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PRESERVED STRUCTURE CONSTANTS FOR RED REFINEMENTS OF PRODUCT ELEMENTS

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Abstract: In this paper we discuss some strategy for red refinements of product elements and show that there are certain structure characteristics (d-sines of angles formed by certain edges in the initial partition) which remain constant during refinement processes. Such characteristics are directly related to the so-called maximum angle condition, the validity of which is a strongly desired property in interpolation theory and finite element analysis.

1. Introduction and basic definitions

Red refinement is one of the most popular meshing techniques used in various branches of numerical mathematics. However, it is usually applied to simplicial partitions only, see e.g. [1, 14, 15, 17, 18], and various aspects of regularity of the generated meshes are then analysed.

In practice, depending on the shape of the domain over which we construct the meshes, one may prefer to construct some initial mesh consisting of so-called product elements, and only after that refine it into simplices. The simplest illustration in this direction would be the case of cylindric-type domains first naturally split into right prisms and then into tetrahedra, see e.g. [13].

In this work, we consider a more general case of product elements of any dimensions and red refinement techniques used independently for each factor of the product. Some regularity properties of resulting simplicial meshes are discussed. Products of simplices, and their triangulations, have been studied in many contexts, in particular for two factors. See e.g. [2, 4, 6, 7, 9].

In 1978, F. Eriksson proposed the following concept of the d-dimensional sine of angles in $\mathbf{R}^d$. In terms of the simplex S, for any of its vertices A_i, $i = 0, \dots, d$, the d-dimensional sine of the angle of S at A_i, denoted by $\hat{A}_i$, is defined as follows (see (3) in [5, p. 72]):

$$\sin_d(\hat{A}_i | A_0 A_1 \dots A_d) = \frac{d^{d-1}\,(vol\,S)^{d-1}}{(d-1)!\,\prod_{j=0, j\neq i}^{d} vol\,F_j}, \qquad (1)$$

where vol denotes the measure of (simplex or its facets) of relevant dimension. We will apply $\sin_d$ to a set of d vectors, which then will mean that we choose A_i at the common point of origin of the vectors and the remaining vertices $A_0, \dots, A_{i-1}, A_{i+1}, \dots, A_d$ as the corresponding endpoints. Write vol(a set of vectors) for the hypervolume of the generalized parallelotope spanned by the vectors. We then have the convenient formula

$$\sin_d(v_i,\ldots,v_d) = \frac{vol(v_1,\ldots,v_d)^{d-1}}{\prod_{i=1}^d vol(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_d)}.$$

See also [12].

Definition 1.1 ([10]). A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions of some polytope into simplices is said to satisfy the *maximum angle condition* if there exists $C_0 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S = \mathrm{conv}\{A_0,\ldots,A_d\} \in \mathcal{T}_h$ we can always find d edges of S, which when considered as vectors, constitute a (higher-dimensional) angle whose d-sine is bounded from below by the constant C_0. Here, we let h denote the maximal diameter of the simplices in a partition.

Simplicial partitions satisfying the maximum angle condition are highly desired in numerical analysis for various interpolation and finite element convergence proofs, see e.g. [17, 10]. There is another (equivalent) definition of the maximum angle condition in [12]. However, we prefer the one from Definition 1.1 in this paper, since it is more suitable for our geometric considerations in what follows.

2. Red refinement strategy and its properties

Recall a standard triangulation of the hypercube (see e.g. Freudenthal [8]).

Theorem 2.1. *Consider the unit hypercube in* $\mathbf{R}^d$ *with* 2^d *vertices* $(0,0,\ldots,0)$ *to* $(1,1,\ldots,1)$ *(all possible combinations of* 0 *and* 1*). For any path from* $(0,0,\ldots,0)$ *to* $(1,1,\ldots,1)$ *consisting of the standard unit vectors* e_i*,* $i=1,\ldots,d$*, in some order, there is a corresponding d-simplex given as the convex hull of the vertices of the hypercube along the path.*

a) *By varying the path over all possible orderings of the vectors* e_i*, this gives a triangulation of the hypercube consisting of* $d!$ *elements, each sharing the common edge from* $(0,0,\ldots,0)$ *to* $(1,1,\ldots,1)$*.*
b) *We can subdivide the unit cube into* 2^d *smaller cubes, with coordinates given by* $0, 1/2$ *or* 1*, and apply the same construction to each of them.*

The resulting triangulation from Theorem 2.1 b) refines the original triangulation, thus defining a *red refinement scheme* for the hypercube by iteration. By making consistent choice of diagonals, we get a conformity always for free when we start refining a single simplex as above.

By embedding a simplex into the hypercube we can produce a consistent red refinement scheme by intersecting with the subdivision of the hypercube as presented in Theorem 2.1, see Figure 1.

More formally, given any d-dimensional simplex S, we can choose a path (or an ordering of the vertices) $A_0A_1\ldots A_d$. There is then a unique affine transformation T that maps A_0 to $(0,0,\ldots,0)$, and A_i to $(1,1,\ldots,1,0,0,\ldots,0)$ (1 in the first i positions), $i=1,\ldots,d$. Then $T(S)$ is one of the simplices in the triangulation of the hypercube described in the theorem, corresponding to the path $e_1,e_2,\ldots,e_d$, which we will use as a reference simplex. To describe the set by inequalities, it is the set

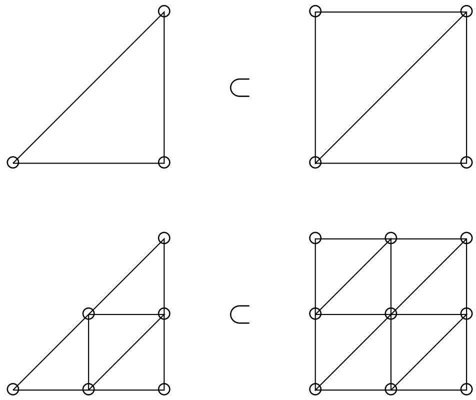

FIGURE 1. By embedding a simplex in a hypercube, we construct a red refinement of the simplex by restriction of the red refinement of the hypercube

$1 \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$. Since we will usually only consider these inequalities in the unit hypercube, we will suppress 1 and 0 at the ends of the string of inequalities. Using the refinement from Theorem 2.1 b), and then translating back with T^{-1}, gives a red refinement of S.

Theorem 2.2. *Consider a simplex S and the red refinement defined above. Then*

a) *each sub-simplex resulting from the red refinement step has a path consisting of the vectors $\{\frac{1}{2}f_i\}$ in some order, where $\{f_i\}$ are the vectors along the chosen path in S,*

b) *repeated red refinements using the paths parallel to the chosen path in S will produce only finitely many similarity types of simplices (up to scaling and rigid transformations),*

c) *each sub-simplex produced by a repeated red refinement has a path such that $\sin_d$ applied to the vectors along the path is equal to $\sin_d(f_1, \dots, f_d)$.*

Proof. Part a) immediately follows from Theorem 2.1 and standard properties of affine transformations. Part b) follows since each simplex produced is congruent to a simplex whose vertices are connected by a path consisting of $\{\frac{1}{2}f_i\}$ in some order. Part c) follows since $\sin_d$ remains unchanged if one of the vectors is multiplied by a non-zero constant, and also after permutation of the inputs (see [5, 12]). □

Remark 2.3. The statements a) and b) in the above theorem are well known (see e.g. [1]), and are included only for completeness. Statement c) of Theorem 2.2 has been considered for tetrahedra in [15].

3. MAIN RESULTS

In what follows we will need a result about the higher-dimensional sines when the arguments are orthogonal to each other.

Lemma 3.1. *Suppose that* $\{v_1, \ldots, v_{d_1}\}$ *and* $\{u_1, \ldots, u_{d_2}\}$ *are two sets of vectors that are orthogonal to each other. Then*

$$\sin_{d_1+d_2}(v_1, \ldots, v_{d_1}, u_1, \ldots, u_{d_2}) = \sin_{d_1}(v_1, \ldots, v_{d_1}) \sin_{d_2}(u_1, \ldots, u_{d_2}).$$

In case d_1 *and/or* d_2 *is equal to one, we use the natural convention* $\sin_1(u_1) = 1$ *(sine of a single nonzero vector is one).*

Proof. We assume that each of the $n = d_1 + d_2$ vectors is a unit vector. Then

$$\sin_n(v_1, \ldots, v_{d_1}, u_1, \ldots, u_{d_2}) = \frac{vol(v_1,\ldots,v_{d_1},u_1,\ldots,u_{d_2})^{n-1}}{\prod_{i=1}^{d_1} vol(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{d_1},u_1,\ldots,u_{d_2}) \prod_{i=1}^{d_2}(v_1,\ldots,v_{d_1},u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{d_2})}.$$

By orthogonality of the two sets of vectors, each hypervolume factors as a product of two lower-dimensional hypervolumes and we get:

$$\begin{aligned}
&\sin_n(v_1, \ldots, v_{d_1}, u_1, \ldots, u_{d_2}) \\
&= \frac{vol(v_1,\ldots,v_{d_1},u_1,\ldots,u_{d_2})^{n-1}}{\prod_{i=1}^{d_1} vol(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{d_1},u_1,\ldots,u_{d_2}) \prod_{i=1}^{d_2}(v_1,\ldots,v_{d_1},u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{d_2})} \\
&= \frac{vol(v_1,\ldots,v_{d_1})^{n-1} vol(u_1,\ldots,u_{d_2})^{n-1}}{\prod_{i=1}^{d_1} vol(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{d_1}) vol(u_1,\ldots,u_{d_2})^{d_1} vol(v_1,\ldots,v_{d_1})^{d_2} \prod_{i=1}^{d_2}(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{d_2})} \\
&= \frac{vol(v_1,\ldots,v_{d_1})^{d_1-1}}{\prod_{i=1}^{d_1} vol(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{d_1})} \frac{vol(u_1,\ldots,u_{d_2})^{d_2-1}}{\prod_{i=1}^{d_2}(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{d_2})} \\
&= \sin_{d_1}(v_1, \ldots, v_{d_1}) \sin_{d_2}(u_1, \ldots, u_{d_2}).
\end{aligned}$$

□

Let $S_1, \ldots, S_k$ be simplices of dimensions $d_1, \ldots, d_k$, respectively. Let $n = \sum_{i=1}^k d_i$. Now we can independently choose a path (or equivalently a vertex ordering) in each S_i, and embed it in the unit d_i-cube by an affine transformation T_i as discussed before Theorem 2.2. By this choice, $T_i(S_i)$ is the reference d_i-simplex. The product polytope $\Delta = \prod S_i$ of the simplices is embedded into $\mathbf{R}^n$ by the product map T of the maps T_i:

$$T : \Delta = \prod_{i=1}^k S_i \to \prod_{i=1}^k T_i(S_i) \subset \prod_{i=1}^k \mathbf{R}^{d_i} = \mathbf{R}^n.$$

Theorem 3.2. *Consider the product of simplices* $\Delta = \prod S_i$ *as defined above. Then:*

a) *The product* Δ *can be triangulated by simplices, each of which contains a path with the property that* $\sin_n$ *of the path vectors is the product of the* $\sin_{d_i}$ *of the chosen paths of the factors.*
b) *There is a red refinement scheme so that all simplices occuring by repeated refinement has a path with* $\sin_n$ *equal to the* $\sin_n$ *in part a), and that only contains a finite number of similarity types of product elements.*

c) *There are also red refinement schemes where the factors are refined at different rates, so that each simplex occuring by repeated refinement has a path with* $\sin_n$ *equal to the* $\sin_n$ *in part a). In this case, the family may contain an infinite number of similarity types of product elements.*

Proof. We apply the affine transformation T defined above, solve the problem in the unit n-cube, and transform using T^{-1} to get back to the original polytope Δ. Look at the coordinates of $T(\Delta)$. For each factor,

$$T_i(S_i) = \{(x_1^{(i)}, \dots, x_{d_i}^{(i)}) | x_1^{(i)} \geq x_2^{(i)} \geq \dots \geq x_{d_i}^{(i)}\} \cap [0,1]^{d_i} \subset \mathbf{R}^{d_i}.$$

The product $T(\Delta) = \prod T_i(S_i)$ can then be described as a product of these sets with the given coordinates. More precisely, if we use

$$x_1^{(1)}, \dots, x_{d_1}^{(1)}, x_1^{(2)}, \dots, x_{d_2}^{(2)}, \dots, x_1^{(k)}, \dots, x_{d_k}^{(k)}$$

as coordinates for $\mathbf{R}^n$, we can use the k sets of the same inequalities to describe $T(\Delta)$. We then consider the triangulation of the hypercube in $\mathbf{R}^n$ from Theorem 2.1. A triangulation of $T(\Delta)$ then uses only those simplices in the triangulation of the hypercube that always satisfy these inequalities. The index set of these simplices can be given by *shuffles*, i.e. permutations of $\{1, 2, \dots, n\}$ that preserve the order of $\{1, \dots, d_1\}$, $\{d_1+1, \dots, d_1+d_2\}$, $\dots$, $\{d_1+\dots+d_k+1, d_1+d_2+\dots+d_k\}$ separately.

The results in parts a) and b) now follow immediately as in Theorem 2.2, using only that if a given set of n vectors splits as a union of sets of cardinality $d_1, d_2, \dots, d_k$ such that all the subsets are orthogonal to each other, the corresponding $\sin_n$ is the product of the corresponding $\sin_{d_i}$ (by Lemma 3.1 and an obvious induction).

For part c), introduce the notation $f_j^{(i)}$ for the preimage under T of the standard basis vectors $e_j^{(i)}$ on $\mathbf{R}^n$, so that e.g. $f_1^{(1)}, f_2^{(1)}, ..., f_{d_1}^{(1)}$ are the vectors along the chosen path of S_1. We can now perform the refinement on a single factor, keeping the remaining factors unchanged. Without loss of generality, we can assume that we only refine the first factor S_1. Then any simplex in the refined triangulation has a path with vectors $\frac{1}{2}f_1^{(1)}, \frac{1}{2}f_2^{(1)}, ..., \frac{1}{2}f_{d_1}^{(1)}$ in some order, and keeping the vectors $f_j^{(k)}$ for all j and for $k \geq 2$. Since $\sin_n$ is unchanged when arguments are scaled, this does not change its value. We can then refine another factor, and keep doing this in any order, with any number of repetitions of factors. This process can in general produce infinitely many shapes of the product elements and of their simplicial subdivisions. □

Example 3.3. We illustrate the main theorem by the simplest nontrivial example, namely a rectangular prism, i.e. a cartesian product of a triangle and an interval. It can be split into three tetrahedra in various ways. Let the chosen path in the triangle be R_{00}, R_{10}, R_{20} (see Figure 2). Each of the three tetrahedra in the figure contains a path with one edge for each of the two edges in the chosen path and a third vertical edge. When we apply $\sin_3$ to these three edges, we get the same value as $\sin_2$ of the two edges in the triangle, which is the same as the ordinary sine of the angle at R_{10}. This value of $\sin_3$ will be preserved by the red refinement scheme described in part c) of Theorem 3.2, i.e. when subdividing the height and the triangular base at independent rates.

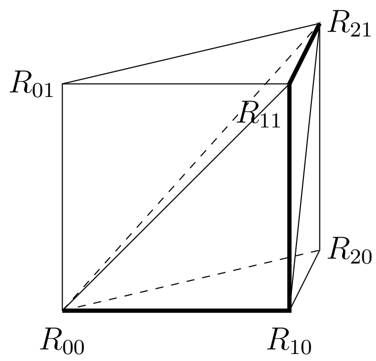

FIGURE 2. The prism is split into three tetrahedra, each with a path containing the same three edge vectors in some order. This path is marked in bold for the middle tetrahedron.

Remark 3.4. The maximum angle condition from Definition 1.1 will be satisfied for all refinements of a product of simplices produced by Theorem 3.2. Since we have used the formulation with $\sin_n$ this is obvious. As the restricted process in part b) of the theorem only produces a finite number of similarity types, that process will even preserve the standard regularity property, see [3]. In the general process in part c), the factors can shrink at different rates, and therefore the regularity condition can be violated. We refer to [11] for more information about the case of prisms in this context.

Remark 3.5. We are currently working on extending the ideas from [13] to guarantee overall conformity of a mesh of product elements, not only for a single product element as considered in the present paper.

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