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Motivations and analysis of the capacitated lot-sizing problem with setup times and minimum and maximum ending inventories

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Abstract

This paper first analyzes the negative impact of the end-of-horizon effect when solving the capacitated multi-item lot-sizing problem with setup costs and times on a rolling horizon. Maximum ending inventories for items and a global minimum ending inventory are considered to define a new optimization problem whose optimal solutions are much less impacted by the end-of-horizon effect. Then, a generation scheme is proposed to create new instances with initial inventories and ending inventories. This scheme relies on the analysis of the cyclical production planning problem to derive relevant parameters. Computational experiments are carried out to compare the solutions obtained for original instances of the literature and for the new instances, and to analyze the relevance of the new instances on a rolling horizon.

Keywords: Manufacturing, Multi-item lot sizing, setup times, ending inventories, rolling horizon.

1. Introduction

Lot-sizing problems aim at determining a production or distribution plan to satisfy demands on a time horizon discretized into periods and that minimizes the total cost. Since the middle of the 20^{th} century, lot sizing has been a very active research field, in particular because of its numerous applications in manufacturing and logistics. Yet, the first results on lot sizing come from the beginning of the 20^{th} century, where Harris (1913) developed the notion of Economic Order Quantity (EOQ) for a cyclical and stationary single-item production planning problem. A global overview on various types of lot-sizing problems can be found in Pochet and Wolsey (2006).

Throughout the years, the focus is increasingly on finding ways to model industrial problems as close to the reality as possible (Jans and Degraeve (2008)). Additional constraints in single-item lot-sizing problems have been extensively considered (Brahimi et al. (2017)). Although some dynamic lot-sizing problems are polynomial, the first one being studied in Wagner and Whitin (1958), they are generally NP-Hard, and many heuristics have been proposed in the literature to find good feasible solutions for single-item and multi-item problems. Lagrangian relaxations (see e.g. Brahimi et al.

(2006) and Süral et al. (2009)) and partial LP-relaxations (see e.g. Absi and Kedad-Sidhoum (2007) and Helber and Sahling (2010)) are two of the most popular methods.

To model the fact that, in many industrial contexts, starting a new product incurs a fixed time to configure the resource, Trigeiro et al. (1989) consider the notion of setup times in the multi-item capacitated lot-sizing problem (CLSP). Usually industrial lot-sizing problems are solved on a rolling horizon. In this context, only the decisions for the immediate periods are taken, after which the horizon is rolled forward and the model is applied once more with updated inventory, demand and capacity parameters. Using this approach enables for each period to be optimized several times and updated according to new information on future demands. However, because each optimization problem only considers a finite time horizon, an end-of-horizon effect can occur. As described in Fisher et al. (2001), most lot-sizing problems have in common that there exists a solution with a zero-ending-inventory property, meaning that there is no inventory at the end of the time horizon. The fact that the ending inventory is 0 for an optimal solution raises some issues, and can affect the production plan during the first periods in such a way that the quality of the solution decreases over time. Stadtler (2000), Fisher et al. (2001), van den Heuvel and Wagelmans (2005) propose ways to either define an adequate length of the time horizon or modify the objective function in order to cope with this end-of-horizon effect. However, they only consider single item uncapacitated lot-sizing problems. They deduce inventory valuations based only on the cost, using indicators such as the *Economic Order Quantity* (Harris (1913)). In addition, the proposed approaches do not apply since capacity is not taken into account when evaluating the ending inventory. These methods do not apply to the CLSP because they assume the zero inventory ordering (ZIO) property (Wagner and Whitin (1958)) and extend the dynamic programming algorithm proposed by Wagner and Whitin (WW). However the CLSP does not have the ZIO property, so an update on the cost coefficients when solving the problem using the WW algorithm as proposed by Stadtler (2000), van den Heuvel and Wagelmans (2005) cannot be used to determine the ending inventory of a multi-item capacitated lot-sizing problem. In such problems, the obtained solutions will not respect the capacity constraints. We also quote the work of Fisher et al. (2001) that provides a valuation for the ending inventory in the objective function, however this cost is not linear and cannot be solved by a linear solver.

Studying the impact of the end-of-horizon effect on a multi-item capacitated lot-sizing problem with setup times cannot be neglected. Yet, in the literature, the ending-horizon effect on the CLSP is very rarely considered. As illustrated in Section 2, independently of the number of periods of the planning horizon, the decisions in the first periods might be impacted by the zero-ending-inventory property. To the best of our knowledge, this phenomenon has never been studied in the literature. Clark and Clark (2000) considered a rolling horizon setting for a capacitated lot-sizing problem with multiple machines and setup carry-over and proposed a new model that modifies production times according to the average demand in order to get better LP-relaxations. Similarly to our approach, their model also takes into account a number of setups per period but the setting of this parameter is left to the user and does not depend on neither the costs nor the capacity at each period. Moreover, they do not consider additional inventory constraints. Campbell and Mabert (1991) justify the fact that cyclical schedules are often preferred in practice, mainly because they can be efficiently implemented. They also point out that cyclical schedules provide good results on average when solving capacitated lot-sizing problems. They impose a cyclical CLSP where cycle lengths are picked among a set of discrete values based on the Time Between Order (TBO) for each item. In our paper, cycle lengths are not predefined, setup times are considered in the TBO calculation, and we use a cyclical sub-problem to define relevant inventory indicators. The numerical results in Campbell and Mabert (1991) show that cyclical schedules are especially relevant for small demand variability, which is consistent with our computational results. Campbell and Mabert (1991) also point out that tighter capacity constraints provide larger gaps between the costs obtained by solving a cyclical problem and a non-cyclical problem. However, this can be caused by the fact that, when considering the set of cycle lengths, capacity is not taken into account in Campbell and Mabert (1991). In our approach, the theoretical cycle lengths also take capacity into account, adjusting the cycle length for each item accordingly.

In Chand et al. (2002), the authors point out the importance and the impact of the length of the time horizon on the solution quality. As pointed out by Carlson et al. (1979), to reduce the nervousness of a material requirement planning problem (MRP), we ideally want to minimize the changes in the production plan on a practical level when we add a period on a rolling horizon. Yet the added information on the new demand might change the optimal order of setups. Federgruen and Tzur (1994) extend the notion of nervousness in the MRP and propose an algorithm to find a minimum *forecast horizon* that is sufficient to not affect the decisions taken over a *planning horizon* for the uncapacitated single-item lot-sizing problem (ULSP). Their numerical results show that the minimum *forecast horizon* varies a lot depending on the parameters but it can be quite high for static costs. Moreover, finding a minimal *forecast horizon* is a problem that is of the same complexity as the ULSP. This importance is even greater when there are setup times and when the capacity is tight. In this case, there is no guarantee that all demands can be satisfied, and lost sales should be allowed and penalized. A common belief among researchers in the field is that extending the planning horizon is enough to ensure that decisions in the first periods are not impacted by the end-of-horizon effect. We show in this paper that this belief is not true for the the CLSP with setup times, and that allowing zero ending inventories might lead to poor decisions on arbitrarily large planning horizons, in particular when planning on a rolling horizon as it is the case in practice.

Hence, in this article, we propose a way to mitigate the end-of-horizon effect by adding to the CLSP with setup times and lost sales (Absi and Kedad-Sidhoum (2007, 2008)) both a maximum ending inventory per item and a global minimum ending inventory to be fulfilled at the end of the horizon. From the generation scheme of Trigeiro et al. (1989) (still used as a benchmark for capacitated lot-sizing problems, see e.g. Absi and Kedad-Sidhoum (2007) and de Araujo et al. (2015)), we propose a new framework to create instances for this new lot-sizing problem where the edge effect is avoided, and that are relevant when solving lot-sizing problems on a rolling horizon where the available information on future demands can be used.

The paper is organized as follows. Section 2 motivates the need to mitigate the end-of-horizon effect by the addition of a global minimum ending inventory as well as maximum inventory levels when solving a CLSP with setup times. In Section 3, an analysis of optimal solutions in a capacitated cyclical configuration is performed in order to evaluate relevant inventory levels on a rolling horizon, in a way similar to the definition of the TBO (Harris (1913)) but taking the capacity into account. Section 4 addresses some extensions of the CLSP for which the same analysis can be applied. In Section 5, an extension of the CLSP with setup times and global minimum ending inventory for all items is introduced. A new generation scheme which extends the one of Trigeiro et al. (1989) to create more relevant instances is then proposed. The generation scheme is extended in Section 4 to more general cases. A computational analysis is carried out in Section 6.1 to compare the solutions obtained by solving the original and the new instances, and the effect of planning on a rolling horizon. Some conclusions and perspectives can be found in Section 7.

2. Motivations

Section 2.1 recalls the multi-item CLSP model with setup times and lost sales. Section 2.2 illustrates the impact of a global minimum ending inventory, and Section 2.3 shows how the end-of-horizon effect can affect decisions taken in the first periods when planning on a rolling horizon, and thus the limits of the model and the instances of Trigeiro et al. (1989). Section 2.4 focuses on the impact of initial inventories.

2.1. Problem formulation

We consider the capacitated lot-sizing problem with setup times and lost sales, where N items have to be produced over a planning horizon of T periods. The discrete demand of each item i is given by d_{it} at period t. Each unit of item i produced at period t induces a production time b_{it} as well as a fixed setup time s_{it}. We aim at finding an optimal production plan, i.e. a production plan complying with the capacity restriction $c_t^{\max}$ for each period t while minimizing the total cost. This cost comprises the fixed and unitary production costs to be incurred each time a production takes place, the inventory holding costs for all the items as well as the lost-sales costs penalizing the unsatisfied demand. The cost parameters are the unitary production p_{it}, fixed setup f_{it} and unitary inventory holding h_{it} costs for item i at period t. The lost-sales costs penalizing each unit of unsatisfied demand of item i at period t is defined by l_{it}. We recall the mathematical formulation of the problem that can be found in Trigeiro et al. (1989) (without lost sales) and Absi and Kedad-Sidhoum (2008).

Let us define the decision variables as follows:

$X_{it} \geq 0$: Quantity of item i produced at period t,
$Y_{it} \in \{0, 1\}$: Setup variable equals to 1 if there is an order for item i at period t, and 0 otherwise,
$I_{it} \geq 0$: Inventory of item i at the end of period t,
$L_{it} \geq 0$: Quantity of lost sales for item i at the end of period t.

We extend the definition of I_{it} with $t = 0$ to describe the initial inventory of item i. Moreover, we use $\bar{\cdot}$ to define the average value of a parameter over all items and all periods, e.g. $\bar{f} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} f_{it}$.

The formulation of the CLSP with setup times and lost sales is given below:

$$\min \sum_{i=1}^{N}\sum_{t=1}^{T}(f_{it}Y_{it} + p_{it}X_{it} + h_{it}I_{it} + l_{it}L_{it}) \tag{1}$$

$$I_{i,t-1} + X_{it} + L_{it} = d_{it} + I_{it}, \qquad \forall i \in 1,\ldots,N,\ \forall t \in 1,\ldots,T \tag{2}$$

$$\sum_{i=1}^{N}(s_{it}Y_{it} + b_{it}X_{it}) \leq c_t^{max}, \qquad \forall t \in 1,\ldots,T \tag{3}$$

$$X_{it} \leq M_{it}Y_{it}, \qquad \forall i \in 1,\ldots,N,\ \forall t \in 1,\ldots,T \tag{4}$$

$$L_{it} \leq d_{it}, \qquad \forall i \in 1,\ldots,N,\ \forall t \in 1,\ldots,T \tag{5}$$

$$Y_{it} \in \{0, 1\}, \qquad \forall i \in 1,\ldots,N,\ \forall t \in 1,\ldots,T \tag{6}$$

$$X_{it}, I_{it}, L_{it} \geq 0, \qquad \forall i \in 1,\ldots,N,\ \forall t \in 1,\ldots,T \tag{7}$$

The objective function (1) minimizes the total production, setup, inventory and lost sales costs of all items over the planning horizon. Constraints (2) are the flow conservation constraints that balance, for each item, the inventory at period $t-1$ and the production and lost sales quantities at period t with the inventory and the demand at period t. Constraints (3) ensure that the capacity consumed by setup and production times does not exceed the maximum production capacity. Constraints (4) link the continuous production variables with the binary setup variables, M_{it} being an upper bound on the optimal production quantity (e.g. $M_{it} = \min(\sum_{k=t}^{T} d_{ik}, c_t^{\max} - s_{it})$). Constraints (5) state that the quantity of lost sales cannot exceed the demand. Constraints (6) and (7) define the domain of the variables.

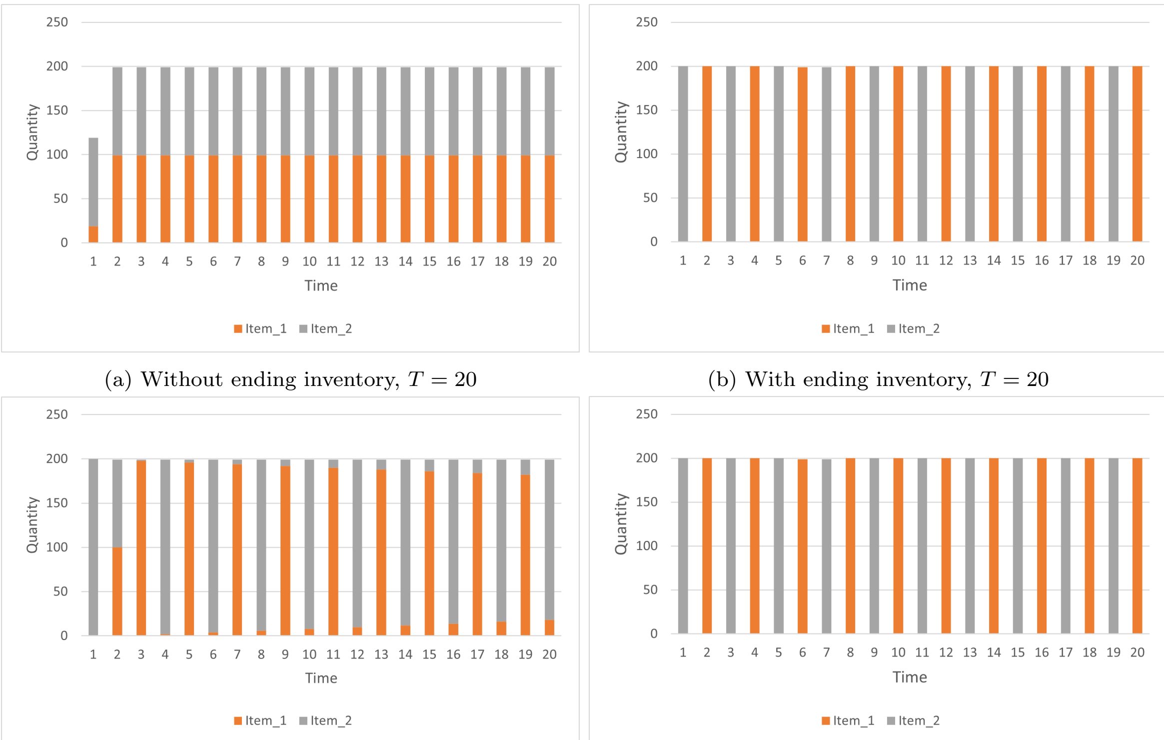
As in Trigeiro et al. (1989), we only consider the case where there are no production costs, and where the cost parameters are constant over the horizon. Thus, the index t is removed in the cost parameters.

2.2. Impact of global minimum ending inventory

In this section, we show through an illustrative example, that the end-of-horizon effect can affect the capacity consumption in the first periods even for large horizon lengths. On a rolling horizon, as discussed in Section 2.3, this can lead to significant lost sales. The example shows that the addition of a global minimum ending inventory can mitigate the end-of-horizon effect.

To illustrate the impact on the first periods of a production plan of considering an ending inventory, let us consider the optimal solution of an instance of the problem with 2 items, i.e. $N = 2$, and $T = 20$. The demand is constant over time and is set to 100, and the holding costs, unitary production times and setup times are set to 1. No setup and production costs are considered. The available capacity is $c^{\max} = 201$ in each period. In addition, the initial inventory $I_{1,0}$ is set to 100 for the first item.

Figure 1a (resp. 1b) shows the optimal solution obtained without (resp. with) a global minimum ending inventory set to 100 using the method proposed in Section 5.2, while Figure 1c (resp. 1d) shows the optimal solution for the first 20 periods when solving the problem with $T = 101$ (resp. $T = 200$).



(a) Without ending inventory, $T = 20$

(b) With ending inventory, $T = 20$

(c) Without ending inventory, $T = 101$

(d) Without ending inventory, $T = 200$

Figure 1: Production quantities in the first 20 periods

The addition of the global minimum ending inventory enables the decisions in the first periods to match the optimal decisions observed over a very long horizon. Even with $T = 20$ and only two items (Figures 1a and 1b), not considering inventory constraints at the end of the horizon directly impacts the decisions taken in the first periods. Without a global minimum ending inventory, Figure 1a shows that the capacity at the end of the horizon is used to add additional setups at each period.

This leads to a poor capacity utilization in the first period, where only a little over half the capacity is consumed. On the opposite, Figure 1b shows that capacity is better used with the ending inventory constraints. As shown in Figures 1c and 1d, the optimal production plan over longer horizon tends to the production plan of Figure 1b.

In order to illustrate the impact of the end-of-horizon effect on an instance with more items, Figure 6a shows the optimal plan for an instance of Trigeiro et al. (1989) with 10 items, where each color corresponds to an item. All the optimal solutions of the instances of Trigeiro et al. (1989) share the same shape, with small production lots in the first periods and an under-utilization of the capacity in the last periods of the horizon, as discussed in more details in Sections 2.3 and 2.4.

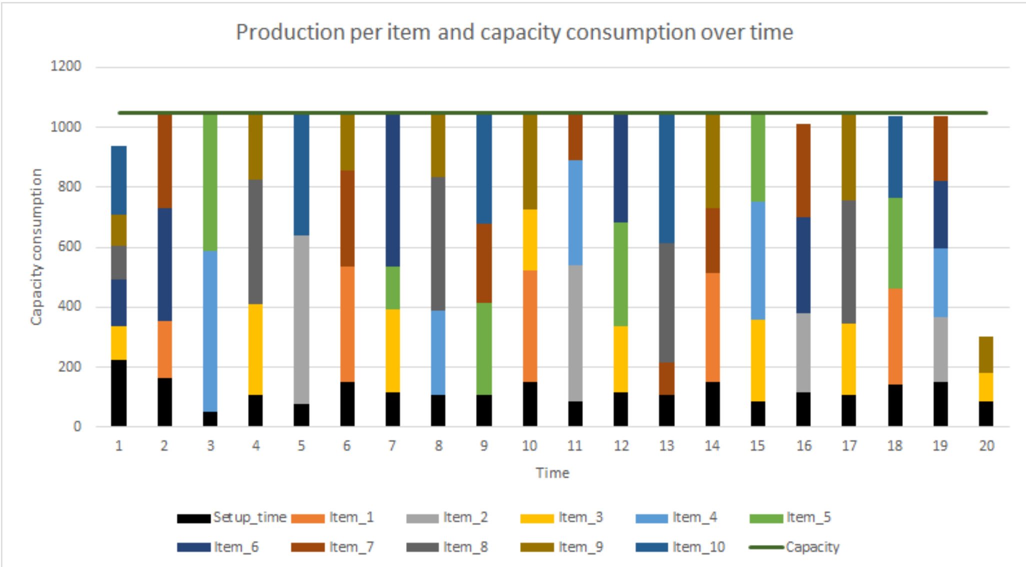


Figure 2: Optimal solution for an instance of Trigeiro et al. (1989).

Enough inventories should be available at the end of horizon, in particular on a rolling horizon, to make better use of production capacity in the first periods. The benefits of considering ending inventories on a rolling horizon are discussed in the following section.

It should also be pointed out that the number of periods affected by the edge effect (both the first and last periods of the horizon) can vary depending on the capacity and cost parameters. The impact of this effect is especially hard to evaluate for the CLSP with setup times, where regular indicators such as the TBO or the EOQ used for instance in Fisher et al. (2001) cannot be applied to capacitated problems. This impact is illustrated in Section 2.3, where even large time horizons cannot cope with the end-of-horizon effect.

2.3. Planning on a rolling horizon

In this section, we consider the process of planning on a rolling horizon, where $\tau \leq T$ is the number of first periods in which decisions are fixed after optimizing the production plan. Let us denote by T the number of periods of each planning horizon and by Ω the number of periods of the total time horizon ($T \leq \Omega$). Let us consider an instance of the problem with $N = 4$, $T = 10$ and $\tau = 1$. The demand is constant over time and is set to 100. The holding costs and unitary production times are set to 1. Setup times are fixed to 50. No setup and production costs are considered. The available capacity is $c^{\max} = 450$ for each period of the horizon. An initial inventory of 300 units is considered for each item. Note that, since no setup costs are considered, the best policy, only guided by the holding costs, is to have the lowest possible inventory levels. Ideally, no inventory would be carried and 100 units of both items would be produced at each period. However this production plan is not possible because of the limited capacity at each period. In this example, we assume that lost sales are highly penalized.

Figure 3 (resp. Figure 4) shows for this instance the inventory levels and lost sales (resp. the capacity consumption) on a rolling horizon with or without ending inventory constraints as proposed in Section 5.2. Let us first analyze the two different cases in Figure 3:

- Figure 3a shows the inventory levels and lost sales in the first 16 periods when no ending inventory constraints are considered. Without ending inventory constraints, the additional capacity provided at the end of the horizon, because of the zero-inventory policy, enables the inventory in the first periods of the rolling horizon to be immediately consumed and not kept to satisfy later demands. At each step of the rolling horizon process, the initial inventory decreases until it reaches a point where it is no longer possible to find a feasible solution without lost sales due to capacity limitations. We then get a cyclic production plan where 50 units are lost every two iterations.

- Figure 3b shows the inventory and lost sales evolution for the same instance with the addition of a global minimum ending inventory of 600 units and a maximum ending inventory of 300 units for each item (using the method proposed in Section 5.2). The ending inventory constraints force the capacity to be fully used throughout the planning horizon, and that the decisions of the first periods are not impacted by the unused capacity at the end of the horizon. We can see that no lost sales are observed in this case.

When lost sales are not allowed we get an infeasible production plan when there are no ending inventory constraints. When lost sales are penalized and not forbidden, because we have a cyclic production plan after 7 periods for the case without ending inventory and 2 periods with ending inventory, we can calculate the optimal cost over a rolling horizon of $\Omega \geq 7$ periods. Assuming l as a unit lost sales cost parameter, we get for the instance of the problem without ending inventory (Figure 3a) an optimal cost of:

$$C_1 = 2550 + (400 + 50l)\lfloor\frac{\Omega - 5}{2}\rfloor + 350\lceil\frac{\Omega - 5}{2}\rceil$$

and

$$C_2 = 1450 + (\Omega - 2)600$$

for the instance of the problem with ending inventory (Figure 3b). In this case, for $\Omega = 100$, we get that $C_1 \geq C_2$ for:

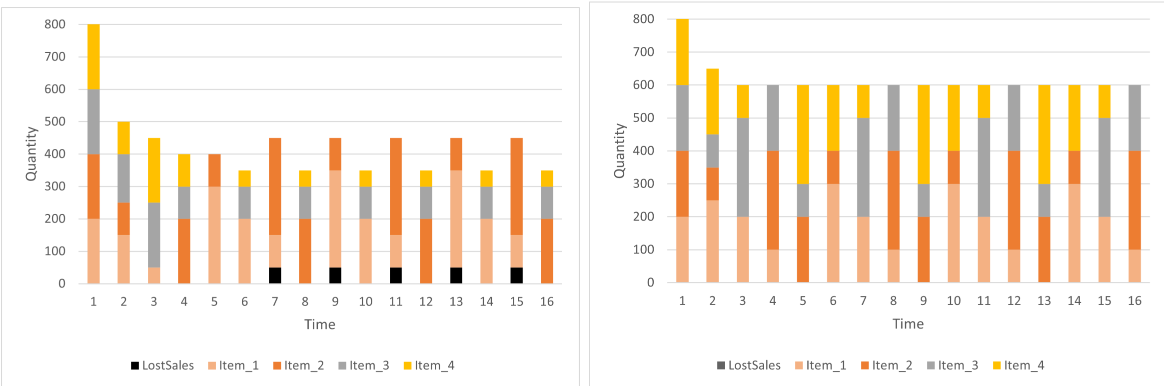
$$38150 + 2350l \geq 60250$$

Thus if $l \geq 9.5$ the production plan in Figure 3b becomes less costly than the plan in Figure 3a, and the difference increases with l. In many industrial applications, lost sales costs are the primary objective to optimize. Note that there are multiple optimal production plans when there are no ending inventory constraints.

Let us now analyze the capacity consumption for the two considered cases in Figure 4:

- In Figure 4a, because there are no ending inventory constraints, more capacity is allocated to setup times since the initial inventory is used in the first periods, and then not enough inventory is kept to fully satisfy part of the demands, leading to the lost sales observed in Figure 3a.

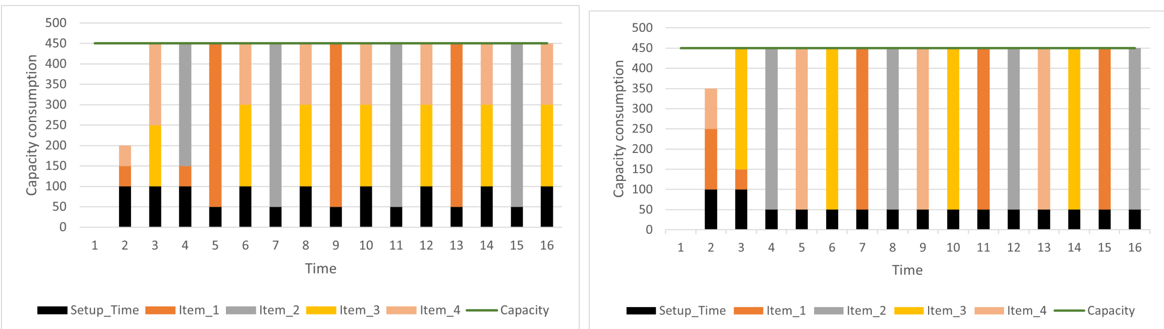
- Figure 4b shows that ending inventory constraints better allocate setup times throughout the horizon. This explains why there are no lost sales in Figure 3b.



(a) Without ending inventory

(b) With ending inventory

Figure 3: Inventory evolution on a rolling horizon



(a) Without ending inventory

(b) With ending inventory

Figure 4: Capacity consumption on a rolling horizon

One can consider that the production plan patterns in Figures 3b and 4b could be obtained by significantly increasing the number of periods of the planning horizon. However, this leads to some negative effects. Indeed, the introduction of non accurate demands, because of the lack of information at the end of the horizon, should impact the quality of the obtained solutions. Moreover, increasing the number of periods negatively affects the computational efficiency of solution approaches. As the capacity gets tighter, the number of periods T that needs to be considered on a rolling horizon increases. In contrast, setting a global minimum ending inventory based on future demand predictions on a rolling horizon allows capacity to be better used. Moreover, contrarily to studies on uncapacitated lot-sizing problems (e.g. Carlson et al. (1979)), there is no theoretical guarantee that, for capacitated problems with setup times, there exists a forecast horizon ensuring that the decisions in the first periods will not be affected by demands outside of the planning horizon.

2.4. Initial inventories

Figure 6a shows that there are 5 setups in the first period while, from period 2 to 18, the number of setups oscillates between 2 and 3. Note also that the fraction of the capacity consumption taken by setup times is larger in the first period than in the following ones. This is because, when there are neither lost sales nor initial inventory, as it is the case for the CLSP, a setup will occur for every item before or at the period corresponding to its first positive demand. The first production periods are not impacted by production and setup costs, which explain the difference in the number of setups. This leads the optimization process to focus on packing the first production quantities to meet the demands of the first periods as well as making full use of the capacity constraints.

The feasibility of the problem highly depends on whether or not the capacity in the first periods is large enough to cover the demands of the first periods. As the capacity is constant over the planning horizon, in order to avoid infeasibility for these instances due to the required capacity for covering the demands during the first periods, 25% of the demands in the first four periods were set to 0 in the instances provided by Trigeiro et al. (1989). This choice was arbitrarily made to guarantee feasibility, and has no practical reality. Optimizing the production quantities in the first production periods, which increases the computational complexity of the optimization problem, also does not make much sense when planning on a rolling horizon.

Initial inventories can also have a significant impact on the feasibility of the solution when planning on a rolling horizon, as the initial inventories are linked to the decisions that are taken in the first periods.

Following this discussion, we propose to mitigate the end-of-horizon effect by adding inventory constraints at the end of the last period of the planning horizon. Adding inventory constraints is a relevant way to ensure that ending conditions are satisfied. Indeed, it makes sense from a practical point of way to always keep a minimum inventory level of all items. This global minimum inventory level differs from the minimum inventory level per item in each period that is typically associated to safety stocks. To avoid that the ending inventory only includes a single item, typically the one with the smallest holding cost, and to balance the ending inventory among items, a maximum ending inventory level for each item is also considered.

3. Inventory levels for the capacitated lot-sizing problem with setup times on a rolling horizon

The main goal of this section is to define new indicators to characterize relevant inventory levels for the considered capacitated lot-sizing problem on a rolling horizon. To this end, we use similar arguments that the ones used to define the Time Between Order and the Economic Order Quantity (Harris (1913)). To do that, we define in Section 3.1 a new problem that enables us to find approximated analytical values whose relevance will be discussed in the numerical analysis of Section 6. We show in Section 3.2 that this problem is relevant compared to the CLSP, and more specifically to the problem with static costs and the parameters considered in Trigeiro et al. (1989). This simplified model will be used in Section 5.2 to update the instance generation scheme proposed in Trigeiro et al. (1989), in order to create instances whose optimal plans will not be affected by the end-of-horizon effect.

3.1. Multi-item cyclical production planning with bounded average capacity consumption

Let us consider the multi-item lot-sizing problem with setup times that consists in finding optimal cycle lengths on a rolling horizon. All costs and demands are static. Let us denote by $\phi_i \in \mathbb{N}^*$ the cycle length of an item i, *i.e.* the number of periods between two production periods. The production cycle length $\bar{\phi} \in \mathbb{N}^*$ is defined as the minimum number of periods, such that each item has an integer number of cycles, *i.e.* the least common multiple of the cycle length of all the items. Additionally, the cycle lengths of each item should be such that the average capacity consumed in each period of a production cycle should not exceed the maximum capacity $c^{\max}$. Figure 5 illustrates the item and the production cycle length for an instance with four items and a horizon of ten periods, where $\phi_1 = \phi_2 = 3$, $\phi_3 = \phi_4 = 2$ and $\bar{\phi} = 6$.

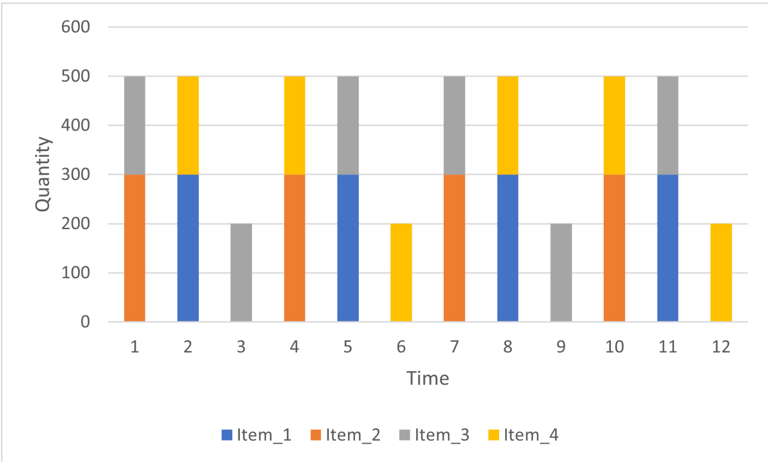


Figure 5: Example of cyclical production with 4 items.

The total inventory cost for one cycle of length ϕ_i is $\bar{h}d_i \sum_{t=1}^{\phi_i-1} t = \bar{h}d_i\phi_i\frac{\phi_i-1}{2}$. The total inventory cost in a production cycle of length $\bar{\phi}$ is then:

$$\sum_{i=1}^{N} \frac{\bar{\phi}}{\phi_i}\bar{h}d_i\phi_i\frac{\phi_i-1}{2} = \bar{h}\bar{\phi}\sum_{i=1}^{N} d_i\frac{\phi_i-1}{2}$$

The average inventory at each period $\tilde{I^{inf}}$, as well as the maximum inventory I_i^{sup} for item i, can be defined as:

$$\tilde{I^{inf}} = \sum_{i=1}^{N} d_i\frac{\phi_i-1}{2} \tag{8}$$

$$I_i^{sup} = (\phi_i - 1)d_i \tag{9}$$

We denote $\phi = \{\phi_1, \cdots, \phi_N\}$. The optimization problem (P_C) that minimizes the cost per period in a production cycle $\bar{\phi}$ can be defined as follows:

$$\min_{\phi \in \mathbb{N}^{*N}} \quad \sum_{i=1}^{N} \frac{f_i}{\phi_i} + \bar{h} \sum_{i=1}^{N} d_i \frac{\phi_i - 1}{2} \tag{10}$$

$$s.t. \quad \sum_{i=1}^{N} \frac{s_i}{\phi_i} \leq c^{\max} - N\bar{b}\bar{d} \tag{11}$$

The objective function (10) minimizes the average setup and inventory costs for a period of the cycle length. In a cyclical configuration, the average production time per period is $N\bar{b}\bar{d}$, which means that, on average, the capacity available for setup times is $c^{\max} - N\bar{b}\bar{d}$. Constraint (11) thus imposes an upper bound on the average number of setups.

3.2. Similarities between (P_C) and the optimization problem of Trigeiro et al. (1989)

To consider a problem with similar data than the instances generated in Trigeiro et al. (1989), average values for costs and demands are set identical for all items. These values correspond to the average values of the parameters defined in Section 2.1, except for the production costs that are equal to 0 and the lost sales that are not allowed.

The expected number of setups per period $k_i \in \mathbb{R}_+^*$ for item i corresponds to the reciprocal of the cycle length ϕ_i:

$$\frac{1}{\phi_i} = k_i$$

This implies that the average number of setups $k \in \mathbb{R}_+^*$ in each period of a production cycle is related to the cycle length of each item:

$$k = \sum_{i=1}^{N} k_i = \sum_{i=1}^{N} \frac{1}{\phi_i}$$

The optimization problem $(\bar{P_C})$ that minimizes the cost per period in a production cycle $\bar{\phi}$ can be derived from problem (P_C) and is defined as follows:

$$\min_{(k,\phi) \in \mathbb{R}_+^* \times \mathbb{N}^{*N}} \quad \bar{f}k + \bar{h}\bar{d} \sum_{i=1}^{N} \frac{\phi_i - 1}{2} \tag{12}$$

$$s.t. \quad \sum_{i=1}^{N} \frac{1}{\phi_i} = k \tag{13}$$

$$k \leq \frac{c^{\max} - N\bar{b}\bar{d}}{\bar{s}} \tag{14}$$

The objective function (12) as well as constraint (14) are similar to (10) and (11). Constraint (13) links the number of setups per period with the cycle length of each item.

The cyclical approximation is a simple but relevant simplification of the model in order to get a general idea of the behaviour of a solution as it provides insights on the role of the costs and the capacity. In the Trigeiro et al. (1989) generation scheme, all costs are static. The costs can vary between each item, but they are all generated by doing slight changes around common average values. Even though the demands are dynamic, they are also generated around the same common value. In $(\bar{P_C})$, we only consider the case where the average capacity consumed in each period is lower than $c^{\max}$, which is a relaxation of the initial problem where there is an upper bound on the capacity in each period. However, with N production cycle lengths, we assume that there is one configuration such that the capacity used in each period is close to the average capacity consumption. Furthermore, in this paper, we are only interested in approximating the inventories in each period. The idea is not to consider the dynamics of production. Thus, analyzing the cyclical multi-item problem, where all costs and demands are averaged and where the average capacity consumption is bounded, should provide enough insight on the shape of an optimal solution of the problem addressed in Trigeiro et al. (1989) on a rolling horizon. The relevance of our assumptions is analyzed in the computational experiments of Section 6.1. The study of the continuous relaxation of $(\bar{P_C})$ helps to evaluate the average total inventory as well as the maximum inventory per item.

3.3. Analysis of the continuous relaxation of $(\bar{P_C})$

When the integrality constraints on variables ϕ_i are relaxed in $(\bar{P_C})$, the following non-linear problem $(\tilde{P_C})$ can be derived:

$$-\frac{N\bar{h}}{2}\bar{d} + \min_{\{k \in \mathbb{R}_+^* | k \leq \frac{c^{\max} - N\bar{d}\bar{b}}{\bar{s}}\}} (\bar{f}k + \frac{\bar{h}}{2}\bar{d} \min_{\{\phi \in \mathbb{R}_+^{*N} | \sum_{i=1}^N \frac{1}{\phi_i} = k\}} \sum_{i=1}^N \phi_i),$$

as the set $\{\phi \in \mathbb{R}_+^{*N} | \sum_{i=1}^N \frac{1}{\phi_i} = k\}$ is non-empty for all $k \in \mathbb{R}_+^*$.

Let us show that the continuous relaxation of $(\bar{P_C})$ can be analytically solved to optimality. Providing an easy-to compute analytical formula might give insights on the links between the costs and the capacity for an instance of the problem.

Property 1. *If* $\frac{c^{\max} - N\bar{d}\bar{b}}{\bar{s}} \geq N\sqrt{\frac{\bar{h}\bar{d}}{2\bar{f}}}$*, then the optimal solution is reached for* $k^* = N\sqrt{\frac{\bar{h}\bar{d}}{2\bar{f}}}$*, and the optimal value is:* $-\frac{\bar{h}\bar{d}N}{2} + N\sqrt{2\bar{h}\bar{d}\bar{f}}$*. Otherwise, the optimal solution is reached for* $k^* = \frac{c^{\max} - N\bar{d}\bar{b}}{\bar{s}}$*, and the optimal value is:* $\frac{\bar{h}\bar{d}N}{2}(\frac{N\bar{s}}{c^{\max} - N\bar{d}\bar{b}} - 1) + \bar{f}\frac{(c^{\max} - N\bar{d}\bar{b})}{\bar{s}}$*.*

Proof. Let us first show that:

$$\min_{\{\phi \in \mathbb{R}_+^{*N} | \sum_{i=1}^N \frac{1}{\phi_i} = k\}} \sum_{i=1}^N \phi_i = \frac{N^2}{k}$$

Using the Euclidian norm and its corresponding scalar product, the Cauchy-Schwarz inequality states that, for $\phi_i \in {\mathbb{R}_+^*}^N$:

$$(\sum_{i=1}^N \sqrt{\frac{\phi_i}{\phi_i}})^2 \leq (\sum_{i=1}^N \frac{1}{\phi_i})(\sum_{i=1}^N \phi_i)$$

Thus, by positivity:

$$\sum_{i=1}^N \phi_i \geq \frac{N^2}{\sum_{i=1}^N \frac{1}{\phi_i}}$$

So that:

$$\min_{\{\phi \in \mathbb{R}_+^{*N} | \sum_{i=1}^N \frac{1}{\phi_i} = k\}} \sum_{i=1}^N \phi_i \geq \frac{N^2}{k}$$

When $\phi_i = \bar{\phi} = \frac{N}{k}$, for all i, we have $\sum_{i=1}^N \frac{1}{\phi_i} = k$ and $\sum_{i=1}^N \phi_i = \frac{N^2}{k}$, and then:

$$\min_{\{\phi \in \mathbb{R}_+^{*N} | \sum_{i=1}^N \frac{1}{\phi_i} = k\}} \sum_{i=1}^N \phi_i = \frac{N^2}{k}$$

The continuous relaxation of $(\bar{P_C})$ is equivalent to:

$$-\frac{\bar{h}\bar{d}N}{2} + \min_{\{k\in\mathbb{R}_+^* | k \leq \frac{c^{\max}-N\bar{d}\bar{b}}{\bar{s}}\}} (\bar{f}k + \frac{\bar{h}\bar{d}N^2}{2k})$$

Let us consider the function $g(x) = \bar{f}x + \frac{\bar{h}\bar{d}N^2}{2x}$ on $\mathbb{R}_+^*$. This function of $x \in \mathbb{R}_+^*$ is decreasing until $x^* = N\sqrt{\frac{\bar{h}\bar{d}}{2\bar{f}}}$, and then increasing. If k^* denotes the optimal average number of setups and as $k^* \leq \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$, then:

- If $\frac{c^{\max}-N\bar{d}\bar{b}}{\bar{s}} \geq N\sqrt{\frac{\bar{h}\bar{d}}{2\bar{f}}}$, then $k^* = N\sqrt{\frac{\bar{h}\bar{d}}{2\bar{f}}}$ and the optimal value is: $-\frac{\bar{h}\bar{d}N}{2} + N\sqrt{2\bar{h}\bar{d}\bar{f}}$,
- Otherwise, $k^* = \frac{c^{\max}-N\bar{d}\bar{b}}{\bar{s}}$, and the optimal value is: $\frac{\bar{h}\bar{d}N}{2}(\frac{N\bar{s}}{c^{\max}-N\bar{d}\bar{b}} - 1) + \bar{f}\frac{(c^{\max}-N\bar{d}\bar{b})}{\bar{s}}$.

□

Let us introduce $k_{capa} = \frac{c^{\max}-N\bar{d}\bar{b}}{\bar{s}}$ and $k_{cost} = N\sqrt{\frac{\bar{h}\bar{d}}{2\bar{f}}}$. In a cyclical configuration, the number of setups k in each period is close to:

$$k = \min(k_{cost}, k_{capa}) \tag{15}$$

and the cycle length ϕ_i is close to $\frac{N}{k}$ for all items.

If the capacity constraints are not binding, the production cycles follow the time between order $TBO = \sqrt{2\frac{\bar{f}}{\bar{d}\bar{h}}}$ for each item. On average, $k = k_{cost}$ items will be produced in each period, consuming a capacity of $\frac{N}{k}\bar{s} + N\bar{d}\bar{b}$.

By applying the same analysis to the original Trigeiro et al. (1989) instances, we find that, for some of these instances, $k_{capa} < 1$. This would imply that these instances are only feasible because of the extra capacity freed by the demands randomly set to 0 in the first periods of the planning horizon. For the other instances, we have $k_{capa} > N$, which is not relevant when the capacity is constrained. Moreover, if the costs are defined such that $\frac{N}{k}$ is integer, then the optimal solution of the relaxed problem $(\tilde{P_C})$ is an optimal solution of problem $(\bar{P_C})$. Indeed, because of the relaxation of the integrity property of the ϕ variables we have:

$$\min_{\{\phi\in\mathbb{R}_+^{*N} | \sum_{i=1}^{N}\frac{1}{\phi_i}=k\}} \sum_{i=1}^{N}\phi_i = \frac{N^2}{k} \leq \min_{\{\phi\in\mathbb{N}_+^{*N} | \sum_{i=1}^{N}\frac{1}{\phi_i}=k\}} \sum_{i=1}^{N}\phi_i$$

For $\phi_i = \frac{N}{k} \in \mathbb{N}^*$, the value of $\frac{N^2}{k}$ is reached for problem $(\bar{P_C})$.

If the cycle length ϕ_i of item i is integer, then we have already established that the average inventory of item i is equal to $\bar{d}\frac{\phi_i - 1}{2}$. The maximum inventory is $\bar{d}(\phi_i - 1)$. We then apply these formula to deduce the approximate values of the total average and the maximum inventory for each item. As $\phi_i = \frac{N}{k}$, we get the following values:

$$\tilde{I^{inf}} = \bar{d}\sum_{i=1}^{N}\frac{\frac{N}{k}-1}{2} = \frac{N(N-k)}{k}\frac{\bar{d}}{2} \tag{16}$$

$$I_i^{sup} = I^{sup} = \frac{N-k}{k}\bar{d} \tag{17}$$

.

We can define analytical values for $I^{\tilde{inf}}$ and I^{sup} even when $\phi_i = \frac{N}{k}$ is not an integer. We show in Section 6.1 that this indicator is effective on a rolling horizon.

3.4. Lost sales costs

Considering the problem defined in Section 3.1 where we allow for a fraction of the demand to not be satisfied, we can find an analytic value for lost sales costs which would ensure that there exists an optimal solution without lost sales. The detailed analysis for this problem leading to (18) can be found in the appendix and shows that there is an optimal solution without lost sales if we set the lost sales cost parameter for item i as:

$$l_i = \max\left(\frac{N\bar{s}\bar{h}c^{\max}}{2(c^{\max} - N\bar{b}\bar{d})^2}, \sqrt{\frac{2\bar{f}\bar{h}}{\bar{d}}}, \frac{\bar{h}}{2}(\frac{N\bar{s}}{c^{\max} - N\bar{b}\bar{d}} - 1) + \frac{\bar{f}}{Nd\bar{s}}(c^{\max} - N\bar{b}\bar{d})\right) \tag{18}$$

This proposed definition of the lost sales cost better integrates the cost and the capacity parameters of the instances. Expression (18) relies on the analysis of the cyclical problem, where the possibility for only a fraction of the demand to be satisfied is considered, and the characterization of sufficient conditions for the optimal cycle to have no lost sales.

4. Extensions to more general cases

This section considers some extensions of the CLSP for which the same rationale can be applied and analytical values for the optimal cycle lengths can be deduced from the relaxed problem. The first extension addresses the case where the average demand varies between items, and the second extension the case where, in addition, the setup times and costs are linearly dependent.

4.1. Average demand per product

Let us consider the case where the average demand d_i over time is different for each item. Let us define an optimal cyclical production plan where the average capacity consumption does not exceed $c^{\max}$ by solving the following problem:

$$\min_{(k,\phi)\in\mathbb{R}^*\times\mathbb{N}^{*N}} \quad \bar{f}k + \bar{h}\sum_{i=1}^{N} d_i \frac{\phi_i - 1}{2} \tag{19}$$

$$\text{s.t.} \quad \sum_{i=1}^{N}\frac{1}{\phi_i} = k \tag{20}$$

$$k \leq \frac{c^{\max} - N\bar{b}\bar{d}}{\bar{s}} \tag{21}$$

Constraint (21) is still valid for the model above, since the average capacity consumed in each period by the production setup time is $\sum_{i=1}^{N} d_i\bar{b} = N\bar{b}\bar{d}$. Hence, the capacity available for setup times is still $c^{\max} - N\bar{b}\bar{d}$. The continuous relaxation of the problem can be written as:

$$-\bar{h}\sum_{i=1}^{N}\frac{d_i}{2} + \min_{\{k\in\mathbb{R}^*_+ | k \leq \frac{c^{\max}-N\bar{d}\bar{b}}{\bar{s}}\}} (\bar{f}k + \frac{\bar{h}}{2}\min_{\{\phi\in\mathbb{R}^{*N} | \sum_{i=1}^{N}\frac{1}{\phi_i}=k\}}\sum_{i=1}^{N} d_i\phi_i)$$

Property 2. *If* $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} \geq \sqrt{\frac{\bar{h}}{2\bar{f}}}\sum_{i=1}^{N}\sqrt{d_i}$*, then the optimal solution is reached for* $k^* = \sqrt{\frac{\bar{h}}{2\bar{f}}}\sum_{i=1}^{N}\sqrt{d_i}$*, and the optimal value is:* $\sqrt{2\bar{f}\bar{h}}\sum_{i=1}^{N}\sqrt{d_i} - \bar{h}\sum_{i=1}^{N}\frac{d_i}{2}$*. Otherwise, the optimal solution is reached for* $k^* = \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$ *and the optimal value is:* $\frac{\bar{f}}{\bar{s}}(c^{\max} - N\bar{b}\bar{d}) + \frac{\bar{s}\bar{h}}{2}\frac{(\sum_{i=1}^{N}\sqrt{d_i})^2}{c^{\max}-N\bar{b}\bar{d}} - \bar{h}\sum_{i=1}^{N}\frac{d_i}{2}$*.*

The proof of Property 2 follows similar arguments than the proof of Property 1.

4.2. Average demand per item and correlated setup costs and times

Let us now assume that the setup times and costs are correlated, i.e. there exists $\lambda \in \mathbb{R}^*$ such that $f_i = \lambda s_i$, $\forall i \in \{1, \dots, N\}$, and also that the average demand d_i is not the same for all items. We set $\tilde{k} = \sum_{i=1}^{N} \frac{s_i}{\phi_i}$, which corresponds to the average setup time per period. The average setup cost per period is then $\sum_{i=1}^{N} \frac{f_i}{\phi_i} = \lambda \sum_{i=1}^{N} \frac{s_i}{\phi_i} = \lambda \tilde{k}$.

Similarly, the following cyclical production planning problem is solved:

$$\min_{(\tilde{k},\phi)\in\mathbb{R}^*\times\mathbb{N}^{*N}} \quad \lambda\tilde{k} + \bar{h}\sum_{i=1}^{N} d_i \frac{\phi_i - 1}{2} \tag{22}$$

$$s.t. \quad \sum_{i=1}^{N} \frac{s_i}{\phi_i} = \tilde{k} \tag{23}$$

$$\tilde{k} \leq c^{\max} - N\bar{b}\bar{d} \tag{24}$$

The continuous relaxation of the problem can be written as:

$$-\bar{h}\sum_{i=1}^{N} \frac{d_i}{2} + \min_{\{\tilde{k}\in\mathbb{R}^*_+ | \tilde{k} \leq c^{\max} - N\bar{d}\bar{b}\}} \quad (\lambda\tilde{k} + \frac{\bar{h}}{2} \min_{\{\phi\in\mathbb{R}^{*N} | \sum_{i=1}^{N} \frac{s_i}{\phi_i} = \tilde{k}\}} \sum_{i=1}^{N} d_i\phi_i)$$

Property 3. *If* $c^{\max} - N\bar{b}\bar{d} \geq \sqrt{\frac{\bar{h}}{2\lambda}} \sum_{i=1}^{N} \sqrt{d_i s_i}$*, then the optimum is reached for* $\tilde{k}^* = \sqrt{\frac{\bar{h}}{2\lambda}} \sum_{i=1}^{N} \sqrt{d_i s_i}$*, and the optimal value is:* $\sqrt{2\lambda\bar{h}} \sum_{i=1}^{N} \sqrt{d_i s_i} - \bar{h}\sum_{i=1}^{N} \frac{d_i}{2}$*. Otherwise, the optimum is reached for* $\tilde{k}^* = c^{\max} - N\bar{b}\bar{d}$*, and the optimal value is:* $\lambda(c^{\max} - N\bar{b}\bar{d}) + \frac{\bar{h}}{2} \frac{(\sum_{i=1}^{N}\sqrt{d_i s_i})^2}{c^{\max} - N\bar{b}d} - \bar{h}\sum_{i=1}^{N} \frac{d_i}{2}$.

The proof of Property 3 follows similar arguments than the proof of Property 1.

5. New instance generation scheme

The new generation scheme proposed in this paper is based on the one proposed in Trigeiro et al. (1989) with additional enhancements and parameters. The original instance generation scheme is recalled in Section 5.1, and the new generation scheme is outlined in Section 5.2. The parameters are described and analyzed in details in Section 5.3.

5.1. Original generation scheme

In the CLSP instances of Trigeiro et al. (1989), the cost and capacity parameters are constant over time. The number of items varies from 10 to 30, and the production costs are equal to 0. The instances were built as follows:

- **Demand range.** Demands are dynamic with an average value $\bar{d} = 100$. Half of the instances have demands following a uniform probability distribution in the range $[75, 125]$, the other half in the range $[0, 200]$. In addition, 25% of the demands in the first four periods are set to 0.

- **Time Between Order (TBO).** The time between order, defined as $TBO = \sqrt{2\frac{\bar{f}}{\bar{d}\bar{h}}}$ (Harris (1913)), is in $\{1, 2, 4\}$. In all the original instances, $\bar{h} = 1$.

- **Production and Setup times.** Half of the instances have an average setup time of $\bar{s} = 11$, and of $\bar{s} = 43$ for the other half. All unitary production times are set to $\bar{b} = 1$.

- **Capacity tightness.** For each instance, an average capacity use per period is computed following the EOQ of Harris (1913). This capacity consumption is divided by a factor $\rho \in \{0.75, 0.85, 0.95\}$ to define the instance capacity per period: $c^{\max} = \frac{N}{\rho}(\frac{\bar{s}}{TBO} + \bar{b}\bar{d})$.

- **Variability between items.** Setup times as well as inventory and setup costs for each item are generated based on their average values multiplied by coefficients taking values uniformly in the range $[0.5, 1.5]$.

5.2. New generation scheme

In this section, we propose a new generation scheme integrating the features discussed in Section 2. Since we consider an extension of the CLSP with lost sales, new related parameters will be defined. Lost sales are allowed but at a very high cost.

In order to obtain optimal solutions with limited edge effects at the beginning or at the end of the planning horizon, we solve a new mixed integer linear problem based on the CLSP formulation (1-7) with the following additional parameters and constraints:

- A global minimum ending inventory I^{inf}, so that the inventory level is not equal to 0 at the end of the horizon, subject to the following constraint:

$$\sum_{i=1}^{N} I_{iT} \geq I^{inf}$$

- An upper bound on the final inventory I^{sup} of each item in order to have enough item diversity in the ending stock, subject to the following constraint:

$$I_{iT} \leq I^{sup}, \quad \forall i \in [1, N]$$

- An initial inventory per item I_{i0} to have enough stock to satisfy the first demands.

The tricky point is the set up of the new parameters I^{inf}, I^{sup} and I_{i0}, so that they will be in line with the practical considerations discussed in Section 2. The parameters that do not appear in the outline of the generation scheme below, are generated according to the original scheme described in Section 5.1. For the new parameters, a reference to the section with the detailed analysis is provided.

- **Demand range.** Demands are dynamic with an average value $\bar{d} = 100$. Half of the instances have demands following a uniform probability distribution in the range $[75, 125]$, the other half in the range $[0, 200]$. Contrary to Trigeiro et al. (1989), no demands in the first four periods are set to 0.

- **TBO, Setup times.** As defined in Section 5.1.

- **Lost sales cost for item i.** As defined in Section 3.4.

- **Maximum inventory per unit.** $I^{sup} = \frac{N-k}{k}\bar{d}$ (see (17) derived in Section 3.3), where $k = \min(k_{cost}, k_{capa})$ (see (15) derived in Section 3.3), with $k_{cost} = N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}}$ and $k_{capa} = \frac{c^{\max} - N\bar{d}\bar{b}}{\bar{s}}$.

- **Global minimum ending inventory.** $I^{inf} = \sum_{i=1}^{N} I^*_{iT}$, where the values of I^*_{iT} are obtained by solving the Mixed Integer Linear Program (MILP) (P_f) in Section 5.3.

- **Initial inventory per item.** $I_{i0} = I^*_{i0}$, where the values of I^*_{i0} are obtained by solving (P_f) in Section 5.3.

- **Capacity tightness.** $c^{\max} = k\bar{s} + N\bar{d}\bar{b} + c_o^*$, where the value of c_o^* is obtained by solving (P_f) in Section 5.3.

Section 3 showed how the last four parameters of the new generation scheme were derived. These parameters are fitted in Section 5.3. In Section 3.3, we defined an approximate global minimum ending inventory I^{inf}, and a maximum inventory per item, denoted by I^{sup}, based on the value of a time between order deduced from the average value of the demands, the average holding and setup costs and the maximum capacity per period. From these parameters, we then define in Section 5.3 a new Mixed Integer Linear Program (MILP) which, given a global inventory value I^{inf} as well as a maximum inventory per item I^{sup}, determines feasible initial inventory values for each item as well as a capacity limit. The value of I^{inf} is also fitted in order to follow the dynamic nature of the demand.

5.3. Fitting I^{inf} and setting initial inventories

In Section 3, a global minimum ending inventory and a maximum inventory based on a static cyclical model were proposed. In order to find fitted values for the initial inventories of the dynamic CLSP with lost sales, we solve a MILP where all the constraints of the original model (2)-(7), as well as additional global minimum ending inventory and maximum inventory constraints, are considered. The initial inventory of each item must be set so that the total initial inventory should be close to $\tilde{I}^{inf}$, yet individually each initial inventory should be lower than I^{sup}. This comes from the fact that, ideally, the total inventory is constant throughout the time horizon, and individual inventories should not exceed the value of I^{sup} deduced in Section 3.3.

We want an inventory configuration at the end of the horizon that is similar to the one at the beginning of the horizon. Therefore, the goal is to find a feasible solution that minimizes the absolute value of the difference between the total initial inventory and the total inventory at the end of the planning horizon. Let us denote by (P_f) the following MILP:

$$\min\ Kc_o + \delta \tag{25}$$

$$I_{i,t-1} + X_{it} = d_{it} + I_{it}, \qquad \forall i \in 1,\dots,N,\ \forall t \in 1,\dots,T \tag{26}$$

$$\sum_{i=1}^{N}(s_{it}Y_{it} + b_{it}X_{it}) \leq \tilde{c}^{max} + c_o, \qquad \forall t \in 1,\dots,T \tag{27}$$

$$0 \leq I_{i0} \leq I^{sup}, \qquad \forall i \in 1,\dots,N \tag{28}$$

$$0 \leq I_{iT} \leq I^{sup}, \qquad \forall i \in 1,\dots,N \tag{29}$$

$$\gamma\tilde{I}^{inf} \leq \sum_{i=1}^{N} I_{i0} \leq \tilde{I}^{inf}, \tag{30}$$

$$\delta \geq \sum_{i=1}^{N}(I_{i0} - I_{iT}), \tag{31}$$

$$\delta \geq \sum_{i=1}^{N}(I_{iT} - I_{i0}), \tag{32}$$

$$X_{it} \leq M_{it}Y_{it}, \qquad \forall i \in 1,\dots,N,\ \forall t \in 1,\dots,T \tag{33}$$

$$Y_{it} \in \{0,1\}, \qquad \forall i \in 1,\dots,N,\ \forall t \in 1,\dots,T \tag{34}$$

$$X_{it} \geq 0, \qquad \forall i \in 1,\dots,N,\ \forall t \in 1,\dots,T \tag{35}$$

$$I_{it} \geq 0, \qquad \forall i \in 1,\dots,N,\ \forall t \in 0,\dots,T \tag{36}$$

$$c_o, \delta \geq 0 \tag{37}$$

where $\tilde{c}^{max}$ is defined using average values on the costs and capacity, $\tilde{c}^{max} = k\bar{s} + N\overline{db}$ with k defined as shown in Section 3.3.

The capacity limit is the sum of the fixed capacity $\tilde{c}^{max}$ and the overtime c_o. The fixed parameter represents the estimated capacity consumption in the cyclical configuration. Overtime is added to guarantee the feasibility of the problem with dynamic demands, but is highly penalized in the objective function (25) by the parameter K. The fixed capacity is defined according to the EOQ provided either by the average costs or by the capacity $c^{\max}$. Variable δ is the gap between the initial inventory and the ending inventory. This gap has to be minimized as well. Constraints (27) are the capacity constraints. Constraints (28) and (29) set bounds on the initial inventories and the ending inventories. Constraint (30) ensures that the total initial inventory is close to the global minimum ending inventory, and parameter γ defines the tightness in Constraint (30), where $0 \leq \gamma \leq 1$. Constraints (31) and (32) link δ with the inventory gap, while Constraints (33) connect the production and setup variables. Finally, the domains of the variables are given by Constraints (34)-(37).

In the proposed generation scheme, we define $c^{\max} = \tilde{c}^{max} + c_o^*$, where c_o^* is the optimal value obtained by solving (P_f). In addition, the global minimum ending inventory is fitted to guarantee that a feasible solution without lost sales can be found, $I^{inf} = \sum_{i=1}^{N} I_{iT}^*$, where I_{iT}^* is the optimal ending inventory obtained by solving (P_f). In our case, we want to avoid adding overtime to the analytical capacity unless to avoid infeasible instances when lost sales are not allowed. We set K, the penalty per unit of overtime, to an order of magnitude higher that the unit penalization of the gap between the initial and the ending inventories. Regarding the γ parameter, ideally it should be close to 1 to keep the same global inventory at the beginning and at the end of the horizon. However, in order to allow some slacks due to the variability of the costs and the parameters, we can set a slightly lower value. In our computational experiments, we set $K = 100$ and $\gamma = 0.95$.

The initial inventory values I_{i0} are considered as decision variables because we want to create instances that are relevant on a rolling horizon. However, a possible extension could be to assume that the initial inventories are known parameters and to define an adequate minimum ending inventory level I^{inf} that is close to $\tilde{I}^{inf}$ but takes into account the potential lack of inventory during the first periods and the impact of the capacity tightness. Ideally the ending inventory should be close to $\tilde{I}^{inf}$ without lost sales or extra capacity. The analytical minimum inventory target $\tilde{I}^{inf}$ might be reached after a few iterations over the rolling horizon even with a lack of initial inventory. If we assume I_{i0} are known parameters, we can remove Constraints (28) and (30) and modify Constraints (31) and (32) from (P_f) as follows:

$$\delta \geq \tilde{I}^{inf} - \sum_{i=1}^{N} I_{iT}, \tag{38}$$

$$\delta \geq \sum_{i=1}^{N} I_{iT} - \tilde{I}^{inf}, \tag{39}$$

Constraints (38) and (39) defines δ as the absolute value of the difference between the ending inventory and the analytical ending inventory $\tilde{I}^{inf}$.

6. Computational experiments

The original instances of Trigeiro et al. (1989) are compared with our new instances in Section 6.1 while, in Section 6.2, the relevance of considering ending inventories on a rolling horizon is shown.

6.1. Comparison of original and new instances

As a benchmark for the creation of the new proposed instances, we used the 180 instances of Trigeiro et al. (1989) with $N = 10$ items and $T = 20$ periods, and the 180 instances with $N = 30$ items. All the instances have an average demand of 100 units per period and per item and, for a given instance, all items have the same cost and demand pattern. Half of the instances have low demand variability (demand between 75 and 125), and the other half have high demand variability (demand between 0 and 200). We first modified the demands that were originally set to 0 in the first periods to ensure the feasibility of the instances, by assigning them a random value generated as the strictly positive demand (see Section 5.1). The original instances will be referred as "Orig.", and the instances created by applying the new generation scheme summarized in Section 5.2 as "New".

The mathematical models are solved using IBM ILOG CPLEX 12.7 on a computer with 2.6 GHz PC, 64 GB of RAM and 2 processors, with a maximum running time of 600 seconds for each instance, except for Table 3 where the time limit is set to 100 seconds.

Adding the initial inventory and the ending inventory constraints is supposed to mitigate the edge effect, whose potential main impacts are a drop of production in the last periods and a large number of setups in the first periods to satisfy the initial demands. A way to measure this edge effect is to analyze the variation of the number of setups and of the production between periods. Indeed, for the original instances, this variation is high because of the edge effects, as observed in Figure 6a. The initial and ending inventories are established using a cyclical sub-model, where we assume a constant production and number of setups over time. Even if the CLSP model with minimum and maximum ending inventories is not cyclical, we expect to find an optimal production plan with low variability between periods, which should lead to a reduction of the impact of the edge effects (Figure 6b). In order to show that the constraints on the minimum and maximum ending inventories do not make the model easier to solve, the optimality gaps obtained within the same time limit of 100 seconds were computed for both the original model and the new model.

Let us define a variability coefficient as the ratio between the standard deviation and the average value over all the periods. For the original and new instances, Table 1 provides the variability coefficients for the number of setups per period, the total quantity produced per period and the total inventory per period. The results are classified according to different parameters: Number of items (N), Time Between Orders (TBO), demand range, average setup time and capacity tightness. Except for the capacity tightness, the classification parameters are not affected by the modification of the original instances using the new generation scheme. It is worth noticing that the capacity tightness relies on the computation of EOQ, and hence on the average demand. The following point is interesting to note about the original instances. Because the capacity is defined by dividing the capacity required for an EOQ production with a coefficient that is smaller than 1, in the problem studied in Section 3.1, the shape of the relaxed solution is only guided by the costs, that is $k = k_{cost}$. By following the new generation scheme in Section 5, it should also be the case for the new instances. However, in all the original instances of Trigeiro et al. (1989), the capacity was deduced by taking an average demand of 90 to compute the EOQ. This should not be the case as Trigeiro et al. (1989) state that the average demand is equal to 100. However, unlike described in their generation scheme, not 25% but 50% of the demands in the first four periods were set to 0 when generating the original instances, and the demands that were removed were not balanced among the demands at other periods. When recomputing k_{capa} with the same capacity but with an average demand of 100 for the new instances, there are cases where $k_{capa} < k_{cost}$, thus an optimal solution guided by the capacity.

Table 1 shows that the expected behavior is observed, i.e. the variability is greatly reduced when adding the initial inventories and the global minimum ending inventory. The variability of the inventory is larger than the variability of the setup and production, but always significantly lower for the new instances than for the original ones. On average, the variability coefficients for the quantity

		Variability coefficient					
		Setup		Production		Inventory	
		Orig.	New	Orig.	New	Orig.	New
N	**10**	0.29	0.15	0.26	0.04	0.70	0.40
	30	0.26	0.11	0.25	0.03	0.59	0.27
TBO	**1**	0.22	0.09	0.23	0.06	1.17	0.75
	2	0.24	0.13	0.25	0.03	0.42	0.18
	4	0.37	0.16	0.28	0.01	0.35	0.08
Demand range	**[75;125]**	0.27	0.09	0.25	0.02	0.76	0.32
	[0;200]	0.28	0.16	0.25	0.05	0.53	0.35
Average setup time	**11**	0.28	0.14	0.25	0.02	0.68	0.34
	43	0.27	0.11	0.26	0.04	0.61	0.33
Capacity tightness$^{(*)}$	**EOQ/0.75**	0.31	0.12	0.31	0.04	0.66	0.37
	EOQ/0.85	0.27	0.12	0.26	0.04	0.76	0.46
	EOQ/0.95	0.25	0.14	0.18	0.02	0.52	0.17

Table 1: Comparison of the variability for the original and new instances.
$^{(*)}$This classification only applies on the original instances.

produced are between 5 to 28 times smaller when the ending inventory constraints are added (from 0.28 to 0.01 for the instances with a TBO of 4). This implies that the deviation from the average quantity produced at each period is much smaller for the new instances. As illustrated in Figure 6a, this variability in the produced quantities was mostly caused by the edge effects. Note that the setup range, defined by the difference between the maximum and the minimum number of setups in a period, is also lower for the new instances.

		Capacity Utilization			
		Mean (%)		Variability coefficient	
		Orig.	New	Orig.	New
N	**10**	84.0	98.2	0.25	0.03
	30	84.1	98.4	0.24	0.02
TBO	**1**	83.0	95.8	0.22	0.05
	2	84.2	99.3	0.24	0.02
	4	85.0	99.8	0.28	0.00
Demand range	**[75;125]**	84.6	99.4	0.25	0.01
	[0;200]	83.5	97.2	0.25	0.04
Average setup time	**11**	84.5	98.9	0.24	0.02
	43	83.6	97.7	0.25	0.03
Capacity tightness$^{(*)}$	**EOQ/0.75**	74.4	97.6	0.31	0.03
	EOQ/0.85	84.5	98.1	0.26	0.03
	EOQ/0.95	93.2	99.3	0.17	0.01

Table 2: Capacity utilization mean and variability.
$^{(*)}$This classification only applies on the original instances.

Let us now analyze the capacity utilization in more detail. Table 2 shows the mean capacity utilization over all periods and the standard deviation between the periods. The mean capacity utilization is defined as the average value of the ratio between the consumed capacity and the available capacity over the time horizon. The capacity parameters of the original instances were generated by taking an average value of the capacity required to have a production based on the EOQ of each item and by dividing it by a coefficient of 0.75, 0.85 and 0.95 (Trigeiro et al. (1989)). It is clear that the mean capacity utilization is much larger for the new instances, always larger than 95% and most often close or larger than 98%. On the opposite, the mean capacity utilization for the original instances is nearly always smaller than 85%, and is even equal to 74.4% when the capacity tightness is equal to EOQ/0.75. The results are even more impressive when considering the variability, which is never larger than 0.05 in the new instances, whereas it is always larger than 0.17 in the original instances with a peak at 0.31, again when the capacity tightness is equal to EOQ/0.75.

Tables 3 and 4 compare the average optimality gaps and the average computational times of the original and new instances. With a maximum computational time of 100 seconds, the average optimality gap for the new instances is as large as 2.3% when TBO is equal to 4, whereas it is never larger than 0.9% for the original instances. When the maximum computational time is increased to 600 seconds, the differences between the optimality gaps remain large, up to 1.7% when TBO is equal to 4. Average computational times are also much larger when solving the new instances.

Finally, Tables 3 and 4 show that the new instances are harder to solve than the original ones, i.e. adding initial inventories and a global minimum ending inventory does not make the problem easier to solve and raises issues as how to solve the new instances efficiently.

		Gap (%)		**MaxGap (%)**		**Time (sec.)**	
		Orig.	**New**	**Orig.**	**New**	**Orig.**	**New**
N	**10**	0.4	1.7	4.9	6.6	34.5	71.7
	30	0.2	0.8	1.4	8.1	38.7	78.2
TBO	**1**	0.1	0.3	1.9	4.3	25.6	34.1
	2	0.2	1.0	2.0	5.8	31.7	90.7
	4	0.5	2.3	4.9	8.1	52.6	100.2
Demand range	**[75;125]**	0.4	1.5	4.9	8.1	41.9	78.2
	[0;200]	0.2	0.9	3.4	4.6	31.3	71.8
Average setup time	**11**	0.3	1.5	4.9	8.1	39.0	81.6
	43	0.3	0.9	2.7	4.6	34.2	68.3
Capacity tightness$^{(*)}$	**EOQ/0.75**	0.0	1.0	0.0	5.8	1.2	68.0
	EOQ/0.85	0.0	0.9	0.7	6.4	19.2	67.2
	EOQ/0.95	0.9	1.7	4.9	8.1	89.4	89.7

Table 3: Average optimality gaps and computational times for the original and new instances with $T_{lim} = 100$ sec.
$^{(*)}$This classification only applies on the original instances.

Figure 6 displays the optimal plans for an original instance and its associated new instance. Note that the production is relatively constant over time, as is the number of setups in each period. However, in the first and last periods, there is both a decrease in the number of setups and an increase of the capacity utilization, in line with the other periods, in Figure 6b compared to Figure 6a, i.e. in the new instance compared to the original one. By adding initial inventories and a global minimum ending inventory, the capacity is fully consumed in all the periods of the horizon in the optimal production plan of the new instance. Moreover, except for slight variations caused by differences in the costs between items, the fraction of capacity used for setup times is rather stable throughout the planning horizon.

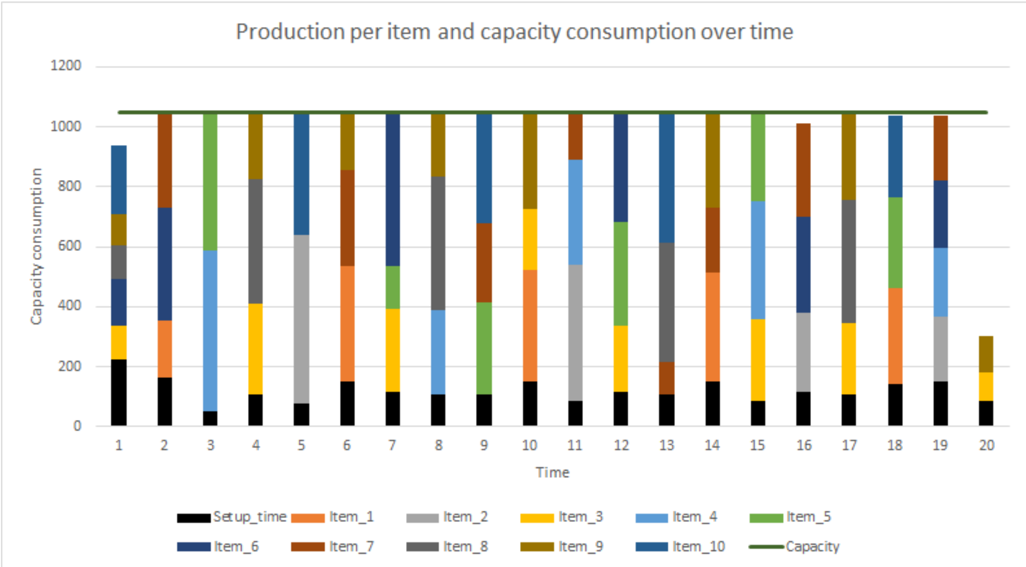
Initial and global minimum ending inventories were deduced from the study of a relaxed version of the problem defined in Section 3.1. In Sections 3.3 and 4, analytical values for objective functions are provided. To validate the study of this simplified problem to deduce values for the CLSP with setup times, Table 5 displays the gaps between the best upper bound obtained by solving the MILP model and the analytical optimal values for each instance.

		Gap (%)		MaxGap (%)		Time (sec.)	
		Orig.	**New**	**Orig.**	**New**	**Orig.**	**New**
N	**10**	0.3	1.3	4.5	5.8	171.7	385.4
	30	0.1	0.5	1.0	2.7	212.5	453.8
TBO	**1**	0.1	0.2	1.1	2.3	138.7	165.5
	2	0.2	0.8	1.8	5.4	180.1	494.6
	4	0.4	1.7	4.5	5.8	257.5	598.6
Demand range	**[75;125]**	0.3	1.2	4.5	5.8	225.4	453.6
	[0;200]	0.2	0.6	3.2	3.1	158.7	385.5
Average setup time	**11**	0.2	1.1	4.5	5.8	206.2	467.8
	43	0.2	0.7	2.4	3.6	178.0	371.4
Capacity tightness$^{(*)}$	**EOQ/0.75**	0.0	0.8	0.0	5.5	5.6	367.9
	EOQ/0.85	0.0	0.7	0.0	5.5	62.9	361.3
	EOQ/0.95	0.7	1.3	4.5	5.8	507.7	529.5

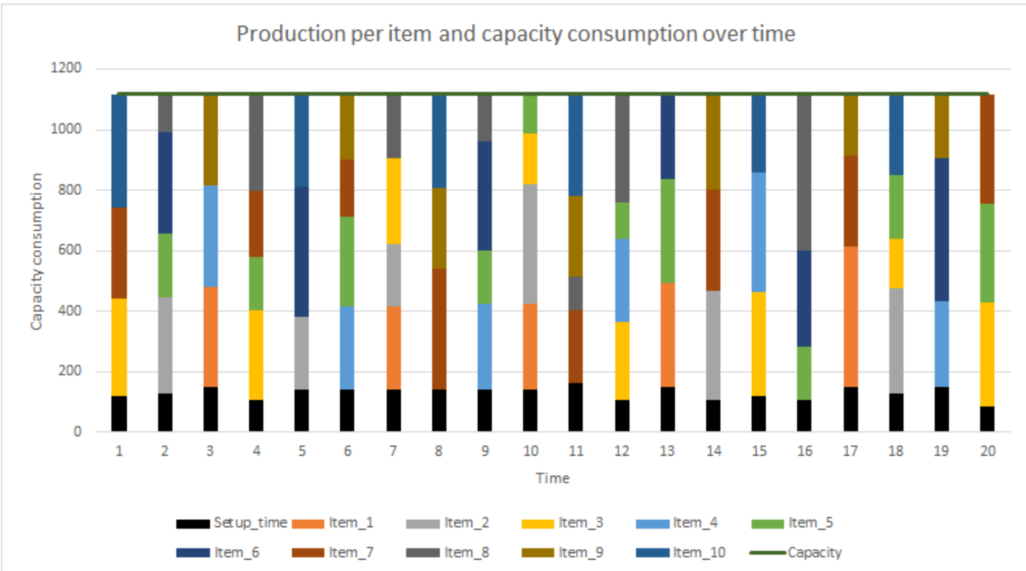
Table 4: Average optimality gaps and computational times for the original and new instances with $T_{lim} = 600$ sec.
$^{(*)}$This classification only applies on the original instances.

		Gap (%)	
		Orig.	**New**
N	**10**	9.7	4.3
	30	10.6	5.9
TBO	**1**	13.3	5.1
	2	10.4	5.2
	4	6.7	5.1
Demand range	**[75;125]**	6.2	2.4
	[0;200]	14.1	7.8
Average setup time	**11**	10.3	5.0
	43	10.0	5.3
Capacity tightness$^{(*)}$	**EOQ/0.75**	11.4	5.1
	EOQ/0.85	10.5	5.2
	EOQ/0.95	8.5	5.1

Table 5: Gap with predicted objective value.
$^{(*)}$This classification only applies on the original instances.



(a) Solution for original instance.



(b) Solution for new instance.

Figure 6: Comparing the optimal plans of two related original and new instances.

Table 5 shows that, for the original and new instances, the gap between the predicted and the optimized objective values is on average equal to 5.1%, which implies that the problem defined in Section 3.1 is a good relaxation of the CLSP with setup times. When the demand range is small, the approximation is even better. The gap is equal to 2.4% for a small demand range, whereas it increases to 7.8% when the demand has a larger range. That makes sense because the smaller the demand range, the closer each demand is to its average value. For the original instances, the average gap is equal to 10.2%, so the approximation is less precise. That can in part be explained by the fact that the original instances have neither initial nor ending inventories, hence approximating the original problem by a problem on a rolling horizon might be too constraining.

Note also that, in Table 5, the quality of the approximation does not seem to depend on the TBO for the new instances. The TBO is theoretically linked to $k = k_{capa} = k_{cost}$ by the formula $k = \frac{N}{TBO}$ in the new generated instances. This is an interesting point as the average number of setups per period k is the main factor shaping the production plan. Consequently, the optimal production plan varies greatly depending on the TBO but the approximation remains of the same quality. In the original instances of Trigeiro et al. (1989), the analytical optimal value better approximates the best upper bound for the problem as the TBO increases (13.3% of average gap for a TBO of 1 to 6.7% for a TBO of 4). The poor evaluation of the optimal objective value for smaller TBO can be partly explained by the fact that, in the original instances of Trigeiro et al. (1989), 50% of the demands in the first 4 periods are set to 0. For the instances with a TBO close to 1, when the capacity is not constraining, the number of setups during the first periods can be reduced compared to the analytical average number of setups. This leads to an overevaluation of the optimal objective value. For the instances with tight capacity, some of the later demands need to be satisfied during the first periods where demands were removed. This leads to additional inventory costs and an underevaluation of the optimal objective value.

6.2. Analysis on a rolling horizon

To test the impact of the global minimum ending inventory on a rolling horizon, we extend the instances of Trigeiro et al. (1989) with $N = 30$ by using the same generation scheme to create instances with $\Omega = 100$. The global minimum ending inventory and maximum ending inventory per item are generated by solving the continuous relaxation of the problem (P_C) defined in Section 4 and set the minimum and maximum ending inventories using (8) and (9). The continuous relaxation of (P_C) is:

$$\min_{(k_i,\phi)\in\mathbb{R}_+^{*N}\times\mathbb{R}_+^{*N}} \quad \sum_{i=1}^{N} f_i k_i + \bar{h}\sum_{i=1}^{N} d_i \frac{\phi_i - 1}{2} \tag{40}$$

$$s.t. \quad \sum_{i=1}^{N} s_i k_i \leq c^{\max} - N\bar{b}\bar{d} \tag{41}$$

$$k_i \phi_i \geq 1, \forall i \in 1, \ldots, N \tag{42}$$

This problem is a *Quadratic Constraint Problem* that can be solved to optimality using the barrier algorithm of CPLEX. The analytical expressions ((16), (17)) correspond to the specific case where the average costs and demands over time are the same for all items. The main advantage of using the quadratic model to fix the inventory levels is that it can be applied to instances with high variability on the item parameters as it provides a different maximum ending inventory (9) for each item. However, the analytical expressions do not require a solver and provide a very good approximation of the maximum ending inventories for instances with low variability on the item parameters. The obtained minimum and maximum inventory levels remain constant through the rolling horizon. Three settings for the time horizon T are considered ($T \in \{5, 10, 20\}$) in order to fix the decisions period by period.

Compared to the original instances, we modified the instances and the capacity by taking $k_{capa} \in \{3.75, 7.5, 15\}$ and set $c_t^{\max} = N\bar{b}\bar{d} + k\bar{s}$, where $k = \min(k_{capa}, N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}})$. The initial inventory for each item has been set as half the maximum ending inventory for each item ($I_{i0} = \frac{I_i^{sup}}{2}$), which corresponds to the average inventory level based on the cycle length determined in Section 4.

When optimizing on a rolling horizon, Constraint (5.2) was slightly modified to allow lost sales on the global minimum ending inventory:

$$\sum_{i=1}^{N} I_{iT} \geq I^{inf} + l_{T+1}.$$

The new parameter l_{T+1}, which is the unit cost of lost sales for the ending inventory, is defined so that it is less costly to have lost sales in the last period of the horizon than in previous periods.

		Fraction of lost sales (%)					
		T=5		T=10		T=20	
		Minimum ending Inventory		Minimum ending Inventory		Minimum ending Inventory	
		w/o	with	w/o	with	w/o	with
		Avg — Max	Avg — Max	Avg — Max	Avg — Max	Avg — Max	Avg — Max
Demand range	[75;125]	6.74—9.68	0.61—1.87	4.14—5.85	0.09—0.40	2.13—3.02	0.13—0.65
	[0;200]	6.20—10.65	0.93—3.45	3.87—6.45	0.52—2.29	1.99—3.28	0.38—2.36
Average setup time	11	4.38—6.24	0.25—1.36	2.93—4.46	0.22—1.29	1.60—3.06	0.19—2.36
	43	8.56—10.65	1.29—3.45	5.08—6.45	0.40—2.29	2.52—3.28	0.33—1.19

Table 6: Fraction of lost sales (%) for $N = 30$ and $k_{capa} = 3.75$

		Fraction of lost sales (%)					
		T=5		T=10		T=20	
		Minimum ending Inventory		Minimum ending Inventory		Minimum ending Inventory	
		w/o	with	w/o	with	w/o	with
		Avg — Max	Avg — Max	Avg — Max	Avg — Max	Avg — Max	Avg — Max
Demand range	[75;125]	5.06—7.16	0.24—0.72	2.78—3.58	0.17—0.70	1.23—1.61	0.19—0.47
	[0;200]	4.56—7.71	0.79—1.89	2.53—4.15	0.58—1.63	1.04—1.77	0.32—1.12
Average setup time	11	3.56—5.13	0.47—1.89	2.22—3.57	0.39—1.63	1.06—1.77	0.22—1.12
	43	6.05—7.71	0.56—1.88	3.09—4.15	0.36—1.37	1.21—1.71	0.28—0.66

Table 7: Fraction of lost sales (%) for $N = 30$ and $k_{capa} = 7.5$

		Fraction of lost sales (%)					
		T=5		T=10		T=20	
		Minimum ending Inventory		Minimum ending Inventory		Minimum ending Inventory	
		w/o	with	w/o	with	w/o	with
		Avg — Max	Avg — Max	Avg — Max	Avg — Max	Avg — Max	Avg — Max
Demand range	[75;125]	1.71—2.87	0.15—0.47	0.75—1.54	0.14—0.40	0.21—0.65	0.07—0.20
	[0;200]	1.39—3.06	0.47—2.15	0.57—1.85	0.28—1.43	0.16—0.89	0.11—0.58
Average setup time	11	1.48—3.06	0.44—2.15	0.79—1.85	0.29—1.43	0.29—0.89	0.11—0.58
	43	1.62—3.01	0.18—0.74	0.53—1.05	0.13—0.40	0.08—0.24	0.07—0.20

Table 8: Fraction of lost sales (%) for $N = 30$ and $k_{capa} = 15$

Tables 6, 7 and 8 compare the fraction of lost sales on a rolling horizon on instances classified according to their demand range and average setup time. Each table presents the results for a specific value of k_{capa} without or with the global minimum ending inventory and for different planning horizons

(T). The influence of each parameter is similar in each table, even if it can be noticed that the average fraction of lost sales seems to increase when the capacity becomes tighted. With $k_{capa} = 3.75$, the average fraction of lost sales is equal to 6.74% for $T = 5$ and no ending inventory constraint while, for the same instances and $k_{capa} = 15$, the average fraction of lost sales decreases to 1.96%.

For a specific value of k_{capa}, several remarks can be raised. In terms of lost sales, the results are much better with ending inventory constraints. In Table 6 with $T = 5$, the average lost sales of 6.47% without ending inventory constraints drops to 0.77%. In Table 7, the average lost sales decreases from 4.81% to 0.52% and, in Table (8), from 1.55% to 0.31%. The average lost sales without ending inventory constraints is almost always larger than 1%, except when both the planning horizon and the capacity are large. With ending inventory constraints, even with a small planning horizon and a constraining capacity, the average lost sales is almost always smaller than 1%. With ending inventories, there are on average both less lost sales and less setup times. In Table 6 for $T = 10$, the average lost sales are equal to 2.93% with $\bar{s} = 11$ and 5.08% with $\bar{s} = 43$. This makes sense because, when setup times are small compared to the available capacity, the impact they can have on the feasibility of the problem is less relevant. When setup times are large, up to a certain point, it becomes more difficult to recover from a lack of production in a previous period, leading to an increase of the lost sales. However, this seems to no longer be the case when setup times reach a given threshold. This is because the number of setups per period becomes fixed, and the decisions in the first periods do not lead to more lost sales in later periods.

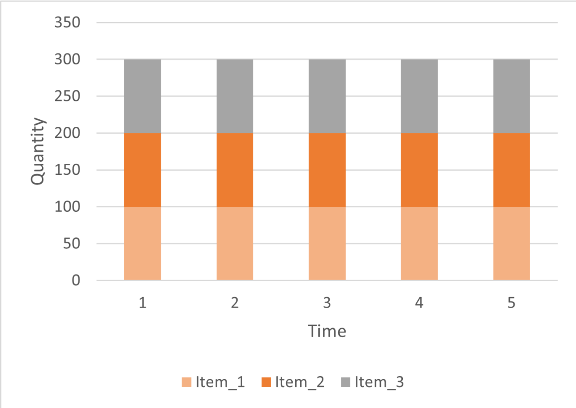
To illustrate the previous point, let us consider the case with $N = 3$, $T = 5$, $d = 100$, and no setup and production costs. The holding costs and unitary production times are set to 1. Let us consider different values for the setup times, $\bar{s} \in \{0, 10, 100\}$. Let us set $k_{capa} = 1$ and define $c^{max} = 300 + k_{capa}\bar{s}$. To model the lack of production in a period before the start of the planning horizon, we consider the case where the initial inventory is set to 0 for all items. Lost sales are highly penalized. The optimal production plans can be found in Figure 7.

- For $\bar{s} = 0$: There are no lost sales in the optimal solution. This can be explained by the fact that the initial lack of production can be recovered because all demands at a period can be satisfied by production quantities at the same period, which is not the case for $\bar{s} > 0$.

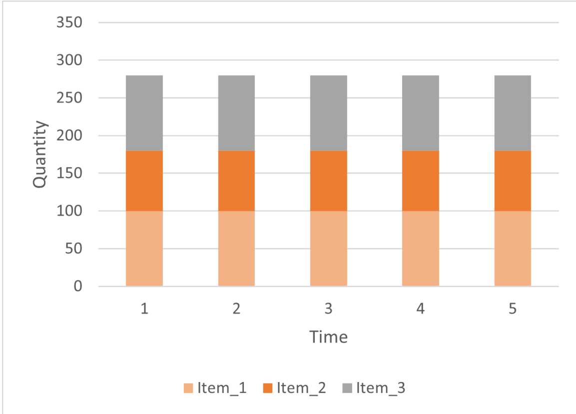
- For $\bar{s} = 10$: There are 80 units of lost sales (20 units of item 2 at each period) in the optimal solution. The production plan is similar to the one obtained for $\bar{s} = 0$. However, when there are 3 setups in a period, only 280 units can be produced, leading to a deficit of 20 units per period.

- For $\bar{s} = 100$: There are 300 units of lost sales (100 units of item 2 at $t = 1$ and at $t = 3$, 100 units of item 3 at $t = 1$) in the optimal solution. Because each setup takes $\frac{1}{4}$ of the capacity, it is suboptimal to have 3 setups in the first period. However, we get a production plan that is optimal on a rolling horizon if one period is fixed in each optimization run, leading to less lost sales compared to Figure 7b.

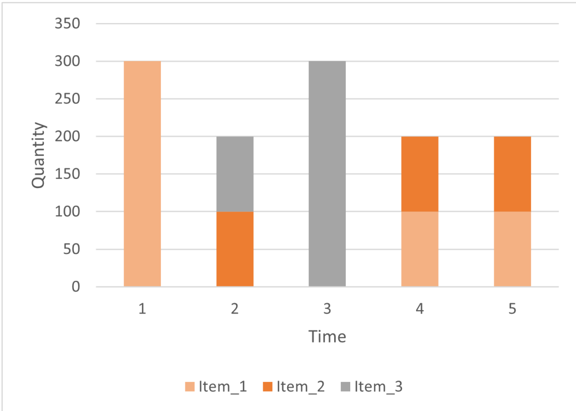
- For $\bar{s} \geq 300$: Each setup takes more than half the capacity available during a period, so the optimal production plan consists in having exactly one setup in each period. There are also 300 units of lost sales in the optimal solution. Because the production capacity forces to have at most one setup per period, the resulting production plan is the optimal production plan on a rolling horizon.



(a) $\bar{s} = 0$



(b) $\bar{s} = 10$



(c) $\bar{s} = 100$

Figure 7: Production quantities for different values of $\bar{s}$

Note that, without inventory levels, the results depend a lot on the length of the time horizon. The fluctuation between the fraction of lost sales when the length of the planning horizon increases is larger without ending inventories. Table 7 shows that, for $\bar{s} = 43$, the fraction of lost sales decreases from 6.05% to 3.09% when T increases from 5 to 10. With ending inventories, the best results are obtained when there is a small variability in demands. For instance, for $T = 10$ in Table 7, the fraction of lost sales drops from 0.79% for instances with a large demand range to 0.24% for instances with a small demand variability. This also makes sense because the computation of the ending inventories are based on models using averages. When looking more specifically at the evolution of the inventories in each period, note that the ending inventories enable for inventories to be kept throughout the horizon rather than being depleted in the first periods.

7. Conclusions

In this paper, we first highlight the issues associated with classical instances of the literature for the capacitated lot sizing problem with setup times. Our analysis shows that the lack of ending inventories creates inconsistencies with the industrial reality where planning is performed on a rolling horizon. In particular, even when using a long planning horizon, decisions in the first periods might be negatively impacted. Building on this analysis and to mitigate the end-of-horizon effect, we propose to modify the problem by considering a global minimum ending inventory and maximum ending inventories for items. A new scheme, extending the one of Trigeiro et al. (1989), is proposed to generate instances with initial and ending inventories. Numerical results on the new instances show the practical relevance of the new problem.

Although we believe inventory constraints at the end of the planning horizon are the most relevant and simple mechanism to avoid the end-of-horizon effect, other ideas could be investigated. For instance, minimum capacity consumption constraints could be added in the last periods, to ensure that the resources are fully used until the end of the planning horizon. However, this might lead to unnecessary setups. Another idea is to set, as proposed by Fisher et al. (2001), negative holding costs at the end of the last period to counterbalance the ZIO property. However, as we have shown, defining updated linear holding costs is not straightforward for capacitated lot-sizing problems. A third idea could be to relax the minimum and maximum ending inventory constraints and penalize their non-satisfaction in the objective function. Again, defining such penalties is not an easy task.

We are now working on solving large instances of the new CLSP with setup times and ending inventories. Parallelized relax-and-fix heuristics are being developed, together with a Lagrangian heuristic inspired by the one proposed in Trigeiro et al. (1989). As future research perspectives, we believe it is interesting to study how constraints on ending inventories impact various extensions of the CLSP with setup times, such as demand and production time windows or inventory capacity constraints. Another interesting research topic would be to pursue the work started at the end of Section 5.3, namely investigating approaches to determine relevant ending inventories from known initial inventories and probably also demand forecasts.

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Appendix: Study on lost sales costs

We want to study a special case of the cyclical problem defined in Section 3.1, where only a fraction of the demand can be satisfied in each period. Let us denote by $\gamma \in [0, 1]$ the fraction of the average demand that is satisfied in a cyclical production process, and by $\bar{l}$ the average lost sales cost over all items. We want to find sufficient conditions on $\bar{l}$ in order to get $\gamma^* = 1$, and thus an optimal solution where all demands are satisfied. We assume that $\frac{N\bar{h}}{2\bar{l}+\bar{h}} \leq \frac{c^{\max}}{\bar{s}}$ otherwise Formula (47) implies that it is optimal to lose all demands ($\gamma^* = 0$).

The mathematical formulation of the problem is given by:

$$\min_{(k,\phi,\gamma)\in\mathbb{R}^*\times{\mathbb{N}^*}^N\times[0,1]} \quad \bar{f}k + \bar{h}\gamma d\sum_{i=1}^{N}\frac{\phi_i - 1}{2} + N(1-\gamma)\bar{d}\bar{l} \tag{43}$$

$$s.t. \quad \sum_{i=1}^{N}\frac{1}{\phi_i} = k \tag{44}$$

$$k \leq \frac{c^{\max} - N\bar{b}\gamma\bar{d}}{\bar{s}} \tag{45}$$

which can be rewritten as:

$$N\bar{d}\bar{l} + \min_{k\in]0,\frac{c^{\max}}{\bar{s}}]}(\bar{f}k + \bar{d}\min_{\{\gamma\in[0,1]|k\leq\frac{c^{\max}-N\bar{b}\gamma\bar{d}}{\bar{s}}\}}\gamma[\bar{h}\min_{\{\phi\in{\mathbb{N}^*}^N|\sum_{i=1}^{N}\frac{1}{\phi_i}=k\}}(\sum_{i=1}^{N}\frac{\phi_i}{2}) - N\bar{l} - \frac{\bar{h}N}{2}]) \tag{46}$$

The continuous relaxation is equivalent to:

$$N\bar{d}\bar{l} + \min_{k\in]0,\frac{c^{\max}}{\bar{s}}]}(\bar{f}k + \frac{N\bar{d}}{2}\min_{\{\gamma\in[0,1]|\gamma\leq\frac{c^{\max}-k\bar{s}}{N\bar{b}\bar{d}}\}}\gamma[\bar{h}(\frac{N}{k}-1) - 2\bar{l}]) \tag{47}$$

If $\bar{h}(\frac{N}{k}-1) - 2\bar{l} \geq 0 \Leftrightarrow k \leq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$, then $\gamma^* = 0$. The term $\frac{N\bar{d}}{2}\bar{h}(\frac{N}{k}-1)$ represents the average inventory costs per period if all demands are satisfied while the term $N\bar{d}\bar{l}$ corresponds to the cost of losing all demands for a period. For a given cycle with k setups per period, when the average inventory costs are higher than the lost sales costs, it will always be better to lose all demands.

When $k \geq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$, the average inventory costs per period are smaller than the lost sales costs, we try to satisfy as much of the demands as possible. Otherwise $\gamma^* = \min(1, \frac{c^{\max}-k\bar{s}}{N\bar{b}\bar{d}})$.

Characterization of optimal solutions

The optimal solution of the problem in Formula (47) is denoted by F^*. We can divide the problem into three distinct cases depending on the definition domain of the average number of setups k:

- **Case 1.** $K_1 = \{k \in]0, \frac{c^{\max}}{\bar{s}}] | k \leq \frac{N\bar{h}}{2\bar{l}+\bar{h}}\}$.

  In that case we have $\gamma^* = 0$ as shown previously and it is optimal to lose all demands:

$$F^* = N\bar{d}\bar{l} + \min_{k\in K^1}\bar{f}k = N\bar{d}\bar{l}$$

- **Case 2.** $K_2 = \{k \in]0, \frac{c^{\max}}{\bar{s}}] | k \geq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and $k \leq \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}\}$.

  In that case we have $\frac{c^{\max}-k\bar{s}}{N\bar{b}\bar{d}} \geq 1$ and thus $\gamma^* = 1$. It is optimal to satisfy all demands:

$$F^* = N\bar{d}\bar{l} + \min_{k\in K_2}(\bar{f}k + \frac{N\bar{d}}{2}(\bar{h}(\frac{N}{k}-1) - 2\bar{l}))$$

$$F^* = -\frac{N\bar{d}\bar{h}}{2} + \min_{k\in K_2}(\bar{f}k + \frac{N^2\bar{d}\bar{h}}{2k}) \tag{48}$$

The minimum of the function in Formula (48) is either reached for $k^* = N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}}$ if $N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}}$ is in the definition domain (which would correspond to a solution guided only by the costs), or k^* corresponds to one of the domain bounds.

- **Case 2.1.** $\frac{N\bar{h}}{2\bar{l}+\bar{h}} \leq N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}} \leq \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$. We have $k^* = N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}}$ and then:

$$F^* = N(\sqrt{2\bar{f}\bar{d}\bar{h}} - \frac{\bar{d}\bar{h}}{2})$$

- **Case 2.2.** $N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}} \leq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$. F^* is increasing between $\frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$, so we have $k^* = \frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and:

$$F^* = N\bar{d}\bar{h}(\frac{\bar{f}}{\bar{d}(2\bar{l}+\bar{h})} + \frac{\bar{l}}{\bar{h}})$$

  It is worth noticing that this case corresponds to a case where the average inventory costs are equal to the lost sales costs, thus all values of γ^* are equivalent. This means that the solution consisting of satisfying all demands is equivalent to the solution considering all demands as lost sales.

- **Case 2.3.** $N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}} \geq \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$. F^* is decreasing between $\frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$, so we have $k^* = \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$ and then

$$F^* = \frac{N\bar{d}\bar{h}}{2}(\frac{N\bar{s}}{c^{\max} - N\bar{b}\bar{d}} - 1) + \frac{\bar{f}}{\bar{s}}(c^{\max} - N\bar{b}\bar{d})$$

- **Case 3.** $K_3 = \{k \in]0, \frac{c^{\max}}{\bar{s}}] | k \geq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and $k \geq \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$ and $k \leq \frac{c^{\max}}{\bar{s}}\}$.

In that case we have $\frac{c^{\max}-k\bar{s}}{N\bar{b}\bar{d}} \leq 1$ and $\gamma^* = \frac{c^{\max}-k\bar{s}}{N\bar{b}\bar{d}}$. It can be optimal to satisfy only a fraction of the demand.

$$F^* = N\bar{d}\bar{l} + \min_{k \in K_3}(\bar{f}k + \frac{(c^{\max} - k\bar{s})}{2\bar{b}}(\bar{h}(\frac{N}{k} - 1) - 2\bar{l}))$$

Let us define $\tilde{f} = \bar{f} + \frac{\bar{s}(2\bar{l}+\bar{h})}{2\bar{b}}$ and $\tilde{h} = \frac{c^{\max}}{N\bar{b}\bar{d}}\bar{h}$.

$$F^* = N\bar{d}\bar{l} - \frac{(2\bar{l}+\bar{h})c^{\max} + N\bar{h}\bar{s}}{2\bar{b}} + \min_{k \in K_3}(\tilde{f}k + \frac{N^2\bar{d}\tilde{h}}{2k}) \quad (49)$$

This corresponds to a new cyclical problem with updated setup and inventory costs. The minimum of the function defined in Formula (49) is either reached for $k^* = N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$ if this value is in the definition domain (which would correspond to a solution guided only by the updated costs), or k^* corresponds to one of the domain bounds.

- **Case 3.1.** $\frac{c^{max}}{\bar{s}} \leq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$:

  In this case the minimum is reached for $k^* = \frac{c^{max}}{\bar{s}}$, which corresponds to $\gamma^* = 0$ and none of the demand is satisfied.

$$F^* = N\bar{d}\bar{l} + \frac{\bar{f}}{\bar{s}}c^{max}$$

- **Case 3.2.** $\frac{c^{max}}{\bar{s}} \geq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$ and $\frac{N\bar{h}}{2\bar{l}+\bar{h}} \leq \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$:

  Then the lower bound of K_3 is $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$ and the minimum of the function in Formula (49) is reached in $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$ or in $N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$.

  * **Case 3.2.1.** $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} \leq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$, we have $k^* = N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$ and $\gamma^* \leq 1$. It can be optimal to satisfy only part of the demand. We have:

  $$F^* = N\bar{d}\bar{l} - \frac{(2\bar{l}+\bar{h})c^{\max} + N\bar{h}\bar{s}}{2\bar{b}} + N\sqrt{2\tilde{f}\tilde{h}\bar{d}}$$

  * **Case 3.2.2.** $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} \geq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$. We have $k^* = \frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}}$ and $\gamma^* = 1$. It is optimal to satisfy all demands. We have:

  $$F^* = \frac{N\bar{d}\tilde{h}}{2}\left(\frac{N\bar{s}}{c^{\max} - N\bar{b}\bar{d}} - 1\right) + \frac{\bar{f}}{\bar{s}}(c^{\max} - N\bar{b}\bar{d})$$

- **Case 3.3.** $\frac{c^{max}}{\bar{s}} \geq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$ and $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} \leq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$:

  * **Case 3.3.1.** $\frac{N\bar{h}}{2\bar{l}+\bar{h}} \leq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$, we have $k^* = N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$ and $\gamma^* \leq 1$. It can be optimal to satisfy only part of the demand. We have:

  $$F^* = N\bar{d}\bar{l} - \frac{(2\bar{l}+\bar{h})c^{\max} + N\bar{h}\bar{s}}{2\bar{b}} + N\sqrt{2\tilde{f}\tilde{h}\bar{d}}$$

  * **Case 3.3.2.** $\frac{N\bar{h}}{2\bar{l}+\bar{h}} \geq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$, we have $k^* = \frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and $\gamma^* \leq 1$ because $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} < \frac{N\bar{h}}{2\bar{l}+\bar{h}}$. It can be optimal to satisfy only part of the demand. We have:

  $$F^* = N(\bar{d}\bar{l} + \frac{\bar{f}\bar{h}}{2\bar{l}+\bar{h}})$$

Definition of relevant lost sales costs

The goal of this section is to propose sufficient conditions to define lost sales costs that will guarantee that the optimal solution does not have a fraction of demand unsatisfied for the problem defined in Formula (47). With regards to the different cases introduced previously, we want to consider only the cases where $\gamma^* = 1$ is the only optimal value for γ. This corresponds to Cases **2.1**, **2.3** and **3.2.2**. Sufficient conditions for the optimal cycle to have no lost sales can be set as follows:

1. In order to prevent Case **3.1**:

$$\frac{c^{max}}{\bar{s}} \geq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$$

2. In order to prevent Cases **3.1**, **3.3** and **3.2.1**, we want to define $\bar{l}$ such that $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} \geq N\sqrt{\frac{\bar{d}\tilde{h}}{2\tilde{f}}}$

and $\frac{c^{\max}-N\bar{b}\bar{d}}{\bar{s}} \geq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$. By setting:

$$\bar{l} \geq \frac{c^{\max}}{(c^{\max} - N\bar{b}\bar{d})} \frac{N\bar{s}\bar{h}}{2(c^{\max} - N\bar{b}\bar{\bar{d}})}$$

this condition on $\bar{l}$ ensures that both conditions are respected. This condition also implies that $\frac{c^{\max}}{\bar{s}} \geq N\sqrt{\frac{\bar{d}\tilde{\bar{h}}}{2\tilde{\bar{f}}}}$ and $\frac{c^{\max}}{\bar{s}} \geq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$.

3. In order to prevent Case **2.2** and to ensure that Case **2.1** is lower than Case **1**, we want to define $\bar{l}$ such that $N\sqrt{\frac{\bar{d}\bar{h}}{2\bar{f}}} \geq \frac{N\bar{h}}{2\bar{l}+\bar{h}}$ and $N(\sqrt{2\bar{f}\bar{d}\bar{h}} - \frac{\bar{d}\bar{h}}{2}) \leq N\bar{d}\bar{l}$. By setting:

$$\bar{l} \geq \sqrt{\frac{2\bar{f}\bar{h}}{\bar{d}}}$$

this condition on $\bar{l}$ ensures that both conditions are respected.

4. In order to ensure that Cases **2.3** and **3.2.2** are lower than Case **1**, we want to define $\bar{l}$ such that $\frac{N\bar{d}\bar{h}}{2}(\frac{N\bar{s}}{c^{\max}-N\bar{b}\bar{d}} - 1) + \frac{\bar{f}}{\bar{s}}(c^{\max} - N\bar{b}\bar{d}) \leq N\bar{d}\bar{l}$, which implies:

$$\bar{l} \geq \frac{\bar{h}}{2}(\frac{N\bar{s}}{c^{\max} - N\bar{b}\bar{d}} - 1) + \frac{\bar{f}}{Nd\bar{s}}(c^{\max} - N\bar{b}\bar{d})$$

It should be pointed out that because the goal was to find sufficient conditions for $\bar{l}$ that were easy to express, the focus for Points 2. and 3. was not to find the minimum $\bar{l}$ satisfying the required conditions but only a threshold that would guarantee that these conditions are respected.