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# THE ALGEBRA OF OBSERVABLES IN NONCOMMUTATIVE DEFORMATION THEORY

EIVIND ERIKSEN AND ARVID SIQVELAND

ABSTRACT. We consider the algebra  $\mathcal{O}(\mathsf{M})$  of observables and the (formally) versal morphism  $\eta: A \to \mathcal{O}(\mathsf{M})$  defined by the noncommutative deformation functor  $\mathsf{Def}_{\mathsf{M}}$  of a family  $\mathsf{M} = \{M_1, \ldots, M_r\}$  of right modules over an associative k-algebra A. By the Generalized Burnside Theorem, due to Laudal,  $\eta$  is an isomorphism when A is finite dimensional, M is the family of simple A-modules, and k is an algebraically closed field. The purpose of this paper is twofold: First, we prove a form of the Generalized Burnside Theorem that is more general, where there is no assumption on the field k. Secondly, we prove that the  $\mathcal{O}$ -construction is a closure operation when A is any finitely generated k-algebra and M is any family of finite dimensional A-modules, in the sense that  $\eta_B: B \to \mathcal{O}^B(\mathsf{M})$  is an isomorphism when  $B = \mathcal{O}(\mathsf{M})$  and M is considered as a family of B-modules.

# 1. INTRODUCTION

Let k be a field, let A be a finite dimensional associative algebra over k, and let  $M = \{M_1, \ldots, M_r\}$  be the family of simple right A-modules, up to isomorphism. We consider the algebra homomorphism

$$\rho: A \to \bigoplus_{i=1}^r \operatorname{End}_k(M_i)$$

given by right multiplication of A on the family M. By the extended version of the classical Burnside Theorem,  $\rho$  is surjective when k is algebraically closed, and if A is semisimple, then it is an isomorphism. We remark that Artin-Wedderburn theory gives a version of the theorem that holds over any field:

**Theorem** (Classical Burnside Theorem). Let A be a finite dimensional k-algebra, and let  $\{M_1, \ldots, M_r\}$  be the family of simple right A-modules. If  $\operatorname{End}_A(M_i) = k$ for  $1 \leq i \leq r$ , then  $\rho : A \to \bigoplus_i \operatorname{End}_k(M_i)$  is surjective.

In Laudal [3], a generalization called the Generalized Burnside Theorem was obtained. This is a structural result for not necessarily semisimple algebras, and the essential idea of Laudal was to replace  $\rho$  with the versal morphism  $\eta$  defined by noncommutative deformations of modules. Let us recall the construction:

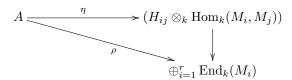
Let A be an arbitrary associative k-algebra, let  $\mathsf{M} = \{M_1, \ldots, M_r\}$  be a family of right A-modules, and consider the noncommutative deformation functor  $\mathsf{Def}_{\mathsf{M}}$ . This functor has a pro-representing hull H and a versal family  $M_H$  if M is a swarm. Following Laudal [3], we define the algebra of observables of a swarm M to be  $\mathcal{O}(\mathsf{M}) = \operatorname{End}_H(M_H) \cong (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$ , and its versal morphism to be the

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algebra homomorphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  given by right multiplication of A on the versal family  $M_H$ . It fits into the commutative diagram



where  $\rho: A \to \bigoplus_{i=1}^{r} \operatorname{End}_{k}(M_{i})$  is the algebra homomorphism given by right multiplication of A on the family M. By Theorem 1.2 in Laudal [3], it follows that  $\eta$  is an isomorphism when A is finite dimensional, M is the family of simple A-modules, and k is algebraically closed. In this paper, we prove a more general version of this result:

**Theorem** (Generalized Burnside Theorem). Let A be a finite dimensional k-algebra, and let M be the family of simple right A-modules, up to isomorphism. The versal morphism  $\eta: A \to \mathcal{O}(M)$  is injective. If  $\operatorname{End}_A(M_i) = k$  for  $1 \le i \le r$ , then  $\eta$  is an isomorphism. In particular,  $\eta$  is an isomorphism if k is algebraically closed.

In case  $D_i = \operatorname{End}_A(M_i)$  is a division algebra with  $\dim_k D_i > 1$  for some simple module  $M_i$ , it is often not difficult to describe the image of  $\eta$  as a subalgebra of  $\mathcal{O}(\mathsf{M})$ , and we shall give examples. As an application of the theorem, we introduce the standard form of any finite dimensional algbra A, given as

$$A \cong \mathcal{O}(\mathsf{M}) = (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$$

when  $\operatorname{End}_A(M_i) = k$  for  $1 \leq i \leq r$ , or as a subalgebra of  $\mathcal{O}(\mathsf{M})$  in general.

Let A be any finitely generated k-algebra and let M be any family of finite dimensional right A-modules. In this more general situation, the versal morphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  is not necessarily an isomorphism. However, we may consider the algebra  $B = \mathcal{O}(\mathsf{M})$  of observables, and M as a family of right B-modules, and iterate the process. We prove that the operation  $(A, \mathsf{M}) \mapsto (B, \mathsf{M})$  has the following *closure property*:

**Theorem** (Closure Property). Let A be a finitely generated k-algebra, let M be a family of finite dimensional A-modules, and let  $B = \mathcal{O}(M)$ . Then the versal morphism  $\eta^B : B \to \mathcal{O}^B(M)$  of M, considered as a family of right B-modules, is an isomorphism.

One may consider a noncommutative algebraic geometry where the closed points are represented by simple modules; see for instance Laudal [4]. With this point of view, one may use versal morphisms  $\eta : A \to \mathcal{O}(M)$  for families M of A-modules to construct noncommutative localization homomorphisms  $\eta_s : A \to A_s$  for any  $s \in A$ . We explain this construction in Section 6. These localization maps are universal S-inverting localization maps, where  $S = \{1, s, s^2, \ldots\}$ , and can be used as an essential building block for structure sheaves on noncommutative schemes.

### 2. Noncommutative deformations of modules

Let A be an associative algebra over a field k. For any right A-module M, there is a *deformation functor*  $\mathsf{Def}_M : \mathsf{I} \to \mathsf{Sets}$  defined on the category  $\mathsf{I}$  of commutative Artinian local k-algebras R with residue field k. We recall that  $\mathsf{Def}_M(R)$  is the set of equivalence classes of pairs  $(M_R, \tau_R)$ , where  $M_R$  is an R-flat R-A bimodule on which k acts centrally, and  $\tau_R : k \otimes_R M_R \to M$  is an isomorphism of right A-modules. Deformations in  $\mathsf{Def}_M(R)$  are called *commutative deformations* since the base ring R is commutative.

Noncommutative deformations were introduced in Laudal [3]. The deformations considered by Laudal are defined over certain noncommutative base rings instead of the commutative base rings in I. In what follows, we shall give a brief account of noncommutative deformations of modules. We refer to Laudal [3], Eriksen [2] and Eriksen, Laudal, Siqveland [1] for further details.

For any positive integer r and any family  $\mathsf{M} = \{M_1, \ldots, M_r\}$  of right A-modules, there is a noncommutative deformation functor  $\mathsf{Def}_{\mathsf{M}} : \mathsf{a}_r \to \mathsf{Sets}$ , defined on the category  $\mathsf{a}_r$  of noncommutative Artinian r-pointed k-algebras with exactly r simple modules (up to isomorphism). We recall that an r-pointed k-algebra R is one fitting into a diagram of rings  $k^r \to R \to k^r$ , where the composition is the identity. The condition that R has exactly r simple modules holds if and only if  $\overline{R} \cong k^r$ , where  $\overline{R} = R/J(R)$  and J(R) denotes the Jacobson radical of R.

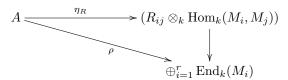
The noncommutative deformations in  $\mathsf{Def}_{\mathsf{M}}(R)$  are equivalence classes of pairs  $(M_R, \tau_R)$ , where  $M_R$  is an *R*-flat *R*-*A* bimodule on which *k* acts centrally, and  $\tau_R : k^r \otimes_R M_R \to M$  is an isomorphism of right *A*-modules with  $M = M_1 \oplus \cdots \oplus M_r$ . In concrete terms, an algebra *R* in  $\mathsf{a}_r$  is a matrix ring  $R = (R_{ij})$  with  $R_{ij} = e_i Re_j$ . By abuse of notation, we write  $e_i$  for the idempotent  $e_i = (0, 0, \ldots, i, \ldots, 0)$  in  $k^r$ , and also for its image in *R* via the structural map  $k^r \to R$ . As left *R*-modules, we have that  $M_R \cong (R_{ij} \otimes_k M_j)$  and its right *A*-module structure is given by an algebra homomorphism

$$\eta_R : A \to \operatorname{End}_R(M_R) \cong (R_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$$

that lifts  $\rho: A \to \bigoplus_i \operatorname{End}_k(M_i)$ . Explicitly, we interpret  $\eta_R(a)$  as a right action of a on  $M_R$  via

$$\eta_R(a) = \sum_i e_i \otimes \rho_i + \sum_{i,j,l} r_{ij}^l \otimes \phi_{ij}^l \quad \Longleftrightarrow \quad (e_i \otimes m_i)a = e_i \otimes (m_i a) + \sum_{j,l} r_{ij}^l \otimes \phi_{ij}^l(m_i)$$

where  $\rho_i : A \to \operatorname{End}_k(M_i)$  is the algebra homomorphism given by the right action of A on  $M_i$ , such that  $\rho = (\rho_1, \ldots, \rho_r)$ , and where  $r_{ij}^l \in R_{ij}$  and  $\phi_{ij}^l \in \operatorname{Hom}_k(M_i, M_j)$ . Deformations in  $\mathsf{Def}_{\mathsf{M}}(R)$  can therefore be represented by commutative diagrams



These deformations are called *noncommutative deformations* since the base ring R is noncommutative.

For any *r*-pointed algebra R, with structural maps  $k^r \to R \to k^r$ , we write  $I(R) = \ker(R \to k^r)$ . Recall that the pro-category  $\hat{a}_r$  is the full subcategory of the category of *r*-pointed algebras consisting of algebras R such that  $R/I(R)^n$  is Artinian for all n and such that R is complete in the I(R)-adic topology.

The family  $\mathsf{M} = \{M_1, \ldots, M_r\}$  is called a *swarm* if  $\dim_k \operatorname{Ext}_A^1(M, M)$  is finite. In this case, the noncommutative deformation functor  $\mathsf{Def}_{\mathsf{M}}$  has a pro-representing hull H in the pro-category  $\widehat{\mathsf{a}}_r$  and a versal family  $M_H \in \mathsf{Def}_{\mathsf{M}}(H)$ ; see Theorem 3.1 in Laudal [3]. The defining property of the miniversal pro-couple  $(H, M_H)$  is that the induced natural transformation

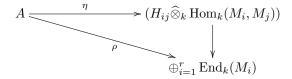
$$\phi : \operatorname{Mor}(H, -) \to \mathsf{Def}_{\mathsf{M}}$$

on  $\mathbf{a}_r$  is smooth (which implies that  $\phi_R$  is surjective for any R in  $\mathbf{a}_r$ ), and that  $\phi_R$  is an isomorphism when  $J(R)^2 = 0$ . The miniversal pro-couple  $(H, M_H)$  is unique up to (non-canonical) isomorphism.

Let M be a swarm of right A-modules, and let  $(H, M_H)$  be the miniversal procouple of the noncommutative deformation functor  $\mathsf{Def}_{\mathsf{M}} : \mathsf{a}_r \to \mathsf{Sets}$ . We define the algebra of observables of M to be

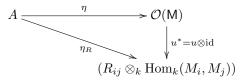
$$\mathcal{O}(\mathsf{M}) = \operatorname{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \operatorname{Hom}_k(M_i, M_j))$$

where  $\widehat{\otimes}$  is the completed tensor product (the completion of the tensor product), and write  $\eta : A \to \mathcal{O}(\mathsf{M})$  for the induced *versal morphism*, giving the right Amodule structure on  $M_H$ . By construction, it fits into the commutative diagram



**Remark 1.** Notice that the diagram extends the right action of A on the family M to a right action of  $\mathcal{O}(M)$ , such that M is a family of right  $\mathcal{O}(M)$ -modules.

**Remark 2.** For any R in  $a_r$  and any deformation  $M_R \in \mathsf{Def}_{\mathsf{M}}(R)$ , there is a morphism  $u: H \to R$  in  $\widehat{a}_r$  such that  $\mathsf{Def}_{\mathsf{M}}(u)(M_H) = M_R$  by the versal property, and the deformation  $M_R$  is therefore given by the composition  $\eta_R = u^* \circ \eta$  in the diagram



In this sense, the versal morphism  $\eta : A \to \mathcal{O}(M)$  determines all noncommutative deformations of the family M.

### 3. Iterated extensions and injectivity of the versal morphism

Let E be a right A-module and let  $r \ge 1$  be a positive integer. If E has a *cofiltration* of length r, given by a sequence

$$E = E_r \xrightarrow{f_r} E_{r-1} \to \dots \to E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 = 0$$

of surjective right A-module homomorphisms  $f_i : E_i \to E_{i-1}$ , then we call E an *iterated extension* of the right A-modules  $M_1, M_2, \ldots, M_r$ , where  $M_i = \ker(f_i)$ . In fact, the cofiltration induces short exact sequences

$$0 \to M_i \to E_i \xrightarrow{J_i} E_{i-1} \to 0$$

for  $1 \leq i \leq r$ . Hence  $E_1 \cong M_1$ ,  $E_2$  is an extension of  $E_1$  by  $M_2$ , and in general,  $E_i$  is an extension of  $E_{i-1}$  by  $M_i$ .

Let  $\mathsf{M} = \{M_1, \ldots, M_r\}$  be a swarm of right A-modules, and let  $\mathsf{Def}_\mathsf{M} : \mathsf{a}_r \to \mathsf{Sets}$  be its noncommutative deformation functor. Then  $\mathsf{Def}_\mathsf{M}$  has a miniversal procouple  $(H, M_H)$ , and we consider the induced versal morphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  and its kernel  $K = \ker(\eta)$ .

We note that Theorem 3.2 in Laudal [3] holds without assumptions on the base field k, since the construction that precedes this theorem works over any field. From this observation, we obtain the following lemma:

**Lemma 3.** Let M be a swarm of right A-modules. For any iterated extension E of the family M, we have that  $E \cdot K = 0$ .

Let A be a finite dimensional k-algebra and let M be the family of all simple right A-modules, up to ismorphism. Then M is a swarm, and we may consider the versal morphism  $\eta : A \to \mathcal{O}(M)$ . If k is algebraically closed, then the versal morphism  $\eta$  is injective by Corollary 3.1 in Laudal [3]. Using Lemma 3, we generalize this result:

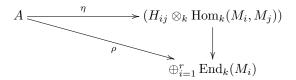
**Proposition 4.** If A, considered as a right A-module, is an iterated extension of a swarm M, then the versal morphism  $\eta : A \to \mathcal{O}(M)$  is injective. In particular,  $\eta$  is injective when A is a finite dimensional algebra and M is the family of simple right A-modules.

*Proof.* If A is an iterated extension of M, then  $1 \cdot K = 0$  by Lemma 3, and this implies that K = 0. If A is finite dimensional, then the right A-module A has finite length, and it is an iterated extension of the simple modules.

We remark that our proof, based on Lemma 3, holds whenever there is an element  $e \in E$  such that  $a \mapsto e \cdot a$  defines an injective right A-module homomorphism  $A \to E$ . This means that  $\eta : A \to \mathcal{O}(\mathsf{M})$  is injective if there is an iterated extension E of  $\mathsf{M}$  such that E contains a copy of  $A_A$ .

### 4. The Generalized Burnside Theorem

Let A be a finite dimensional k-algebra, and let  $\mathsf{M} = \{M_1, \ldots, M_r\}$  be the family of simple right A-modules, up to isomorphism. Then M is a swarm, and we consider the versal morphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  and the commutative diagram



Clearly,  $\rho$  factors through A/J(A), and if  $\operatorname{End}_A(M_i) = k$  for  $1 \leq i \leq r$ , then  $A/J(A) \to \bigoplus_i \operatorname{End}_k(M_i)$  is an isomorphism by the Artin-Wedderburn theory for semisimple algebras. This proves the Classical Burnside Theorem mentioned in the introduction. By Theorem 3.4 in Laudal [3], the versal morphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  is an isomorphism when k is algebraically closed. We generalize this result:

**Theorem 5.** Let A be a finite dimensional k-algebra and let M be the family of simple right A-modules, up to isomorphism. Then  $\eta : A \to \mathcal{O}(\mathsf{M})$  is injective, and it is an isomorphism if  $\operatorname{End}_A(M_i) = k$  for  $1 \leq i \leq r$ . In particular, the versal morphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  is an isomorphism if k is algebraically closed.

Proof. By Proposition 4, the versal morphism  $\eta$  is injective, and it is enough to prove that  $\eta$  is surjective when  $\operatorname{End}_A(M_i) = k$  for  $1 \leq i \leq r$ . Note that  $\eta$  maps the Jacobson radical J(A) of A to the Jacobson radical  $J = (J(H)_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$ of  $\mathcal{O}(\mathsf{M})$ . Moreover, A is J(A)-adic complete since it is finite dimensional, and  $\mathcal{O}(\mathsf{M})$  is clearly J-adic complete. By a standard result for filtered algebras, it is therefore sufficient to show that  $\operatorname{gr}_1(\eta) : J(A)/J(A)^2 \to J/J^2$  is surjective, since  $\operatorname{gr}_0(\eta) : A/J(A) \to \bigoplus_i \operatorname{End}_k(M_i)$  is an isomorphism by the Classical Burnside Theorem. We notice that

$$J/J^2 \cong ((J(H)/J(H)^2)_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)) \cong (\operatorname{Ext}^1_A(M_i, M_j)^* \otimes_k \operatorname{Hom}_k(M_i, M_j))$$

since  $J(H)/J(H)^2$  is the dual of the tangent space  $(\text{Ext}_A^1(M_i, M_j))$  of  $\mathsf{Def}_M$ . We note that Lemma 3.7 in Laudal [3] holds over any field. Hence the map

$$J(A)/J(A)^2 \to (\operatorname{Ext}^1_A(M_i, M_j)^* \otimes_k \operatorname{Hom}_k(M_i, M_j))$$

induced by  $\eta$  is an isomorphism, and this completes the proof.

# 5. The closure property

Let A be a finitely generated k-algebra of the form  $A = k \langle x_1, \ldots, x_d \rangle / I$ , and let  $\mathsf{M} = \{M_1, \ldots, M_r\}$  be a family of finite dimensional right A-modules. Then M is a swarm, since

 $\dim_k \operatorname{Ext}^1_A(M_i, M_j) \le \dim_k \operatorname{Der}_k(A, \operatorname{Hom}_k(M_i, M_j)) \le \dim_k \operatorname{Hom}_k(M_i, M_j)^d$ 

The last inequality follows from the fact that any derivation  $D: A \to \operatorname{Hom}_k(M_i, M_j)$ is determined by  $D(x_l) \in \operatorname{Hom}_k(M_i, M_j)$  for  $1 \leq l \leq d$ . We consider the algebra of observables  $B = \mathcal{O}(\mathsf{M})$  of the swarm  $\mathsf{M}$ , and write  $\eta : A \to B$  for its versal morphism. In general,  $\mathsf{M} = \{M_1, \ldots, M_r\}$  is a family of right *B*-modules via  $\eta$ .

**Lemma 6.** The family  $M = \{M_1, \ldots, M_r\}$  of right B-modules is the simple right B-modules, and it is swarm of B-modules.

*Proof.* It follows from the Artin-Wedderburn theory that  $M = \{M_1, \ldots, M_r\}$  is the family of simple modules over

$$\overline{B} = B/J(B) \cong (H/J(H) \otimes_k \operatorname{Hom}_k(M_i, M_j)) \cong \bigoplus_i \operatorname{End}_k(M_i).$$

Since B and  $\overline{B} = B/J(B)$  have the same simple modules, it follows that M is the family of simple right B-modules. We have that  $\operatorname{Ext}_B^1(M_i, M_j)$  is a quotient of  $\operatorname{Der}_k(B, \operatorname{Hom}_k(M_i, M_j))$ , and any derivation  $D: B \to \operatorname{Hom}_k(M_i, M_j)$  satisfies  $D(J^2) = JD(J) + D(J)J = 0$  when J = J(B) since M is the family of simple B-modules. From the fact that

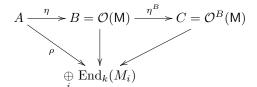
$$B/J^2 \cong ((H/\operatorname{J}(H)^2)_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$$

is finite dimensional, and in particular a finitely generated k-algebra, it follows from the argument preceding the lemma that M is a swarm of B-modules.

In this situation, we may iterate the process. Since M is a swarm of right *B*-modules, the noncommutative deformation functor  $\mathsf{Def}^B_{\mathsf{M}}$  of M, considered as a family of right *B*-modules, has a miniversal pro-couple  $(H^B, M^B_H)$ . We write  $\mathcal{O}^B(\mathsf{M}) = \operatorname{End}_{H^B}(M^B_H) \cong (H^B_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$  for its algebra of observables and  $\eta^B : B \to \mathcal{O}^B(\mathsf{M})$  for its versal morphism.

**Theorem 7.** Let A be a finitely generated k-algebra, let  $\mathsf{M} = \{M_1, \ldots, M_r\}$  be a family of finite dimensional A-modules, and let  $B = \mathcal{O}(\mathsf{M})$ . Then the versal morphism  $\eta^B : B \to \mathcal{O}^B(\mathsf{M})$  of  $\mathsf{M}$ , considered as a family of right B-modules, is an isomorphism.

*Proof.* Since M is a swarm of A-modules and of B-modules, we may consider the commutative diagram



The algebra homomorphism  $\eta^B$  induces maps  $B/J(B)^n \to C/J(C)^n$  for all  $n \ge 1$ , and it is enough to show that each of these induced maps is an isomorphism. For n = 1, we have

$$B/J(B) \cong C/J(C) \cong \bigoplus \operatorname{End}_k(M_i)$$

so it is clearly an isomorphism for n = 1. For  $n \ge 2$ , we have that  $B_n = B/J(B)^n$  is a finite dimensional algebra with the same simple modules as B since  $M_i J^n = 0$ . We may therefore consider the versal morphism of the swarm M of right  $B_n$ -modules, which is an isomorphism by the Generalized Burnside Theorem since  $\operatorname{End}_B(M_i) = k$ for  $1 \le i \le r$ . Finally, any derivation  $D: B \to \operatorname{Hom}_k(M_i, M_j)$  satisfies  $D(J^n) = 0$ when  $n \ge 2$ . Therefore, we have that

$$\operatorname{Ext}_{B_n}^1(M_i, M_j) \cong \operatorname{Ext}_B^1(M_i, M_j)$$

and this implies that  $B/J(B)^n \to C/J(C)^n$  coincides with the versal morphism of the swarm M of right  $B_n$ -modules. It is therefore an isomorphism.

Theorem 7 implies that the assignment  $(A, \mathsf{M}) \mapsto (B, \mathsf{M})$  is a closure operation when A is a finitely generated k-algebra and  $\mathsf{M} = \{M_1, \ldots, M_r\}$  is a family of finite dimensional right A-modules. In other words, the algebra  $B = \mathcal{O}(\mathsf{M})$  has the following properties:

- (1) The family M is the family of simple right *B*-modules.
- (2) The family M has exactly the same module-theoretic properties, in terms of extensions and matric Massey products, considered as a family of *B*-modules and as a family of *A*-modules.

Moreover, these properties characterize the algebra of observables  $B = \mathcal{O}(M)$ .

**Remark 8.** Assume that k is a field that is not algebraically closed. When A is a finite dimensional k-algebra and M is the family of simple right A-modules, it could happen that the division algebra  $D_i = \operatorname{End}_A(M_i)$  has dimension  $\dim_k D_i > 1$ for some simple A-modules  $M_i$ . In this case,  $\eta : A \to \mathcal{O}(M)$  is not necessarily an isomorphism. However, if the subfamily  $M' = \{M_i : \operatorname{End}_A(M_i) = k\} \subseteq M$  is non-empty, we may consider the algebra  $B = \mathcal{O}(M')$ , and it follows from the closure property that  $\eta : B \to \mathcal{O}^B(M')$  is an isomorphism. This means that the Generalized Burnside Theorem holds for the family M' of right B-modules.

### 6. Noncommutative localizations via the algebra of observables

Let A be a finitely generated k-algebra, and denote by X = Simp(A) the set of (isomorphism classes of) simple finite dimensional right A-modules. For any  $s \in A$ , we write

$$D(s) = \{ M \in X : M \xrightarrow{\cdot s} M \text{ is invertible} \} \subseteq X.$$

We note that  $\{D(s)\}_{s \in A}$  is a base for a topology on X, since  $D(s) \cap D(t) = D(st)$ , which we call the *Jacobson topology* on X = Simp(A).

For any inclusion  $\mathsf{M} \subseteq \mathsf{M}'$  of finite subsets of D(s), there is a surjective algebra homomorphism  $\mathcal{O}(\mathsf{M}') \to \mathcal{O}(\mathsf{M})$ . We may consider the algebra homomorphism

$$\eta_s: A \to \varprojlim_{\mathsf{M} \subseteq D(s)} \mathcal{O}(\mathsf{M})$$

where the projective limit is taken over all finite subsets  $M \subseteq D(s)$ . Notice that  $\eta_s(s)$  is a unit, since it is a unit in  $\mathcal{O}(M)$  for any finite subset  $M \subseteq D(s)$ . We define  $A_s$  to be the subring of the projective limit

$$\varprojlim_{\mathsf{M}\subseteq D(s)} \mathcal{O}(\mathsf{M})$$

generated by  $\eta_s(A)$  and  $\eta_s(s)^{-1}$ . By abuse of notation, we write  $\eta_s$  for the algebra homomorphism  $\eta_s: A \to A_s$  into the subring  $A_s$ .

Let S be the multiplicative subset  $S = \{1, s, s^2, ...\} \subseteq A$ . Then  $\eta_s : A \to A_s$  is an S-inverting algebra homomorphism, and it has the following universal property: If  $\phi : A \to B$  is any S-inverting algebra homomorphism, then there is a unique algebra homomorphism  $\phi_s : A_s \to B$  such that  $\phi_s \circ \eta_s = \phi$ . We remark that  $A_s$  is a finitely generated k-algebra, generated by the images of the generators of A and  $\eta_s(s)^{-1}$ . In general, it is not a (left or right) ring of fractions.

## 7. Applications

Let A be a finite dimensional k-algebra. We consider the family  $M = \{M_1, \ldots, M_r\}$  of simple right A-modules. By the Generalized Burnside Theorem, A can be written in *standard form* as

$$A \cong \operatorname{im}(\eta) \subseteq (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)) = \mathcal{O}(\mathsf{M})$$

If  $\operatorname{End}_A(M_i) = k$  for  $1 \leq i \leq r$ , then the standard form of A is  $A \cong \mathcal{O}(M)$ , and in general, it is a subalgebra of  $\mathcal{O}(M)$ .

The standard form can, for instance, be used to compare finite dimensional algebras and determine when they are isomorphic. Let us illustrate this with a simple example. Let k be a field, and let A = k[G] be the group algebra of  $G = \mathbb{Z}_3$ . In concrete terms, we have that  $A \cong k[x]/(x^3-1)$ , and over a fixed algebraic closure  $\overline{k}$  of k, we have that

$$x^{3} - 1 = (x - 1)(x^{2} + x + 1) = (x - 1)(x - \omega)(x - \omega^{2})$$

with  $\omega \in \overline{k}$ . If char $(k) \neq 3$  and  $\omega \in k$ , then the simple A-modules are given by  $\mathsf{M} = \{M_0, M_1, M_2\}$ , where  $M_i = A/(x - \omega^i)$ . Furthermore, a calculation shows that  $\operatorname{Ext}_A^1(M_i, M_j) = 0$  for  $0 \leq i, j \leq 2$ . Hence, the noncommutative deformation functor  $\mathsf{Def}_{\mathsf{M}}$  has a pro-representing hull  $H = k^3$  (it is rigid), and the versal morphism  $\eta : A \to \mathcal{O}(\mathsf{M})$  is an isomorphism. The standard form of A is therefore given

by

$$A = k[\mathbb{Z}_3] \cong k^3 = \begin{pmatrix} k & 0 & 0\\ 0 & k & 0\\ 0 & 0 & k \end{pmatrix}.$$

If char(k) = 3, then  $M_0$  is the only simple A-module since  $x^3 - 1 = (x - 1)^3$ , and we find that  $\operatorname{Ext}_A^1(M_0, M_0) = k$ . In this case, it turns out that  $H \cong k[[t]]/(t^3)$ , and the standard form of A is given by  $A = k[\mathbb{Z}_3] \cong k[t]/(t^3)$ . In both cases, it follows from the Generalized Burnside Theorem that  $\eta$  is an isomorphism, since  $\operatorname{End}_A(M) = k$  for all the simple A-modules M.

If  $\operatorname{char}(k) \neq 3$  and  $\omega \notin k$ , then the simple A-modules are given by  $\mathsf{M} = \{M, N\}$ , where  $M = M_0 = A/(x-1)$  is 1-dimensional, and  $N = A/(x^2 + x + 1) \cong k(\omega) = K$ is 2-dimensional. In this case, we have that  $\operatorname{End}_A(M) = k$  and  $\operatorname{End}_A(N) = K$ , and we find that the standard form of A is given by

$$H = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad \Rightarrow \quad A \cong \operatorname{im}(\eta) = \begin{pmatrix} k & 0 \\ 0 & K \end{pmatrix} \subseteq \mathcal{O}(\mathsf{M}) = \begin{pmatrix} k & 0 \\ 0 & \operatorname{End}_k(K) \end{pmatrix}.$$

It follows from Proposition 4 that  $\eta : A \to \mathcal{O}(\mathsf{M})$  is injective. However, it is not an isomorphism in this case.

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BI NORWEGIAN BUSINESS SCHOOL, DEPARTMENT OF ECONOMICS, N-0442 OSLO, NORWAY *Email address:* eivind.eriksen@bi.no

UNIVERSITY OF SOUTH-EASTERN NORWAY, FACULTY OF TECHNOLOGY, NATURAL SCIENCES AND MARITIME SCIENCES, N-3603 KONGSBERG, NORWAY Email address: arvid.siqveland@usn.no