$ at equilibria, and hence – to the extent that dynamic considerations are applicable – makes this *assumption* of the IFT extraneous.[21]

Consider now the linear approximation of (3) around a fixed point $\bar{x}$ (I temporarily drop the reference to the finite-dimensional parameterization and assume that the fixed point has been translated to the origin):

(5) $\dot{x} = D_x f(\bar{x}) x$

Given LSS and the Proposition, the vector field $D_x f(\bar{x})$ generates a hyperbolic linear flow $\exp(tD_x f(\bar{x}))x$ (Hirsch & Smale [1974] Ch. 5 gives a complete characterization of such flows). The following particularily strong result on topological equivalence near the fixed point is available (see e.g. Perko [2001] Ch. 2.8 for a proof):

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[21] Since $D_x f(\bar{x};\mu)$ real, its characteristic polynomial is real, and its roots are real or appear in conjugate pairs, that is if $\mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda) \in \Lambda(D_x f)$, then $\mathrm{Re}(\lambda) - i\,\mathrm{Im}(\lambda) \in \Lambda(D_x f)$. Hence, by the Proposition (hyperbolicity) $\det D_x f(\bar{x};\mu) = \prod \lambda_i \neq 0$.

Mas-Colell [1986], Ch. 8.3, discusses how sufficient dependence of f on μ in M (via knowledge of rank $D_\mu f$) allows inferences of full rank of $D_x f$ (that is, no *simultaneous* collapse of both the real and imaginary part of the eigenvalues) for "almost all" $\mu \in M$ using transversality theory (see Sec. 4 below).

HARTMAN–GROBMAN THEOREM. Given hyperbolicity of $\bar{x}$, there is a homeomorphism h defined on some neighborhood U of $\bar{x}$ in $\mathbf{R}^n$ locally taking trajectories of the nonlinear flow $\phi^f(t,x)$ of (3) to those of the linear flow $\exp(tD_x f(\bar{x}))x$ of (5). The homeomorphism preserves the orientation of trajectories and can also be chosen to preserve parameterization by time.[22]

I started by highlighting a logical flaw in the CP. However, when laying out the proper setting or hypothesis for the principle (LSS), the Hartman–Grobman theorem shows that in a neighborhood of the fixed point, the behavior of the non-linear system is fully described by its *linear* part. Hence the necessary (and sufficient) conditions for asymptotic stability of (5) are *necessary* (and sufficient) for the stability of the non-linear system (3): the CP "is". Furthermore, the solution to (5) not only yields local asymptotic behavior, but – up to a homeomorphism – also provides the local topological structure of the phase portrait. The latter is important when the dynamics is interpreted as a *real time* movement (slow dynamics).[23]

4 Walrasian Equilibrium Set Structure

I consider finite exchange economies with H households and commodity space $\mathbf{R}^L$. Each household is characterized by a vector of initial endowments and possibly some parameters describing household preferences. In the terminology of the theory, μ is an economy in the space of economies *M,* where $M \subset \mathbf{R}^m$, $m \geq LH$ (economies defined in more general spaces are admittable, see below). I abstract from individual demand functions and assume that preferences and the endowment configuration generate a C^1 aggregate excess demand function $F : P \times M \to \mathbf{R}^L$ ($P = \mathbf{R}^L_+ \setminus \{0\}$ is the price space). Since there are no intermediate goods, desirability assumptions are assumed such that for each μ, $\tilde{p}^T F(\tilde{p};\mu) = 0$ for all $\tilde{p} \in P$ (Walras' Law, WL), and such that the set of Walrasian equilibria $\tilde{E}$ of the economy can be described as the set of strictly positive price systems given by

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[22] $\exp(tD_x f)$ is the exponential matrix defined by $\exp(tD_x f) = \sum_{k=0}^{\infty} \frac{(tD_x f)^k}{k!}$. Since the homeomorphism can be chosen to preserve time, that is $h(\varphi^f(t,x)) = \exp(tD_x f(\bar{x}))h(x)$ for each $t \in \mathbf{R}$, the relationship is called a C^0-conjugacy.

[23] Stable and unstable manifolds will be discussed in Section 4. Note that since the homeomorphism is not necessarily differentiable, nodes and foci are not distinguished. If *f* is analytic and certain non-resonance condi-

(6) $\tilde{E}(\mu)=\left\{\tilde{p}\in \operatorname{int} P \,\middle|\, \hat{F}(\tilde{p};\mu)=0\right\}$.

The $L-1$ equations in (6) arise from *F,* deleting the *L*-th component due to WL. The restriction to strictly positive prices is a local implication of desirability. Since *F* is homogeneous of degree zero on *P*, $\tilde{p}_L$ can be normalized to 1, and the equilibrium set can be defined as ($p=(p_1,...,p_{L-1})$, $f(p;\mu)=\hat{F}(p_1,...,p_{L-1},1;\mu)$)[24]

(7) $E(\mu)=\left\{p\in \mathbf{R}^{L-1}_{++} \,\middle|\, f(p;\mu)=0\right\}$.

DEFINITION (REGULAR ECONOMY). An equilibrium price system $\overline{p}\in E(\mu)$ is regular if the rank of the Jacobian of *f* is full at that point, that is $rank\left(D_p f(\overline{p};\mu)\right)=L-1$. An economy $\mu\in M$ is *regular* if all price systems $\overline{p}\in E(\mu)$ are regular.[25]

Through the full rank property, regular economies have all the persistence properties implied by the IFT (see Sec. 1). Debreu [1970] proved that a regular economy has a finite number of equilibria, and that "almost all" economies in *M* are regular.[26] Using properties of the excess demand function on the boundary of a suitably redefined price space (global implications of desirability), Dierker [1972], drawing on the Poincaré–Hopf Theorem, added structure to the set $E(\mu)$ in a regular economy. Let $\#\mathrm{E}(\mu)$ be its (finite) number of elements, and define the *index I* as

$$I(\overline{p};\mu)\Big|_{\substack{\overline{p}\in E(\mu)\\ \mu\ regular}}=\begin{cases}1 & if \quad \det\left(-D_p f(\overline{p};\mu)\right)>0\\ -1 & if \quad \det\left(-D_p f(\overline{p};\mu)\right)<0\end{cases}.$$

tions are satisfied, there is an analytic change in coordinates in a neighborhood of the stationary point which brings (3) into linear form (see Arnold [1983] for details).

[24] The homogeneity property implies that $\operatorname{rank} D_{\tilde{p}}F(\tilde{p};\mu)<L$. Desirability assumptions typically take the form that the excess demand for a commodity shoots to infinity as its price converges to zero from above. The stated version of WL is called Walras Law in the narrow sense. For omitted details of this section, Mas-Colell, Whinston & Green [1995] is a good textbook reference. Balasko [1988] and Mas-Colell [1985] are exhaustive monographs.

[25] Debreu [1970] defined regular economies in the equilibrium correspondence context, relative to a space of endowments $\omega=(\omega^1,...,\omega^H)\in\mathbf{R}^{LH}$. The stated definition is due to Dierker & Dierker [1972]. Since it does not involve the space of parameters, it readily opens for wider classes of specifications (Dierker [1980] Sec. 3).

[26] The above references refine the statement "almost all", including extensions to economies defined in more general spaces than *M*.

THEOREM (DIERKER). For every regular economy $\mu \in M$, $\#\mathrm{E}(\mu)=2k\text{-}1$ for a $k \in \{1,2,...,K\}$, of which k have $I=1$ and $k-1$ have $I=-1$; $\sum_{\overline{p}\in E(\mu)} I(\overline{p};\mu)=1$.

This theorem, which implies existence, exhausts the structure of $E(\mu)$ in the full generality of Arrow-Debreu models. Further restrictions require additional assumptions. The much studied Walrasian tâtonnement process

(8) $\quad \dot{p}= f(p;\mu)$,

or some C^1 sign preserving transformation (see the comments to (4) and Balasko [1988] Ch. 1.9) has been a source of such restrictions. Dierker [1972], Theorem 2, proposed that a regular economy $\mu \in M$ for which all $p \in E(\mu)$ are locally stable, has a unique equilibrium, arguing that stability of (8) implies that all eigenvalues of $D_p f(\overline{p};\mu)$ have strictly negative real parts.[27] Hence, $\det\left(-D_p f(\overline{p};\mu)\right)$ is positive over all $\overline{p}\in E(\mu)$ and uniqueness follows from the index theorem.[28]

However, since strictly negative real parts of the eigenvalues is not a necessary condition for local asymptotic stability of the non-linear system (8), the index restriction does not follow from the assumed stability. However, as argued above, LSS should be considered a part of the stability hypothesis. If not, arbitrarily small changes in explicit data or in the dynamic process with respect to which the economy is assumed stable, could lead to another economy with essentially different properties, making the assumption of stability at the fixed points uninteresting. By the implied property of hyperbolicity of the fixed point from LSS (see the Proposition), Dierker's link between the dynamics of the system and the structure of the set of equilibria is restored, consistent with the spirit of the CP.

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[27] Dierker [1972] endowed the economy with a more general adjustment process, sharing properties with the excess demand function only around the fixed points (and on the boundary of the price space in his normalization).

[28] From the modified Routh–Hurwitz conditions, a necessary condition for $-Df_p(\overline{p};\mu)$ to only have eigenvalues with strictly negative real roots is that $-Df_p(\overline{p};\mu)=(-1)^{L-1}Df_p(\overline{p};\mu)>0$, see e.g. Murata [1977] p. 92.

It is worth noting that LSS under the assumption of global asymptotic stability of the unique fixed point (required when $L \geq 3$) is sufficient for SS under the assumptions of Dierkers theorem: Using the choice of state space and local parameterization in Varian [1982], Sections 1 and 2, Theorem 1 on p. 314 in Hirsch & Smale [1974] can be applied (see Fuchs [1975] Theorem B for a similar result).

5 Rational Expectations Models

Rational or model-consistent expectations may be introduced in a normal form like (3) – under a real time interpretation – by writing the vector field as $f = (f_x, f_y)$, and

(9) $$\begin{aligned} \dot{x} &= f_x(x, y; \mu) \\ E\left\{\dot{y} \middle| I_t\right\} &= f_y(x, y; \mu), \; t \neq t_I \end{aligned} .$$

The state variables have been partitioned into a vector *y* of n_y "forward looking" or "jump" variables and $n_x = n - n_y$ continuous or "predetermined" variables *x*. *E* is the expectation operator, I_t the information set conditioning expectations formed at time *t*, and $\dot{y}$ the right hand derivative

$$E\left\{\dot{y}(\tau) \middle| I_t\right\} = E\left\{\lim_{\varepsilon \to 0^+} \frac{y(\tau+\varepsilon) - y(\tau)}{\varepsilon} \middle| I_t\right\}.$$

The information set contains all current and past values of *x*, *y* and μ as well as the model structure (9). Interpreting also $\dot{x}$ as a right hand derivative (due to possible discontinuities in *y* or μ, see below), the assumed expectation formation implies that all actual and anticipated rates of change coincide, except possibly with respect to $\dot{y}$ at time points t_I when new information about the parameter system becomes available. The current values of non-predetermined variables are functions of current anticipations of future values of endogenous variables and the parameter system. Accordingly, it is necessary to specify the whole future path for the parameters, and to assume that $E_t \mu(\tau)$ is bounded on $[t, \infty)$. The model assumptions are formulated more completely in Buiter [1984]. I consider a simple example of so-called anticipated event analysis, where (9) is given an autonomous perturbation from a (expected) constant level μ^0 to a constant level μ^1 at time t_I, see Figure 2b.[29]

This class of models "leads to notions of stability and to solution theory that is different than much of that in the natural sciences" (Malliaris & Brock [1989] p. 263). The continuity property of the total dynamic equilibrium $\varphi^f(t;x,\mu^0) = \left(\varphi_x^f(t;x,\mu^0), \varphi_y^f(t;x,\mu^0)\right)$ is broken at time $t = t_I$ with respect to the forward variables *y*; the right-continuous non-predetermined variables are allowed to (possibly) "jump" at t_I to a region in the state space such that an *assumed* resumed continuous movement of the model solution to the fixed point $(\bar{x}^1, \bar{y}^1) = \left(g_x(\mu^1), g_y(\mu^1)\right) = g(\mu)$ is assured under the new parameter system.[30]

The solution methodology can be described as follows: Obtain from (9) the linear approximation system (assuming coordinates have been transformed such that the fixed point is translated to the origin)

$$(10) \qquad \begin{pmatrix} \dot{x} \\ E\left\{\dot{y} \middle| I_t\right\} \end{pmatrix} = D_{(x,y)} f(\bar{x}, \bar{y}; \mu) \begin{pmatrix} x \\ y \end{pmatrix}$$

Divide the spectrum of the Jacobian the fixed point, $\Lambda(D_{(x,y)} f(\bar{x}, \bar{y}; \mu))$, into three parts Λ_s, Λ_c, and Λ_u with $\mathrm{Re}(\lambda) < 0$ if $\lambda \in \Lambda_s$, $\mathrm{Re}(\lambda) = 0$ if $\lambda \in \Lambda_c$, and $\mathrm{Re}(\lambda) > 0$ if $\lambda \in \Lambda_u$. Define the following linear subspaces of $\mathbf{R}^n$

the n_s-dimensional stable subspace $E^s(\bar{x}, \bar{y}) = span\left\{u^1, ..., u^{n_s}\right\}$

the n_c-dimensional center subspace $E^c(\bar{x}, \bar{y}) = span\left\{v^1, ..., v^{n_c}\right\}$

[29] Note that Frisch [1929], writing on dynamic models of exchange (i.e. real time dynamical versions of the models in Sec. 4 above), in emphasising the demand concept as *functional*-theoretic, came close to defining a rational expectations equilibrium: "Should we incorporate the phenomenon of anticipations in an entirely satisfactory manner, we would have to introduce the higher growth rates, or even better, introduce the whole likely future path of the time curve that represents *p* [the Walrasian price vector], and weigh the different alternatives according to certain principles of probability calculus" (p. 341—342; my translation).

[30] For ordinary systems like (3) with continuous solutions as discussed in Section 2.2, at any one moment of time $\frac{\partial \varphi^f(t,x,\mu)}{\partial \mu} = 0$. In the case where *f* has a finite or countable number of discontinuity points, $\dot{x}$ does not exist

the n_u-dimensional unstable subspace $E^u(\bar{x},\bar{y}) = span\{w^1,...,w^{n_u}\}$

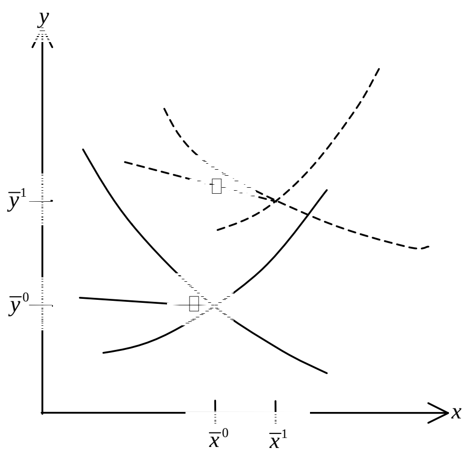
where $u^1,...,u^{n_s}$, $v^1,...,v^{n_c}$ and $w^1,...,w^{n_u}$ are the n_s, n_c and n_u (generalized) eigenvectors that correspond to $\lambda(\mu)\in\Lambda_s$, $\lambda(\mu)\in\Lambda_c$ and $\lambda(\mu)\in\Lambda_u$, respectively. Hirsch & Smale [1974] proves that the spaces form a direct sum decomposition of $\mathbf{R}^n$ and that they are invariant under the linear flow $\exp(tD_{(x,y)}f(\bar{x},\bar{y};\mu))x$, that is, $\exp(\cdot)\big|E^i$ remains on E^i for all *t*, $i = s,c,u$ Furthermore, all solutions that start in E^s (E^u) approach (leave) the fixed point $(\bar{x},\bar{y})$ exponentially fast. The solutions that lie in E^c do neither.[31]

Assuming full rank (regularity) of the Jacobian at $(\bar{x},\bar{y})$ and appealing locally to the linear approximation system, *n* linearly independent boundary conditions are required for a unique solution. In the case where the number of predetermined variables equals the dimension of the stable subspace (or the number of eigenvalues with negative real parts, counting multiplicities), $n_x = n_s$, n_x initial conditions – through the (two-sided) continuity requirement on the predetermined variables – are pinned down by the history of the system. The remaining $n - n_x = n_y$ boundary conditions are obtained from the transversality condition of asymptotic convergence, which – by the properties of the eigenspaces – requires the n_y forward variables to (possibly) "jump" at t_I such that $(\varphi_x^f(t_I,x;\mu^1),\varphi_y^f(t_I,x;\mu^1))\in E^s(g(\mu^1))$, and hence $\varphi^f(t,x;\mu^1)\in E^s(\bar{x}^1,\bar{y}^1)$ for all $t\ge t_I$. There are just enough free variables at time $t = t_I$ to make this happen for any configuration of the continuous (momentarily exogenous) vector *x* (see Fig. 2).[32]

(unless interpreted as a right hand derivative), but a continuous solution curve will, possibly with kinks. See Sydsæter & Seierstad [1987], Theorem A9 and Appendix A respectively, for mathematical details.

[31] For linear (autonomous) systems, asymptotic stability is *equivalent* to exponential asymptotic stability. Solutions in E^c are stable (but never asymptotically stable) if every eigenvalue in Λ_c is of multiplicity one, otherwise they become unbounded.

[32] Buiter [1984] provides an explicit solution. (It might also be useful to observe that eigenvalues and eigenvectors of $A(\mu) = D_x f(g(\mu);\mu) = \left[a_{ij}(\mu)\right]_{n\times n}$ are functions of μ, whose derivatives can be obtained through total differentiation and by the formulas in e.g. Magnus [1985] relating eigenvalues and eigenvector elements to the matrix elements $a_{ij}(\mu) = \partial f^i(g(\mu);\mu)/\partial x_j$.) In the case of $n_y < n_s$, an asymptotically stable solution generally (that is, for any *x*) does not exist. In the case of $n_y > n_s$, a continuum of solutions exist. Buiter discusses how economically motivated linear restrictions on the state variables may ensure uniqueness.



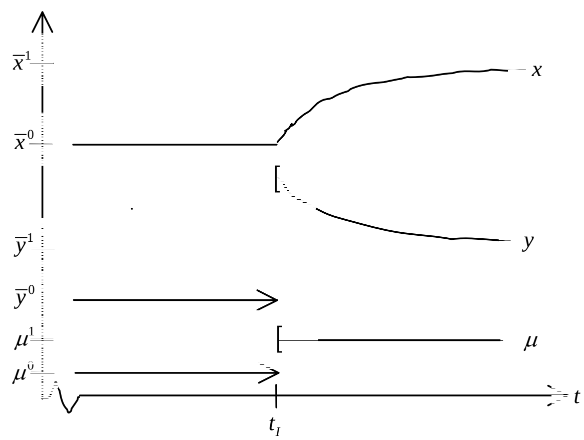


Figure 2a. Phase space analysis, $n_x = n_y = n_s = n_u = 1$. The solid and dashed *curves* (isoclines) are graphs of $(f_x, f_y)(x, y; \mu^0) = 0$ and $(f_x, f_y)(x, y; \mu^1) = 0$, respectively. The solid and dashed *lines* are subsets of $E^s(\bar{x}^0, \bar{y}^0)$ and $E^s(\bar{x}^1, \bar{y}^1)$. $\left(\varphi_x^f(t; x, \mu^0), \varphi_y^f(t; x, \mu^0)\right) \in E^s(\bar{x}^0, \bar{y}^0)$ is assumed to be arbitrarily close to $(\bar{x}^0, \bar{y}^0)$ as $t \to t_I^-$. At $t = t_I$ μ shifts (permanently) to μ^1. By asymptotic stability, $\left(\varphi_x^f(t; x, \mu^1), \varphi_y^f(t; x, \mu^1)\right) \in E^s(\bar{x}^1, \bar{y}^1)$ for $t \in [t_I, \infty)$.

Figure 2b. Sketch of solution trajectories; unanticipated event analysis.

However, analogous to the difficulty with the CP due to non-linearity (Sec. 1), the requirement $\varphi^f(t, x, \mu) \in E^s(g(\mu))$ is *not necessary* for the asymptotic stability of the system (9) even arbitrarily close to the fixed point. This is due to the possible behaviour of (9) on the (local) *center manifold* $W^c(g(\mu))$, the non-linear analogue to the center subspace. As the discussion of the center manifold theorem in Guckenheimer & Holmes [1983] shows, solutions can be expanding or contracting on W^c (see Ch. 3.2; this point is also illustrated by the system in note 7 above). For non-linear systems (9), the solution suggested from the analysis of linear system (10) does not follow from the partition of the state variables into predetermined and non-predetermined and the assumed (conditional) asymptotic stability.

However, the solution and stability concept of rational expectations (implicitly) clearly embodies a notion of LSS; in particular it is assumed that the dimension of the stable and unstable manifolds (denoted W^s and W^u in the non-linear case) are invariant to changes in the explicit parameter system, and that the system converges to the new perturbed stationary state.[33] Applying from LSS the equivalent notion of hyperbolicity (that is, $n_c = 0$ or $W^c = E^c = \varnothing$, see the Propostion in Sec. 4), the following theorem endows the linear approximation system with the qualitative properties of the non-linear system locally around the fixed point, as required for the correctness of the above analysis:

STABLE MANIFOLD THEOREM. Let $f \in C^r(U)$ where U is an open subset of $\mathbf{R}^n$ containing the fixed point hyperbolic $(\bar{x}, \bar{y})$, $r \geq 1$. At the fixed point, there exists an n_s-dimensional stable manifold $W^s(\bar{x}, \bar{y})$ tangent to the (generalized) eigenspace $E^s(\bar{x}, \bar{y})$ of Λ_s at $(\bar{x}, \bar{y})$ and an n_u-dimensional unstable manifold $W^u(\bar{x}, \bar{y})$ tangent to (generalized) eigenspace $E^u(\bar{x}, \bar{y})$ of Λ_u at $(\bar{x}, \bar{y})$. The manifolds are invariant under the flow of *f* and of class C^r.[34]

6 Concluding remarks

This paper has examined implications of a maintained hypothesis of (local) structural stability as part of the correspondence principle assumptions. The local focus has allowed a simple hyperbolicity criterion and a straightforward analysis, but has given important results. In particular, a deficiency in the link envisioned by Samuelson between assumed stability properties of equilibria and the equilibrium set structure, has been repaired.

In models with a real time interpretation, dynamic systems whose laws of motion change with time might be considered.[35] Several issues would also benefit from a global analysis. This applies in particular to the class of rational expectation models, where some of the qualitative

[33] If the system is not LSS, then arbitrarily small changes in the data or functional equations could for instance in the case of $n_x = n_s$ lead to a situation where no solution exists, or a continuum of solutions exist.
[34] Perko [2001] Ch. 2.7 gives a precise definition of a differentiable manifold and a proof of the theorem.
[35] Hahn [1967] Sec. 56 gives some sufficiency theorems on stability under persistent perturbations.

properties that can be seen in two-dimensional structurally stable systems seem to be required as a part of the solution concept. Such challenging questions are, however, beyond the scope of this paper.

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