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How bond risk affects risk parity portfolios

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Master Thesis

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ABSTRACT

We document the performance of risk parity portfolios (RPP) of U.S. equities and government bonds over more than fifty years of daily and monthly data. RPP's strong outperformance compared to 60/40 portfolios has to some extent relied on sub-periods of falling interests rates. RPP have a large tail risk materializing in periods of sharply rising interest rates together with recession or stagflation. In these situations, positive return correlation together with rising rates have a very negative impact on both RPP and 60/40 portfolios, with RPP suffering the larger tail loss. We also analyze how volatility variation of equities and bonds affect RPP's volatility theoretically and empirically.

Keywords: Risk parity portfolio, Government bond, Government bond yield, volatility variation

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1 Introduction

Traditional strategic asset allocation theory is based on mean-variance portfolio optimization framework put forward by Markowitz (1952). The Markowitz methodology, however, is difficult to implement due to measurement error involve estimating expected returns(Merton (1980)). In practice, institutions largely take on 50/50 or 60/40 equity/bond allocation portfolio. However, these portfolios empirically have under-diversified in its risk exposure as its variance is dominated by equity, and most of its returns are earned from equity risk. Alternatively, institutional investors apply in risk parity portfolio (RPP) or equally weighted risk contribution (ERC) portfolio in recent years, which is presumed well-diversified and do not rely on market timing. Large funds such as Black Rock and AQR introduced the strategy into their practical investment baskets. Hurst et al. (2010) showed that the risk parity strategy has shown more consistent long-term performance than traditional 60/40 portfolios in the period from January 1971 to December 2009. However, a long-lasting debate pertains to the robustness of RPP's performance and application of leverage. Inker (2010) questioned whether some asset classes like commodities and T-bills included in risk parity portfolios have risk premiums and whether the portfolio is dependent on falling bond yields. Ray Dalio (Burton and Schatzker (2020)) said investors would be 'crazy' to hold government bonds during the coronavirus pandemic and the next few years since the bond almost gives no interest rate or even negative interest rate when inflation rates are higher than the 10-year or 30-year government bond yields or yields are negative. In this case, RPP may suffer great loss and seems 'strange' to leverage on zero or negative return asset government bond since it puts large amount of money into this kind of asset. Although widely accepted as safe assets, government bonds have the probability of default, and this situation usually happens in countries that have low government bond ratings like Greece, Argentina and Brazil, investing in RPP in these countries may result in great loss. To be clear, investors usually do not consider government bond default risk when investing in developed countries like US or Europe. Besides, there are also alternative criticisms about risk parity portfolios' usage of leverage, introducing new problems like financing costs and liquidity risk in recession period.

This paper starts by describing daily S&P 500 and U.S. long term government bond return data from 1962 to 2020, specifically, we construct long-term government bond daily return index using one-bond portfolios and changing bond held every few calendar years. Using only two assets allows us to expand our datasets to as early as 1962. We then compute rolling correlation between two data sets and document periods of positive and negative relations. Next, we construct unlevered RPP using the two data sets, the formulation of the portfolio is motivated by Maillard et al. (2010). In the formulation, we simply approximate the forecasted volatility by realized volatility, which is summation of high frequency intra-period squared returns. After deciding asset allocation in unlevered RPP, we then apply leverage by targeting RPP's volatility with 60/40 portfolio's full sample average realized volatility. Leverage allows a risk parity investor to build a higher return for the risk taken portfolio, and it is one of the crucial ingredients of risk parity portfolio announced by Asness (2014). After getting historical performances of both RPP and 60/40 portfolio, we then analyze them by splitting sample into different economic situations and compare them to examine factors driving the performance of the RPP strategy and difference between them, specifically, we divide results into several pairs of periods: increasing interest rate periods and falling interest rate periods, positive asset correlation periods and negative asset correlation periods. In order to have a full picture of how RPP performs before 1962 when yield experience both increasing and falling periods, we also use monthly data of the two assets from 1926 to 1962 to construct RPP, and we document similar results in both daily and monthly data. Last, but not least, we analyze how volatility variation (or the change between forecasted and actual volatility) of equities and bonds affect RPP's volatility theoretically and empirically, the formulation of theoretical part is motivated by Ruban and Melas (2011). In the theoretical part, we set up a model to denote the volatility of RPP into 5 factors. In the empirical part, we show the factors in our models and their effects on volatility of RPP from 1962 to 2020.

2 Literature review

Risk parity (or ERC) approach is based on the study of Neukirch (2008), Maillard et al. (2010), who defined the global theoretical issues linked to risk parity. According to its definition, risk parity approach is similar to creating a minimum-variance portfolio subject to the constraint that each asset in portfolio contributes equally to the portfolio overall volatility. Risk parity portfolio is an attractive alternative to minimum variance and equally weighted portfolios, and it not only takes diversification into consideration, but also incorporates single and joint risk contributions of the assets. In two-asset cases, the solution for RPP asset allocation is unique and do not depend on the correlation between assets. Volatility forecasting is an essential part in deciding the weights, Andersen and Bollerslev (1998) documented that realized volatility provides a relatively accurate measure of volatility. French et al. (1987) and Schwert (1989) rely on the sum of daily squared returns in their construction and modelling of monthly U.S. equity volatilities. Comparing to other volatility forecasting models like GARCH, realized volatility approach can be easily implemented by an investor in real time and does not rely on any parameter estimation.

Markowitz (1952) introduced Modern portfolio theory (MPT), which is a mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk. Under the mathematical model, portfolio volatility is a function of the correlations of the component assets, for all asset pairs. So after getting the weights of two assets and correlation between them, we can easily get the portfolio volatility and apply leverage by targeting volatility.

Empirical applications of RPP from Hurst et al. (2010) showed that RPP has overall superior performance than 60/40 portfolio from 1971 to 2009 except in the period of surprise hike of the Fed Funds rate, importantly, RPP worked well throughout several crisis episodes, including the Great Depression, the Great Recession, and 1987 stock market crash. Practical implementation of this strategy from Bridgewater All Weather Fund and AQR also documented higher realized Sharpe ratio than traditional 60/40 portfolio from 1996 to 2017 (Wealthfront Advisers (n.d.)). Most risk parity funds experience this good time for the RPP, but it's too early to conclude that RPP dominates 60/40 portfolio in any time. We will extend our sample period to a further past and examine the robustness and characteristics of this strategy.

3 Data

We use daily S&P 500 and U.S. long term government bond return (LTR) data to construct our risk parity portfolio. Daily S&P 500 is from Bloomberg, and CRSP TREASURIES database from WRDS[1] provides the daily time series of the total return of treasuries[2] for each U.S. treasury since 1961. Ibbotson et al. (2016) described how to generate monthly total return long-term government bond, we use the same method and choose treasuries that have term to maturity of approximately 20 years and a reasonably current coupon for each calendar year to generate daily LTR from 1961 to 2020. The treasuries used to construct the LTR are shown in appendix A. Risk free rate is one-month treasury bill rate from Ibbotson Associates and is obtained from Fama French data library. Besides, we also use monthly data from 1926 to 1962 to expand our sample, and monthly S&P 500, long term rate of returns (LTR) data and long-term yield(lty) are from Amit Goyal's website[3]. Monthly S&P 500 is Center for Research in Security Press (CRSP) month-end values, which are continuously compounded returns on the S&P 500 index including dividends. LTR are total return of long-term government bond from Ibbotson's Stocks, Bonds, Bills and Inflation Yearbook.

We begin with the plot of time series of daily S&P 500 and LTR cumulative log return and excess log return data (see figure 1) and 252-day rolling correlation (see figure 2). After excluding days used for counting correlation and realized volatility, the sample period is from 1962-06-19 to 2020-03-31. Daily log return of S&P 500 and LTR demonstrate distinct dynamic patterns, which is evidenced by 252-day rolling correlation, which shows that S&P 500 and LTR are positively correlated for most of the time,

[1] Wharton Research Data Services

[2] The return in the flat price of bond, where flat price is the bond price plus the accrued coupon

[3] `http://www.hec.unil.ch/agoyal/`

	Ex. return	Std.	SR	VaR	ES	MDD
LTR	0.0275	0.1016	0.2705	-0.0097	-0.0147	-25.52%
S&P 500	0.0108	0.1634	0.0658	-0.0153	-0.0242	-59.99%

Table 1: Performance of S&P 500 and LTR (daily data)

The table shows historical performances of S&P 500 and LTR from 1962.06 to 2020.03. All indicators are calculated from full sample. Ex. return is annualized excess log return, Std. is annualized standard deviation of excess log return, SR is annualized Sharpe ratio obtained by dividing excess log return by its standard deviation. 5% VaR and ES are one day value of log return. MDD is maximum drawdown.

and they are negatively correlated during the period 1962.06-1967.10, 1998.06-1999.09, 2000.10-2006.06 and 2007.04-2020.03. Cumulative return of LTR sees constant growth since 1980, however, S&P 500 experience large drawdowns, in the end, LTR realizes a larger final value, specifically, if we invest $1 from the beginning the S&P 500 realizes around $25.08 at the end of the sample versus about $65.57 for the LTR, showing that the total return of bond is about 2.61 times that of the equity. In addition, figure 1 shows that S&P 500 is more volatile than LTR, with several large drawdowns. Throughout the sample period, LTR realizes higher Sharpe ratio since it has larger cumulative excess log return and lower volatility as shown in table 1. It's worth noting that before 1980, both S&P 500 and LTR underperform the risk-free rate and the former is worse as shown in figure1.b. Using full sample, RPP that use leverage to enhance the return contribution of safer assets, eg. government bonds, in portfolio will certainly realize a considerable Sharpe ratio and return. We will prove this result in following portfolio construction part.

Figure 1: Cumulative (excess) log return of S&P 500 and LTR (daily data)

The figure shows the daily cumulative log return and excess log return of S&P 500 and LTR from 1962.06.19 to 2020.03.31, rb represents long term government bond log return, rs represents S&P 500 log return, excess_rb is rb minus log risk free rate, excess_rs is rs minus log risk free rate.

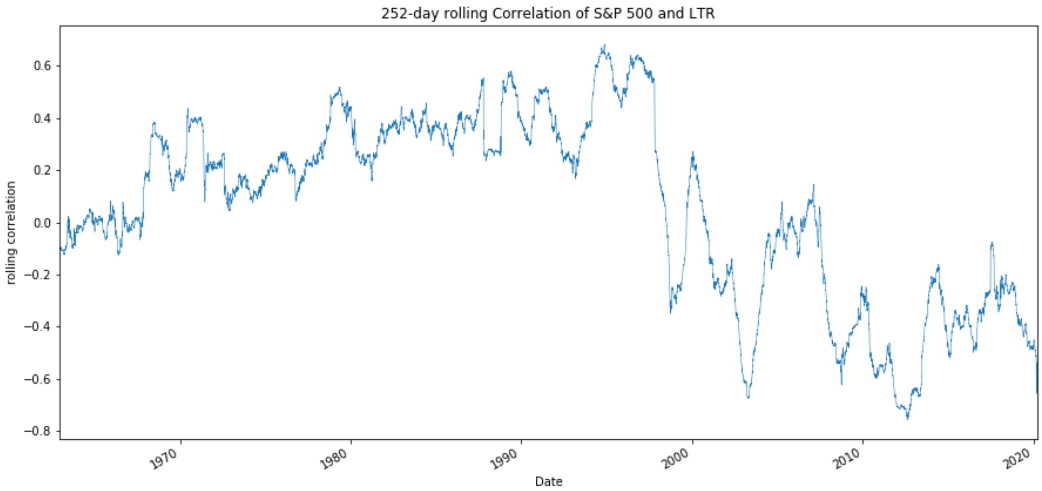

Figure 2: 252-day rolling correlation of daily log return of S&P 500 and LTR

The figure shows the 252-day rolling correlation of daily log return of S&P 500 and LTR from 1962.06.19 to 2020.03.31.

4 Portfolio construction

Consider a portfolio of n assets, $w = (w_1, w_2, \dots, w_n)$ is the weight of assets, σ_i^2 is the variance of asset i, $\sigma_{i,j}$ is the covariance between asset i and j, $\rho_{i,j}$ is the correlation between asset i and j, Σ is the variance covariance matrix of all assets, the volatility of the portfolio σ considered as the risk of the portfolio will be:

$$\sigma = \sqrt{w^T \Sigma \mathrm{w}} = \sqrt{\sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_{i,j}} \tag{1}$$

The marginal risk contributions will be the change in the total risk of the portfolio induced by a change in holdings of asset i, in other words, it is the partial differential of total portfolio risk with respect to the allocation in asset i, that is $\partial_{w_i} \sigma$:

$$\partial_{w_i} \sigma = \frac{\partial \sigma}{\partial w_i} = \frac{w_i \sigma_i^2 + \sum_{j \neq i} w_j \sigma_{i,j}}{\sigma} \tag{2}$$

Sum of risk contribution of asset i ($w_i \times \partial_{w_i} \sigma$) will be equal to the total risk of the portfolio:

$$\sigma = \sum_{i=1}^{n} w_i \times \partial_{w_i} \sigma \tag{3}$$

According to risk parity (or ERC) portfolio's inherent property that risk contributions for all assets in the portfolio are the same (Maillard et al. (2010)), we obtain the following equations:

$$\sum w_i = 1 \tag{4}$$

$$w_i \times \partial_{w_i} \sigma = w_j \times \partial_{w_j} \sigma \text{ for all } i, j \tag{5}$$

Formula (5) can also be rewritten into:

$$w_i^2 \sigma_i^2 + \sum_{k \neq i} w_i w_k \sigma_{i,k} = w_j^2 \sigma_j^2 + \sum_{k \neq j} w_j w_k \sigma_{j,k} \text{ for all } i, j \tag{6}$$

In our two-asset case of $n = 2$, equation (4) and (6) will be:

$$w_1 + w_2 = 1 \tag{7}$$

$$w_1^2\sigma_1^2 + w_1 w_2 \sigma_{1,2} = w_2^2\sigma_2^2 + w_1 w_2 \sigma_{1,2} \tag{8}$$

Solve equation (7) and (8), we obtain weights of two assets:

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \tag{9}$$

$$w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2} \tag{10}$$

After deciding on the asset allocation in the RPP, we next set the leverage to scale the return of portfolio. We choose to target the volatility as most risk parity funds did, that is, constrain the risk parity portfolio to have the target volatility. Most risk parity funds like AQR and Blackstone match volatility because it's good for implementing risk management and reducing tail risk. Here we assume a target annualized volatility of σ_{target}.

We multiply the unlevered RPP's volatility σ with a constant so that the portfolio has the same annualized volatility as target one, and we get:

$$c = \frac{\sigma_{target}}{\sigma} \tag{11}$$

where c is the leverage and σ_{target} could be the realized volatility of another benchmark portfolio or investor's risk tolerance.

According to Ang et al. (2011), we could get the levels of implied leverage able to be obtained in each asset from the typical margin requirements by brokers or other counterparties. For treasuries, the level of leverage is more than 33, and for equities the number is more than 2. They also documented that the empirical average gross leverage[4] across all hedge funds is 2.1. For

[4]for both long and short positions

those funds with less equities, the leverage could be as high as 4.8. In our thesis, the RPP's leverage[5] is restricted into a rational range. Besides, we assume that the leverage is costless in our analysis which will simply our model and is reasonable because leverage cost is not high for portfolios that put most money on treasuries.

Motivated by research of Bollerslev et al. (2018) and Schwert (1989), we use assets' realized variance as a proxy for their variance $\sigma_1^2$and σ_2^2:

$$RV_t = \sum_{i=1}^{1/\Delta} [log(P_{t-1+i\Delta}) - log(P_{t-1+(i-1)\Delta})]^2 \tag{12}$$

where $1/\Delta$ is the number of intra-period observations, for example, if we use daily data to compute annual realized variance, $1/\Delta$ will be about 252. Then we obtain realized volatility by taking square root of realized variance.

Now we apply the empirical data of S&P 500 and LTR to construct RPP. Realized volatility (RV) of each asset is obtained by summing up square of N days daily log return of each asset, which is rs (log return of stock) and rb (log return of bond) respectively. Then the RV of rs and rb (rv_rs and rv_rb) will be used in calculating the weights from equations 9 and 10, and the correlation between them(ρ) will be used in equation 11 to obtain the leverage c, since rv_rs, rv_rb and ρ are not in a stable level for a long time (heterogeneity exists), we need to reset the weights w_1 and w_2 and leverage c in a rational frequency. Taking trading and operating costs into consideration, rebalance portfolio every day is not practical, besides, Schrimpf et al. (2020) pointed out that risk parity portfolios tend to have longer look-back periods and are less responsive to short-lived episodes of financial market volatility, so we choose to rebalance our portfolio every 12 months (252 trading days) and use 6-month historical realized volatility and 12-month historical correlation as the prediction of volatility and correlation in the reset period. To compare with

[5]both treasuries and equities' leverage

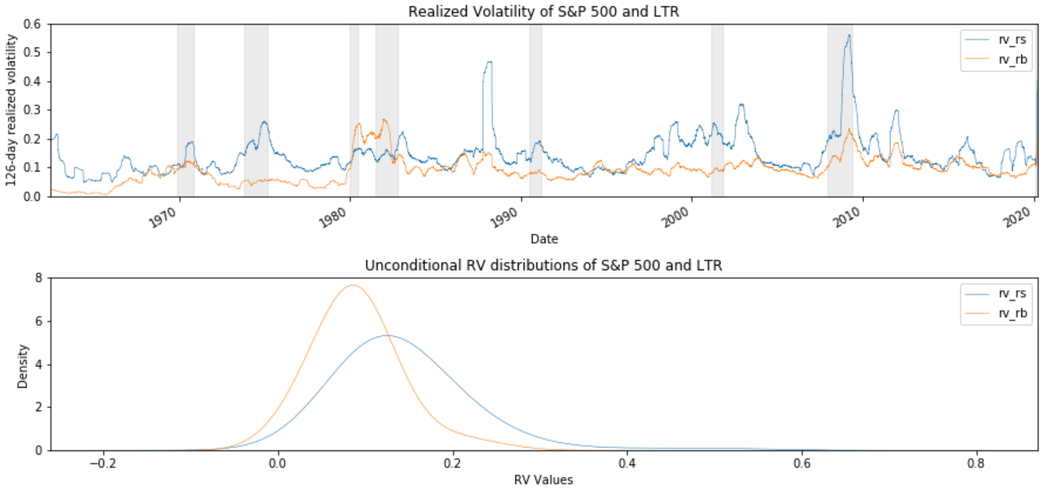

Figure 3: Top is 126-day realized volatility of S&P 500 and LTR, bottom is unconditional daily RV distributions

The top figure shows the 126-day realized volatility of S&P 500 and LTR from 1962-06-19 to 2020-03-31, light shaded bars indicate NBER recessions and show a clear business cycle pattern in volatility. The bottom figure shows the kernel density estimates of the unconditional daily RV (annualized) for the two assets, rv_rb represents RV of long-term government bond, rv_rs represents RV of S&P 500.

traditional portfolios which puts more weight into stocks, we set the leverage to target the volatility of 60/40 portfolio. To be clear, we use the realized volatility calculated from the 60/40 portfolio using full sample period (1960 to 2020) as our σ_{target}, and use equation 11 get the leverage. To avoid an extremely high leverage we limit it to a rational level at less than 6.

Top of figure 3 plots RV of the two assets together and shows that S&P 500 is absolutely more volatile than LTR, which is consistent with the results from figure 1, furthermore, the kernel density estimates of unconditional distribution of 126-day(6 months) realized volatilities(annualized) in bottom of figure 3 for the two assets also support the result. LTR only has larger realized volatility during the period 1980-1986. In spite of the huge difference in the overall absolute levels of the volatilities, the volatilities of the two assets show similarities and commonality in the general patterns, with most of the peaks happening at almost the same time associated with specific economic events. However, their returns do not share the same pattern, as shown is figure 1.

5 Portfolio performance

Figure 4 reports historical performances of RPP and 60/40 portfolio. We see superior cumulative excess log return of RPP from top panel of the figure. Specifically, if we invest \$1 in 1962, the 60/40 strategy gains \$50.51 at the end of the sample and the RPP realizes around \$314.39 which is about 6.22 times the 60/40 portfolio. RPP is dominant from perspective of total excess return. From figure of 1-year rolling log return, RPP outperforms 60/40 strategy for 69.61% of the time periods, but it shows much lower 1-year return in the period 1966.01-1967.01, 1967.10-1968.06, 1969.07-1971.09, 2009.10-2010.08. Coincidentally, periods 1967.10-1968.06 and 1969.07-1971.09 also see positive correlation between S&P 500 and LTR as shown in figure 2, implying that positive correlation of the two assets may have adverse impact on RPP's return. Besides, the RPP experiences 5% larger drawdown in the period 1965.12-1972.06, 1973.06-1973.09, 1978.12-1979.01, 2003.07-2003.12, 2013.06-2013.10 and 2016.11-2017.04. Specifically, average drawdowns in these periods are -17.02%, -16.80%, -10.60%, -9.22%, -7.45% and -7.80% respectively. These periods are also bad time for bond, especially before 1980, so the performance of RPP is certainly poor as it puts on average 62% of capital on bond as shown in the third panel of figure 4. To be clear, actual bond weight deviates from theoretical bond weight calculated from equation 9 or 10 because we rebalance our portfolio every 12 month, in the time interval, bond weights will fluctuate when portfolio gains or loses. RPP has its edge, light shaded NBER recession bars in the second and fourth panel show that after 1981 RPP outperforms traditional 60/40 in recession from perspective of return and drawdown.

Table 2 reports performances of two strategies using full sample on several measurements: annualized excess log return, annualized Sharpe ratio, 5% Value-at-Risk (VaR) of one day, expected shortfall (ES) of one day , average

	Ex. return	**SR**	**VaR**	**ES**	**Avg. DD**	**MDD**
60/40	0.0234	0.2342	-0.0096	-0.0140	-4.80%	-33.12%
RPP	0.0555	0.4519	-0.0110	-0.0171	-6.17%	-41.21%

Table 2: Performance of RPP and 60/40 portfolio (daily data)

The table shows historical performances of two portfolios from 1962.06 to 2020.03. RPP represents risk parity portfolio, 60/40 represents 60/40 portfolio. Avg. DD is average drawdown, other indicators' descriptions are the same with table 1.

and maximum drawdown. RPP realizes a higher annualized excess log return, annualized Sharpe ratio, but a larger one-day VaR, ES, average and maximum drawdown. It's worth noting that maximum drawdown (MDD) of RPP reaches as large as -41.21% which is a devastating loss that few investors can withstand. In detail, figure 5 puts the drawdown of two portfolios together and shows that RPP experiences several large drawdowns more than -30% during the period 1965-1981 and is worse than traditional 60/40 portfolio. This period also sees decline yield and poor bond performance shown in next paragraph. It's highly risky to invest in RPP during these periods.

Our historical RPP results has shown that the strategy put more than half capital into bond, so RPP is more exposed to interest rate changes than traditional 60/40 stock and bond portfolios. Implementation of the RPP strategy began in the late 1990s, coinciding with a period of falling interest rates, as shown in figure 6. In order to examine whether risk parity strategy relies on falling yield, we break the sample into three sub-periods: moderately rising rates (1962.06-1979.09), sharply rising rates (1979.10-1981.09), and falling rates period (1981.10-2020.03) so that we can further explore how RPP and 60/40 portfolios have performed under each scenario. Cumulative excess log return and drawdown of two portfolios in three subperiods are shown in figure 7. Besides, we report how two portfolios perform in 8 NBER recession periods in table 4. When we examine table 3 and 4 together, in moderate rising and sharply rising interest rate periods, although RPP outperforms

60/40 portfolio from perspective of excess log return, it experienced several larger fluctuations in return and larger average drawdown, and the maximum drawdown reaches to -41.55% and -36.41% in the two rising rate periods shown in table 3, which means that RPP has larger tail risk in rising interest rate periods. And recessions before 1981.09 also see worse performance of RPP, especially from 1969.12 to 1970.11 (see table 4), RPP is inferior on all 6 indicators. Therefore, rising interest rate has an adverse effect on two portfolios, no matter it is moderately rising or sharply rising, and it is tougher for RPP and RPP suffers most if accompanied by recession, so it will be painful for investors to invest in RPP during these periods. When interest rates rise, the present value of future cash flows long duration assets, like stocks and bonds, are reduced. When yield increases sharply than expected, as shown in figure 6 that 10-year treasury note yield experience an increase of 633 basis points from 9.51% to 15.84% during the period 1979.10-1981.09, investors would turn to cash or deposit than bonds or stocks. So obviously, yield increases directly hurt fixed income investments such as our RPP which allocates on average 62% into bond. In falling rate periods, both portfolios gain considerable annualized excess returns of 8.89% and 5.35% for RPP and 60/40 respectively, and they have few large drawdowns excluding the Great Recession in 2008. And recessions after 1981.09 (see table 4) see superior performance of RPP almost on all indicators.

Popularity of RPP after 1990s also coincides with the negative correlation between stocks and bond. In order to examine whether risk parity strategy relies on negative correlation between assets, we break the sample into two situations according to figure 2: positive correlation period (1967.11-1998.06), negative correlation periods (1962.06-1967.10 and 1998.07-2020.03). Results in figure 8 and table 5 show that RPP does not always outperforms 60/40 portfolio in negative correlation period, with much worse average and maximum drawdown in period 1962.06-1967.10, overall

	Ex. return	SR	VaR	ES	Avg. DD	MDD
1962.06.20 to 1979.09.30(moderate rising interest rates)						
RPP	0.0088	0.0732	-0.0113	-0.0170	-10.99%	-41.55%
60/40	-0.0209	-0.2696	-0.0078	-0.0105	-6.60%	-33.12%
1979.10.01 to 1981.09.30(sharp rising interest rates)						
RPP	-0.1169	-0.4821	-0.0216	-0.0384	-6.86%	-36.41%
60/40	-0.1506	-1.0805	-0.0153	-0.0205	-4.66%	-16.67%
1981.10.01 to 2020.03.31(falling interest rates)						
RPP	0.0852	0.7481	-0.0109	-0.0164	-3.86%	-23.63%
60/40	0.0521	0.4923	-0.0102	-0.0150	-3.91%	-30.98%

Table 3: Portfolio performances in different interest rate period (daily data)
The table shows historical performances of two portfolios in different interest rate periods, including two rising periods from 1962.06 to 1979.09 and from 1981.10 to 2020.03, and one falling period from 1979.10 to 1981.09. Indicators' descriptions are the same with table 1.

RPP has larger tail risk in this period. This period also sees rising interest rate, so we cannot conclude whether negative assets correlation can boost or impair performance of RPP. In positive correlation periods, RPP has several larger drawdowns before 1980 and smaller drawdowns after 1980 from figure 8, which means that it does not have consistent performance throughout the positive correlation periods. Let's take one step further to split these positive correlation periods into three subperiods with rising and falling interest rate. Table 6 shows that both portfolios realize negative return and Sharp ratio in positive assets correlation together with rising interest rates periods, and RPP suffers larger tail loss. However, in positive assets correlation and falling interest rates periods, RPP is superior from almost all indicators. Correlation between assets is an important factor in building a diversified portfolio, when assets have positive correlation, they move together, so when one asset falls in value, so does the other. Obviously, positive assets correlation has adverse effect on both RPP and 60/40 portfolio, and even strong when yield is increasing. We will further theoretically examine the impact of correlation

	Ex. return	SR	VaR	ES	Avg. DD	MDD
1969.12 to 1970.11 (moderate rising interest rates)						
RPP	-0.1303	-0.8243	-0.0138	-0.0221	-9.21%	-26.79%
60/40	-0.0598	-0.5166	-0.0098	-0.0145	-5.63%	-18.63%
1973.11 to 1975.03 (moderate rising interest rates)						
RPP	-0.1413	-0.9840	-0.0143	-0.020	-13.86%	-29.73%
60/40	-0.1941	-1.5496	-0.0126	-0.015	-14.42%	-28.23%
1980.01 to 1980.07 (sharp rising interest rates)						
RPP	0.2050	0.5227	-0.0413	-0.0555	-8.13%	-30.78%
60/40	0.0250	0.1722	-0.0142	-0.0203	-3.21%	-13.44%
1981.07 to 1982.11 (falling interest rates)						
RPP	-0.0037	-0.0296	-0.0113	-0.0160	-4.05%	-11.36%
60/40	-0.0045	-0.0298	-0.0135	-0.0193	-4.92%	-13.58%
1990.07 to 1991.03 (falling interest rates)						
RPP	0.0350	0.2632	-0.0128	-0.0181	-4.28%	-12.22%
60/40	-0.0052	-0.0416	-0.0124	-0.0166	-5.65%	-13.42%
2001.03 to 2001.11 (falling interest rates)						
RPP	0.0220	0.2308	-0.0096	-0.0120	-1.86%	-5.63%
60/40	0.0060	0.0508	-0.0101	-0.0138	-2.88%	-8.77%
2007.12 to 2009.06 (falling interest rates)						
RPP	-0.1473	-0.8558	-0.0181	-0.0250	-10.85%	-23.63%
60/40	-0.1847	-1.1355	-0.0164	-0.0252	-14.41%	-30.98%
2020.02 to 2020.03 (falling interest rates)						
RPP	0.0983	0.3126	-0.0458	-0.0495	-4.08%	-17.48%
60/40	-0.4718	-1.5329	-0.0352	-0.0457	-5.70%	-16.64%

Table 4: Portfolio performances in NBER recession periods (daily data)

The table shows historical performances of two portfolios in NBER recession periods. Indicators' descriptions are the same with table 1.

	Ex. return	SR	VaR	ES	Avg. DD	MDD
1962.06.20 to 1967.10.31(negative correlation)						
RPP	0.0540	0.4511	-0.0110	-0.0183	-6.55%	-35.63%
60/40	0.0180	0.3193	-0.0058	-0.0083	-2.25%	-14.99%
1967.11.01 to 1998.07.01(positive correlation)						
RPP	0.0372	0.2937	-0.0111	-0.0175	-6.46%	-38.85%
60/40	0.0135	0.1286	-0.0098	-0.0144	-5.50%	-33.12%
1998.07.02 to 2020.03.31(negative correlation)						
RPP	0.0814	0.6916	-0.0115	-0.0174	-4.40%	-23.63%
60/40	0.0385	0.3814	-0.0102	-0.0147	-4.41%	-30.98%

Table 5: Portfolio Performances with assets correlation division (daily data)
The table shows historical performances of two portfolios in different assets correlation periods, including two negative periods from 1962.06 to 1967.10 and from 1998.07 to 2020.03, and one positive period from 1967.11 to 1998.07. Indicators' descriptions are the same with table 1.

between two assets on portfolio volatility in our volatility variation effect part.

Additionally, to have full picture of how RPP performs, we expand our sample period back to 1926 using monthly data. From 1926 to 1962, the government bond yield has both rising and falling periods. We use the same method to construct RPP, the target volatility is about 12.03% annually which is the annualized realized volatility of 60/40 portfolio from 1927 to 2018(12 months data is used for calculating correlation and realized volatility, so the sample starts from 1927). We also analyse the performance by dividing the results into interest rising and falling periods. During this period, Results in table 7 show that both portfolios experience extreme maximum drawdown as -40.35% or -63.77% which every investor cannot withstand, and also go through several NBER recessions as shown in figure 9, however, RPP outperforms 60/40 portfolio on all six indicators. Besides, we report how two portfolios perform in 7 NBER recession periods in table 8. When we examine table 7 and 8

	Ex. return	SR	VaR	ES	Avg. DD	MDD
1967.11.01 to 1979.9.30 (moderate rising interest rates)						
RPP	-0.0117	-0.0964	-0.0115	-0.0164	-10.67%	-38.21%
60/40	-0.0386	-0.4525	-0.0085	-0.0111	-8.51%	-33.12%
1979.10.01 to 1981.09.30 (sharp rising interest rates)						
RPP	-0.1169	-0.4821	-0.0216	-0.0384	-6.86%	-36.41%
60/40	-0.1506	-1.0805	-0.0153	-0.0205	-4.66%	-16.67%
1981.10.01 to 1998.07.01 (falling interest rates)						
RPP	0.0902	0.8291	-0.0103	-0.0150	-3.17%	-18.27%
60/40	0.0699	0.6242	-0.0101	-0.0154	-3.26%	-24.05%

Table 6: Portfolio performance with interest rate division in positive assets correlation periods (daily data)

The table shows historical performances of two portfolios with different interest rate division in positive assets correlation periods, including two rising periods from 1967.11 to 1979.09 and from 1979.10 to 1981.09, and one falling period from 1981.10 to 1998.07. Indicators' descriptions are the same with table 1.

together, before 1941.10 RPP outperforms 60/40 portfolio on all indicators in recessions, however, after 1941.10 RPP in general has larger VaR, ES, average and maximum drawdowns in recessions except in the periods from 1945.02 to 1945.10, in other words, it has larger tail risk in recession accompanied by rising interest rates. Therefore, monthly data and daily data share the same conclusions: In moderate rising rate periods, although RPP will have slight larger annualized risk adjusted return (Sharpe ratio), but it also has much larger drawdown and tail risk and usually underperform 60/40 portfolio in recession; in sharp rising rate period, RPP and 60/40 portfolio both have the worst time with poorest return and extreme drawdown; in falling rate periods, RPP has superior performance on most indicators and also outperforms 60/40 portfolio in recession.

	Ex. return	SR	VaR	ES	Avg. DD	MDD
1927-01 to 1962-07(full monthly data)						
RPP	0.0895	0.6172	-0.0583	-0.0914	-7.44%	-40.35%
60/40	0.0302	0.2172	-0.0626	-0.0981	-14.30%	-63.77%
1927-01 to 1941-10(falling interest rates)						
RPP	0.0746	0.4924	-0.0678	-0.1183	-9.85%	-40.35%
60/40	0.0042	0.0216	-0.0859	-0.1342	-24.79%	-63.77%
1941-11 to 1962-07(moderate rising interest rates)						
RPP	0.1002	0.7132	-0.0565	-0.0719	-5.69%	-32.80%
60/40	0.0487	0.6031	-0.0347	-0.0452	-3.74%	-17.49%

Table 7: Portfolio performances with interest rate division (monthly data)

The table shows historical performances of two portfolios in different interest rate periods from 1927.01 to 1962.07, including one falling periods from 1927.01 to 1941.10, and one rising period from 1941.11 to 1962.07. Indicators' descriptions are the same with table 1. Since we use monthly data, VaR and ES are one-month value of log return.

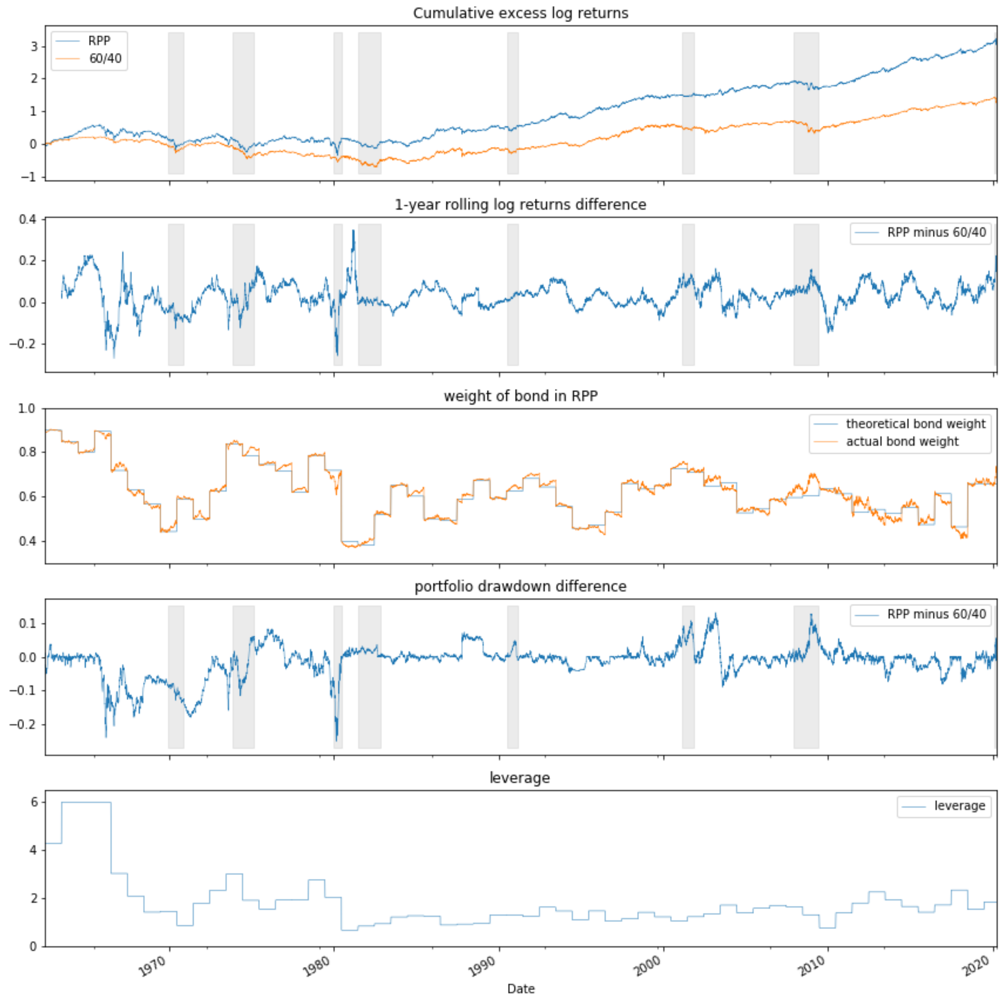

Figure 4: Performance of RPP and 60/40 portfolio (daily data)

The figure shows historical performances of RPP and 60/40 portfolio from 1962 to 2020. The top panel plots the cumulative excess returns of RPP and 60/40 portfolio. The y-axis is on a log scale. The second panel shows the the difference of one-year rolling log returns of the two strategies and the fourth panel shows the difference of drawdown of the two strategies, in both panels, we use values of RPP to minus those of 60/40 portfolio. The third and fifth panel shows weight of bond and leverage in RPP respectively. Light shaded bars indicate NBER recessions. RPP represents risk parity portfolio, 60/40 represents 60/40 portfolio.

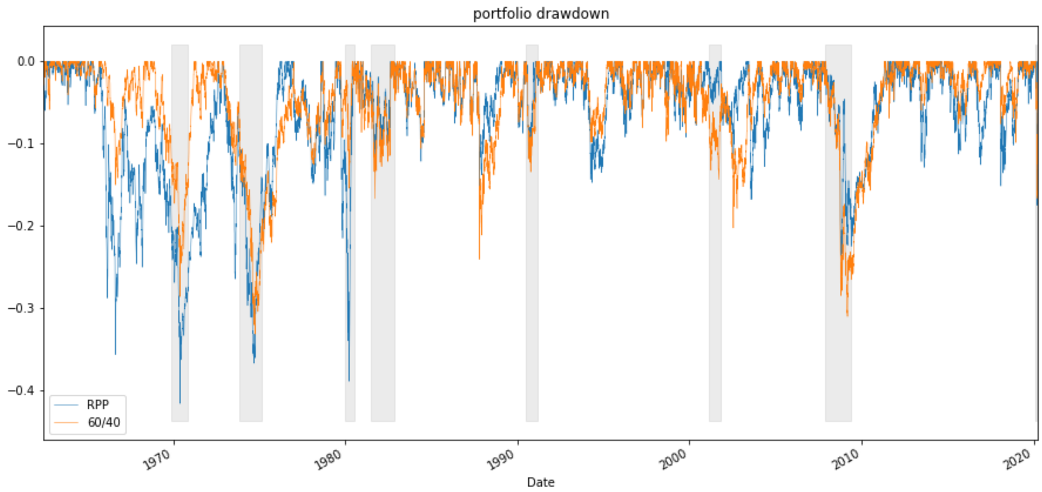

Figure 5: Drawdown of RPP and 60/40 portfolio (daily data)

The figure shows historical drawdowns of RPP and 60/40 portfolio from 1962 to 2020. Light shaded bars indicate NBER recessions and show a clear business cycle pattern in drawdown. Abbreviations are the same with figure 4.

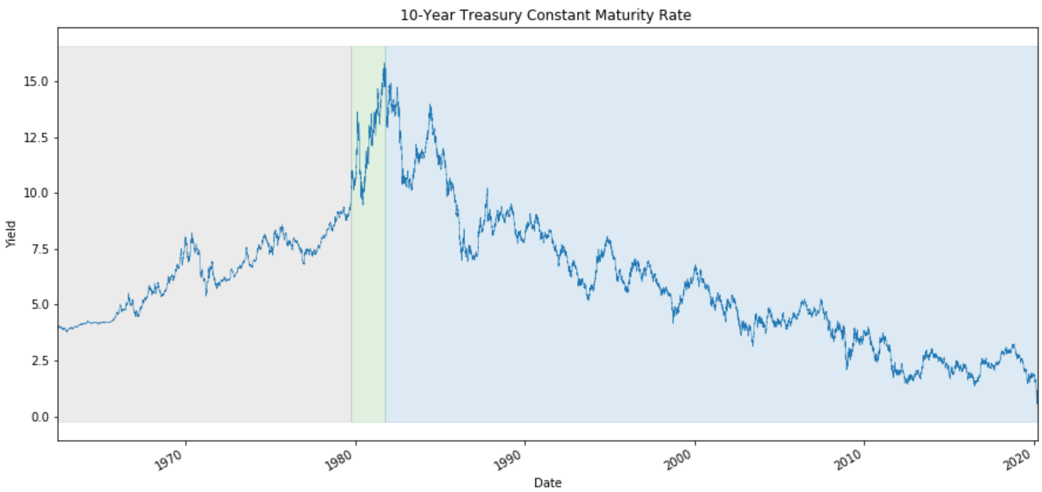

Figure 6: Yield on 10-Year U.S. Treasury Note

The figure shows daily yield on 10-Year U.S. treasury note from 1962.06 to 2020.03, the y-axis is in percent.

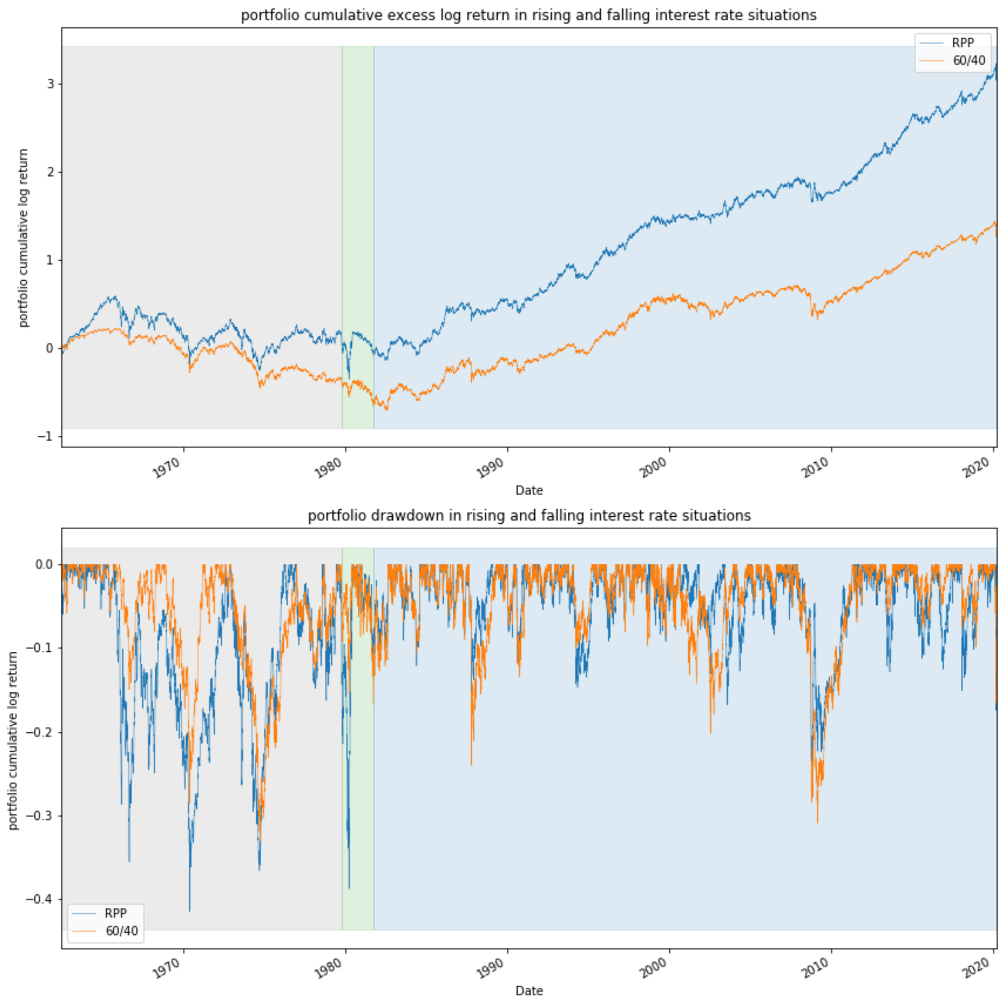

Figure 7: Portfolio performance with interest rate division (daily data)

The figure shows performances of RPP and 60/40 portfolio in different subperiods: moderately rising rates periods (shaded grey), sharply rising rates periods (shaded green), and falling rates period (shaded blue). Abbreviations are the same with figure 4.

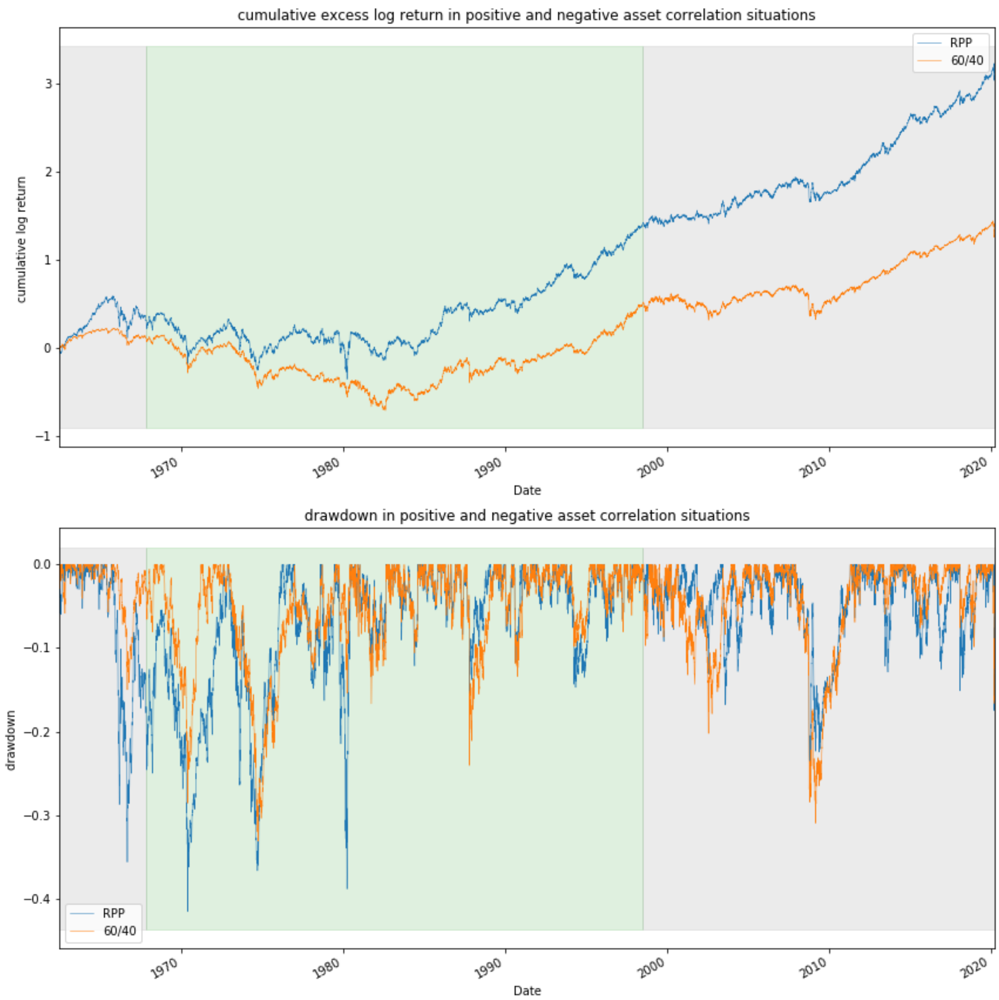

Figure 8: Portfolio performance with assets correlation division (daily data)

The figure shows performances of RPP and 60/40 portfolio in different situations: positive assets correlation situation (shaded green), negative assets correlation situation (shaded grey). Abbreviations are the same with figure 4.

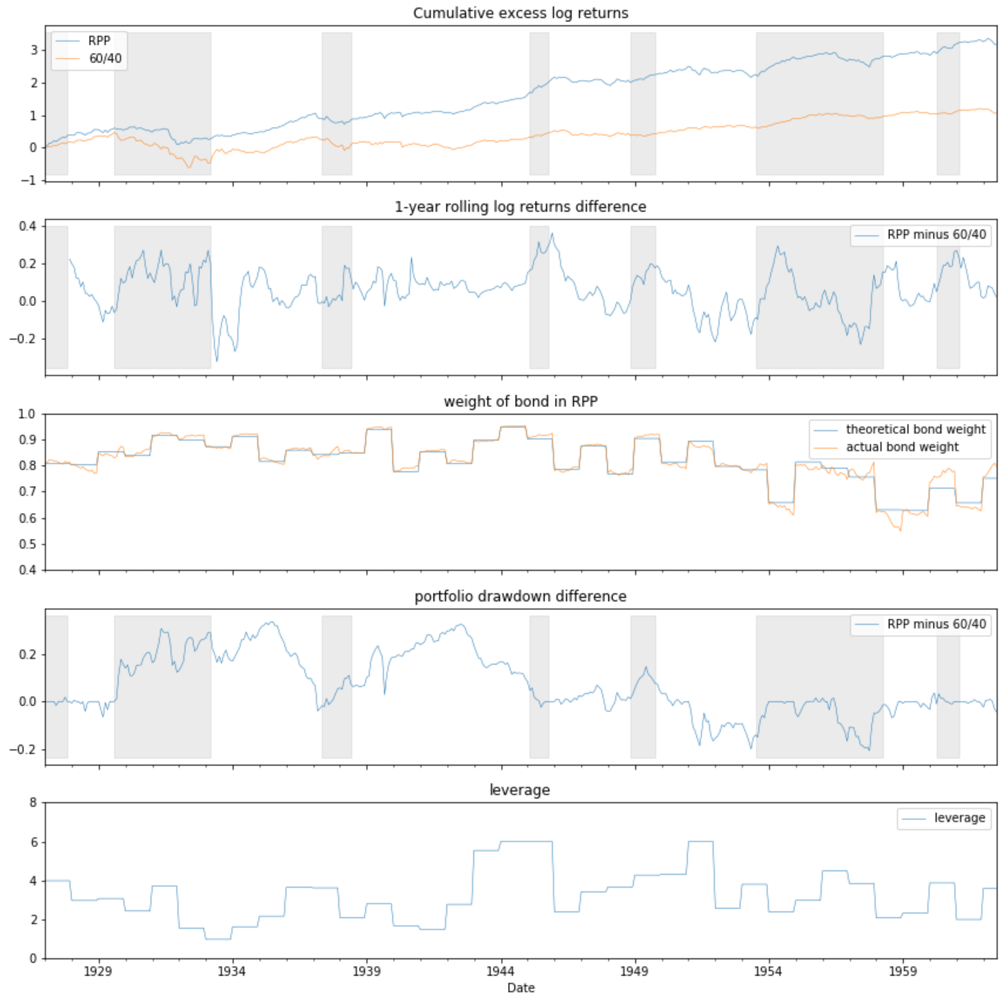

Figure 9: Performance of RPP and 60/40 portfolio (monthly data)

The figure shows historical performances of RPP and 60/40 portfolio from 1927 to 1962 with monthly data. Figure descriptions and abbreviations are the same with figure 4. In the third panel, actual bond weight deviates from theoretical bond weight calculated from equation 9 or 10 because we rebalance our portfolio every 12 month, in the time interval, bond weights will fluctuate when portfolio gains or loses.

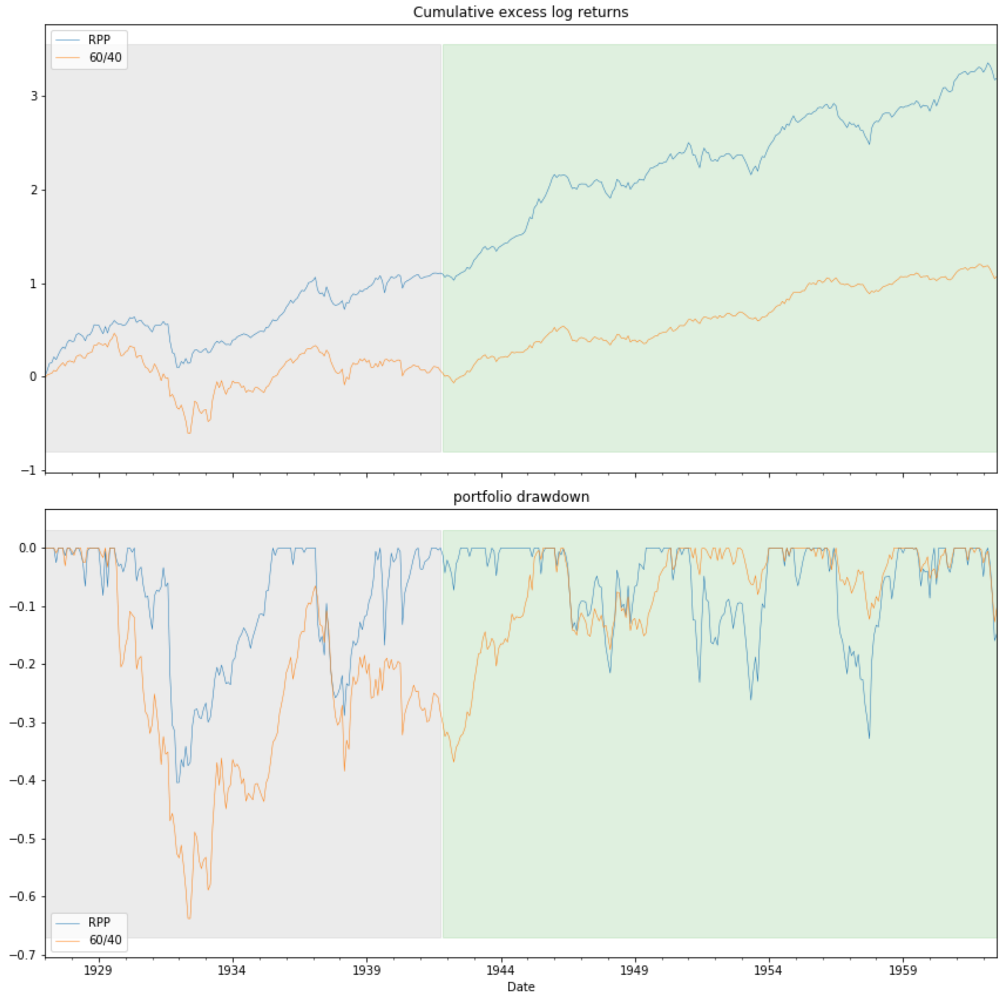

Figure 10: Portfolio performance with interest rate division (monthly data)

The figure shows performances of RPP and 60/40 portfolio in different situations from 1927 to 1962: rising rate periods from 1927.01 to 1941.10 (shaded green), falling rate periods from 1941.11 to 1962.07 (shaded grey). Abbreviations are the same with figure 4.

	Ex. return	SR	VaR	ES	Avg. DD	MDD
1927.01 to 1927.11 (falling interest rates)						
RPP	0.3979	3.2453	-0.0254	-0.0254	-0.34%	-2.51%
60/40	0.1712	2.0316	-0.0310	-0.0310	-0.35%	-3.06%
1929.08 to 1933.03 (falling interest rates)						
RPP	-0.0813	-0.4620	-0.1151	-0.1466	-16.55%	-40.35%
60/40	-0.2360	-0.9273	-0.1286	-0.1544	-35.90%	-63.77%
1937.05 to 1938.06 (falling interest rates)						
RPP	-0.0260	-0.1293	-0.0925	-0.0925	-11.94%	-21.34%
60/40	-0.1344	-0.5195	-0.1688	-0.1688	-15.45%	-30.95%
1945.02 to 1945.10 (moderate rising interest rates)						
RPP	0.4756	2.8641	-0.0454	-0.0454	-0.74%	-4.44%
60/40	0.2004	2.1206	-0.0279	-0.0279	-0.48%	-2.75%
1948.11 to 1949.10 (moderate rising interest rates)						
RPP	0.1721	1.442	-0.0720	-0.0720	-0.12%	-1.06%
60/40	-0.0026	-0.027	-0.0656	-0.0656	-0.83%	-3.43%
1953.07 to 1958.04 (moderate rising interest rates)						
RPP	0.1216	0.7680	-0.0534	-0.0719	-7.39%	-32.80%
60/40	0.0708	0.8722	-0.0344	-0.0346	-2.79%	-12.25%
1960.04 to 1961.02 (moderate rising interest rates)						
RPP	0.2914	1.6841	-0.0646	-0.0646	-0.91%	-4.17%
60/40	0.1005	1.3715	-0.0312	-0.0312	-0.72%	-3.32%

Table 8: Portfolio performances in NBER recession periods (monthly data)

The table shows historical performances of two portfolios in NBER recession periods. Indicators' descriptions are the same with table 1.

6 Volatility variation effect

6.1 Theories

In our two-asset case of $n = 2$, applying equations 9 and 10 into 1, we get

$$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2} = \frac{2\sigma_1^2\sigma_2^2(1+\rho_{1,2})}{\sigma_1^2+\sigma_2^2} \tag{13}$$

$$\sigma^2 = 2w_i^2\sigma_i^2(1+\rho_{1,2}) \text{ for } i = 1, 2 \tag{14}$$

where $\rho_{1,2}$ is the correlation between assets. To reach the target volatility in equation 11, the weights of assets and leverage must satisfy

$$(cw_i)^2 = \frac{\sigma_{target}^2}{\sigma_i^2}\frac{1}{2(1+\rho_{1,2})} \text{ for } i = 1, 2 \tag{15}$$

In our empirical portfolio, σ_i and $\rho_{1,2}$ are historical measures that derive the weights and leverage and these historical values are taken as our forecasted ones. The actual volatilities of portfolio not only depend on weights and leverage, but also depend on actual volatilities of assets and the actual correlation between them, which could be denoted as $\tilde{\sigma}_i$ and $\tilde{\rho}_{1,2}$. Let's go back to equation 1 again, the actual volatility of portfolio $\tilde{\sigma}$ must satisfy[6]

$$\tilde{\sigma}^2 = \sigma_{target}^2 \times \frac{1}{2(1+\rho_{1,2})} \times [(\frac{\tilde{\sigma}_1}{\sigma_1})^2 + (\frac{\tilde{\sigma}_2}{\sigma_2})^2 + \frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\sigma_1\sigma_2}2\tilde{\rho}_{1,2}] \tag{16}$$

Let $\frac{\tilde{\sigma}_1}{\sigma_1} = k_1$ and $\frac{\tilde{\sigma}_2}{\sigma_2} = k_2$, then the actual variance of portfolio in equation 16 can be rewritten as

$$\tilde{\sigma}^2 = \sigma_{target}^2 \times \frac{1}{2(1+\rho_{1,2})} \times (k_1^2 + k_2^2 + 2k_1k_2\tilde{\rho}_{1,2}) \tag{17}$$

[6]note that σ is the volatility of the portfolio without leverage, but $\tilde{\sigma}$ is the volatility of the portfolio with leverage. So $\tilde{\sigma} = \sigma \times c$

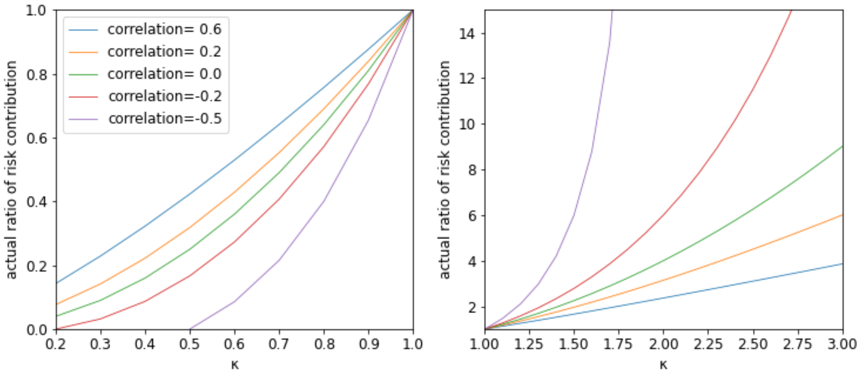

Figure 11: How actual ratio of risk contribution moves on κ with different $\rho_{1,2}$

In this figure, y-axis is the actual ratio of risk contribution (RC) of assets ($w_1 \times \partial_{w_1}\sigma$: $w_2 \times \partial_{w_2}\sigma$), x-axis is $\kappa = \frac{k_1}{k_2} = \frac{\tilde{\sigma}_1\sigma_2}{\sigma_1\tilde{\sigma}_2}$, lines in different colors represent that the correlations between assets are 0.6, 0.2, 0, -0.2 and -0.5 respectively. The left one shows the case that $0.2 < \kappa < 1.0$, when κ is constant, a stronger positive correlation results in a smaller actual ratio. The right one shows the case that $1.0 < \kappa < 3.0$, when κ is constant, a stronger positive correlation results in a larger actual ratio. When correlation between assets is 0, actual ratio is κ^2.

Let $\kappa = \frac{k_1}{k_2}$, we have

$$\tilde{\sigma}^2 = \sigma_{target}^2 \times \frac{k_1 k_2}{1+\rho_{1,2}} \times \frac{\left[(\kappa + \tilde{\rho}_{1,2}) + (\frac{1}{\kappa} + \tilde{\rho}_{1,2})\right]}{2} \tag{18}$$

Similar with the actual volatility of portfolio, the actual RC of assets may not be equal as we have expected. With equation 2 and 3, the actual ratio of RC of assets ($w_1 \times \partial_{w_1}\sigma : w_2 \times \partial_{w_2}\sigma$) is

$$\begin{aligned} \text{actual ratio} &= (\sigma_2^2\tilde{\sigma}_1^2 + \sigma_1\sigma_2\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}) : (\sigma_1^2\tilde{\sigma}_2^2 + \sigma_1\sigma_2\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}) \\ &= (k_1^2 + k_1k_2\tilde{\rho}_{1,2}) : (k_2^2 + k_1k_2\tilde{\rho}_{1,2}) \\ &= (\kappa + \tilde{\rho}_{1,2}) : (\frac{1}{\kappa} + \tilde{\rho}_{1,2}) \end{aligned} \tag{19}$$

The equation means that assets in the portfolio have equal RC if and only if $\kappa = 1$, which is the point where all lines intersect in figure 11. When $\kappa \neq 1$, the actual ratio of RC will be away from our target value $1:1$ or 1. Figure 11 shows that when $\tilde{\rho}_{1,2} = 0$, the actual ratio of RC is κ^2 (the green line). When $\tilde{\rho}_{1,2} < 0$ and κ is constant, the actual ratio of RC is smaller than κ^2 (the purple and red line) and farther than 1; when $\tilde{\rho}_{1,2} > 0$ and κ is constant, the actual ratio is larger than κ^2 (the blue and orange line) and closer to 1. Therefore, a stronger positive correlation between assets is more likely to satisfy equal risk contributions when $\kappa \neq 1$. When $\tilde{\rho}_{1,2} = 1$ or assets are perfectly positive correlated, the actual ratio is exactly κ because $(\kappa+1) : (\frac{1}{\kappa}+1) = \kappa$.

Let's take one step further, we denote π, the geometric mean of k_i as the average ratio between actual and predicted volatility of assets, and denote the heterogeneity of ratio between actual and predicted volatility of assets as λ.

$$\pi = \sqrt{k_1 k_2} = \sqrt{\frac{\tilde{\sigma}_1 \tilde{\sigma}_2}{\sigma_1 \sigma_2}} \tag{20}$$

$$\lambda = \frac{\kappa + \frac{1}{\kappa}}{2} = \frac{k_1^2 + k_2^2}{2k_1 k_2} = \frac{\sigma_2^2 \tilde{\sigma}_1^2 + \sigma_1^2 \tilde{\sigma}_2^2}{2\sigma_1 \sigma_2 \tilde{\sigma}_1 \tilde{\sigma}_2} \tag{21}$$

Then from equation 18 we could get

$$\tilde{\sigma} = \sigma_{target} \times \pi \times \sqrt{\frac{\lambda + \tilde{\rho}_{1,2}}{1 + \rho_{1,2}}} \tag{22}$$

We find that forecasted and actual volatilities of assets do not affect actual volatility of portfolio individually, instead, they show the impact when putting together, as shown in equations 20 and 21 that they determine factors π and λ. From equation 22, actual volatility of portfolio $\tilde{\sigma}$ increases with π, illustrating that the more we underestimate average volatility of assets, the higher portfolio volatility we will get, vice versa; $\tilde{\sigma}$ also increases with $\frac{\lambda+\tilde{\rho}_{1,2}}{1+\rho_{1,2}}$, indicating that the more heterogeneity of ratio between actual and predicted volatility of assets (the higher λ), or the more we underestimate correlations, the higher portfolio volatility we will get, vice versa.

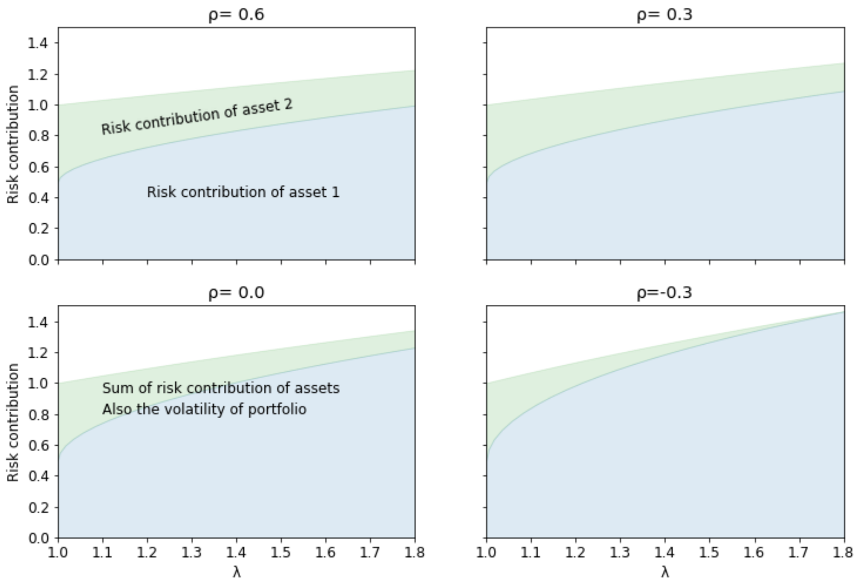

Figure 12: How actual risk contribution moves on λ with different ρ

In this figure, y-axis is the actual RC of assets ($w_i \times \partial_{w_i}\sigma$) and the sum of them (the actual portfolio volatility). Here we assume that $k_1 > k_2$, the actual RC of asset 1 (the blue area) is always no smaller than that of asset 2 (the green area). X-axis is λ. The 4 subfigures show RC of assets when the correlations between assets are 0.6, 0.3, 0 and -0.3 respectively.

From equation 22 we could get

$$\frac{\tilde{\sigma}}{\sigma_{target}} = \pi \times \sqrt{\left(\frac{1+\tilde{\rho}_{1,2}}{1+\rho_{1,2}} + \frac{\lambda - 1}{1+\rho_{1,2}}\right)} \tag{23}$$

Let's assume that on average $\pi \cong 1$ and $\frac{1+\tilde{\rho}_{1,2}}{1+\rho_{1,2}} \cong 1$ (that is to say, we correctly estimate $\tilde{\sigma}_1\tilde{\sigma}_2$ and $\tilde{\rho}_{1,2}$) and they are independent, then we find that when $\lambda = 1$, we will get $\frac{\lambda-1}{1+\rho_{1,2}} = 0$ and $\frac{\tilde{\sigma}}{\sigma_{target}} \cong 1$, that is to say, the actual volatility of portfolio is approximately equal to the target one. But $\lambda > 1$ except in the case $\frac{\tilde{\sigma}_1\sigma_2}{\sigma_1\tilde{\sigma}_2} = 1$,[7] then we get $\frac{\lambda-1}{1+\rho_{1,2}} > 0$ except in the case $\frac{\tilde{\sigma}_1\sigma_2}{\sigma_1\tilde{\sigma}_2} = 1$, which means that the actual volatility of portfolio is higher than the target volatility in most cases. From this perspective, RPP tend to have a higher volatility than target for most of the time.

To intuitively show how λ affects both RC[8] and portfolio volatility, here we assume $\sigma_{target} = 1$, $\pi = 1$, $\rho_{1,2} = \tilde{\rho}_{1,2} = \rho$, then get figure 12, which shows that portfolio volatility increases with λ in an approximately linear way. Every subfigure shows that when λ is greater than 1, RC of two assets is far from equal with asset 1 dominating. When λ is constant, a weaker correlation results in a higher portfolio volatility and a more uneven RC between assets.

6.2 Empirical analysis

With our empirical risk parity portfolio (RPP) results, we obtain figure 13 and figure 14 to show the factors in our models and their effects on volatility of RPP from 1962 to 2020. The bottom of the two figures show that the theoretical (using equation 22) and actual volatilities of RPP are almost the same, the difference comes from the difference between target and actual assets' weight, which is caused by the time interval 12 months to rebalance

[7] $\lambda = \frac{\kappa + \frac{1}{\kappa}}{2}$, and $x + \frac{1}{x} > 2$ unless $x = 1$ for all $x > 1$, so $\lambda > 1$ unless $\kappa = 1$
[8] the formulations for risk contribution are shown in Appendix B

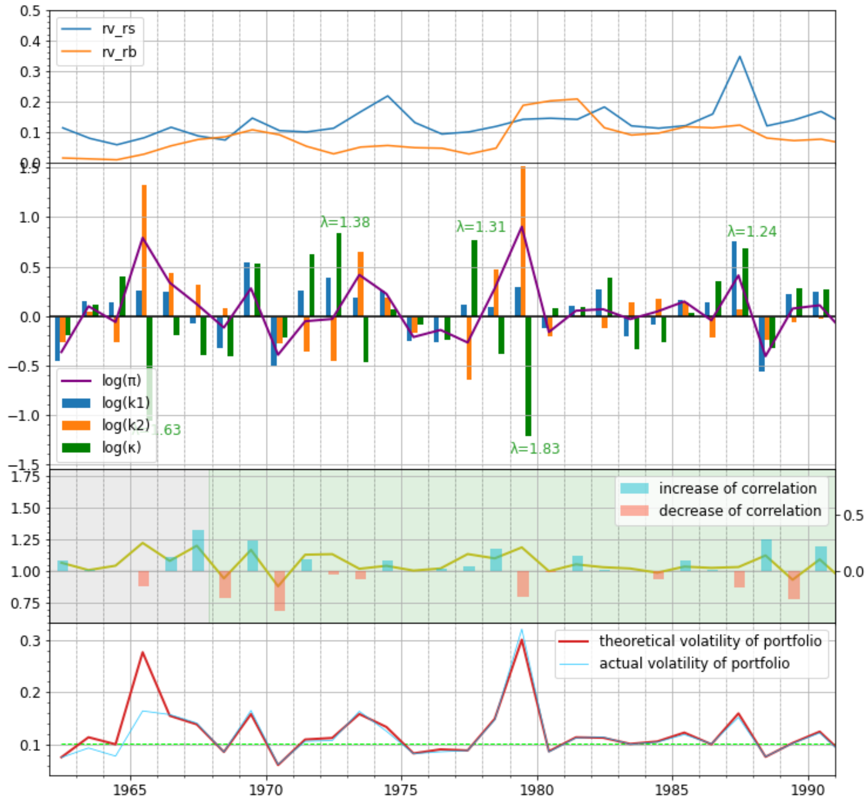

Figure 13: $\tilde{\sigma}_i$, $\log k_i$, $\log \kappa$, $\log \pi$, $(\tilde{\rho}_{1,2} - \rho_{1,2})$ and $\tilde{\sigma}$ from 1962 to 1991

The top panel shows the actual volatilities of S&P 500 and LTR ($\tilde{\sigma}_i$) in every rebalance time. Second panel shows $\log k_i$ ((k_1 for S&P 500, k_2 for LTR)), $\log \kappa$ and $\log \pi$. Because $\log \kappa = \log k_1 - \log k_2$, $\log \pi = \frac{\log k_1 + \log k_2}{2}$, so the relationship among them are quite intuitive: $\log \kappa$ is the difference between $\log k_1$ and $\log k_2$, and $\log \pi$ is their midpoint. Periods when absolute value of $\log \kappa$ is large will have a large λ, and λ in these periods are labelled. Third panel shows change in correlation compared with previous period (y-axis is in the right), and $\sqrt{\frac{\lambda + \tilde{\rho}_{1,2}}{1+\rho_{1,2}}}$ of current period (the yellow line, y-axis is in the left). Besides, periods shaded grey see positive assets correlation, and periods shaded green see negative assets correlation. The bottom panel shows the theoretical volatility of portfolio (using our model) and actual volatility of portfolio in every rebalance time, and the green line is target volatility equal to 0.1030.

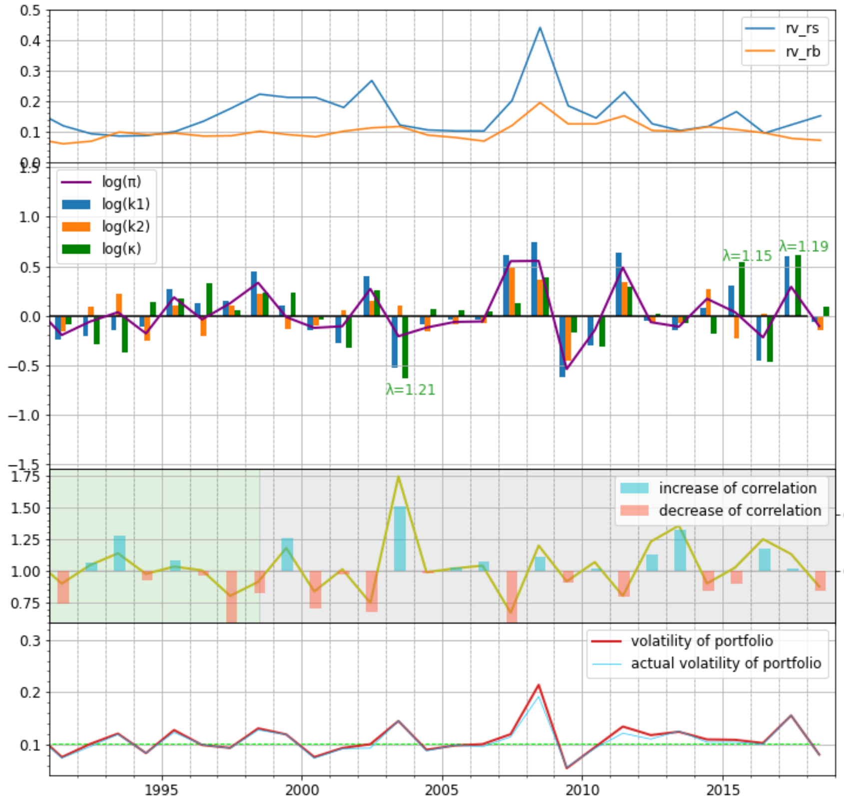

Figure 14: $\tilde{\sigma}_i$, $\log k_i$, $\log \kappa$, $\log \pi$, $(\tilde{\rho}_{1,2} - \rho_{1,2})$ and $\tilde{\sigma}$ from 1991 to 2019

the portfolio[9], and the leverage limit. Recall that in our RPP we use the historical volatilities and correlation as the prediction ones, so the ratio between actual and predicted value could be treated as the ratio between values of current and previous period, therefore, the ratio k_1 and k_2 represent variation in volatility (from previous values to current ones) of S&P 500 and LTR respectively. From figures, we find that π, which represents the average variation in volatility of S&P 500 and LTR, determines most of the variation in portfolio volatility $\tilde{\sigma}$. Before 1990s, there are two extremely large $\tilde{\sigma}$ and π in 1965[10] and 1979, which is the result of large k_2, the increase in volatility of LTR. In the recent 30 years, there's only one extremely large $\tilde{\sigma}$ in 2008, the Great Recession, which is the result of large k_1, the increase in volatility of S&P 500. Therefore, there is a clear shift in core factor determining extreme portfolio volatility before 1990s and after 1990s, variation in bond volatility dominates before 1990s, and variation in equity volatility contributes more after 1990s. It's worth noting that π in 2007 and 2008 are almost the same, but in 2007 there is a large decreasing in assets correlation whereas in 2008 assets correlation slightly increased, therefore, year 2007 do not see a relatively large $\tilde{\sigma}$, indicating that variation in assets correlation clearly affect portfolio volatility. Besides, periods with large λ does not always see large $\tilde{\sigma}$, so heterogeneity of ratio between actual and predicted volatility of assets λ may have impact on portfolio volatility $\tilde{\sigma}$, but not strong as π and ρ.

Figure 15 shows the unconditional distributions of $\log k_i$ ($\log k_1$ for S&P 500, $\log k_2$ for LTR), and the risk contribution (RC) of LTR and S&P 500. Means of $\log k_1$ and $\log k_2$ are close to zero, indicating that on average both bond and equity volatilities do not change much, at least, between two rebalance intervals, so we to some extent forecast the volatility correctly. The average RC of S&P 500 is slightly larger than that of LTR, however, RC of

[9]See the 3rd subfigure of figure 4

[10]To be precise, period from 1965-06-20 to 1966-06-20

	Mean	Min	Max	Skew	Excess Kurt
log(k2) (LTR)	0.0441	-0.6520	1.5103	1.8957	6.0745
log(k1) (S&P 500)	0.0512	-0.6249	0.7500	0.0608	-0.3613
RC of LTR	0.0572	0.0067	0.2465	3.0349	14.2204
RC of S&P 500	0.0607	0.0107	0.1928	1.6352	3.9921

Table 9: Summary statistics for $\log k_1$, $\log k_2$ and risk contributions

The table reports summary statistics for $\log k_1$, $\log k_2$ ($\log k_1$ for S&P 500, $\log k_2$ for LTR) and the risk contribution (RC) of LTR and S&P 500.

LTR sees larger skewness and excess kurtosis as shown in table 9. This results could be explained by k_1 and k_1: the mean of $\log k_1$ (S&P 500) is slightly greater than $\log k_2$, but the skewness and excess kurtosis of $\log k_2$ (LTR) is greater, which means that LTR has more 'extreme' volatility variations and more 'extreme' large RC.

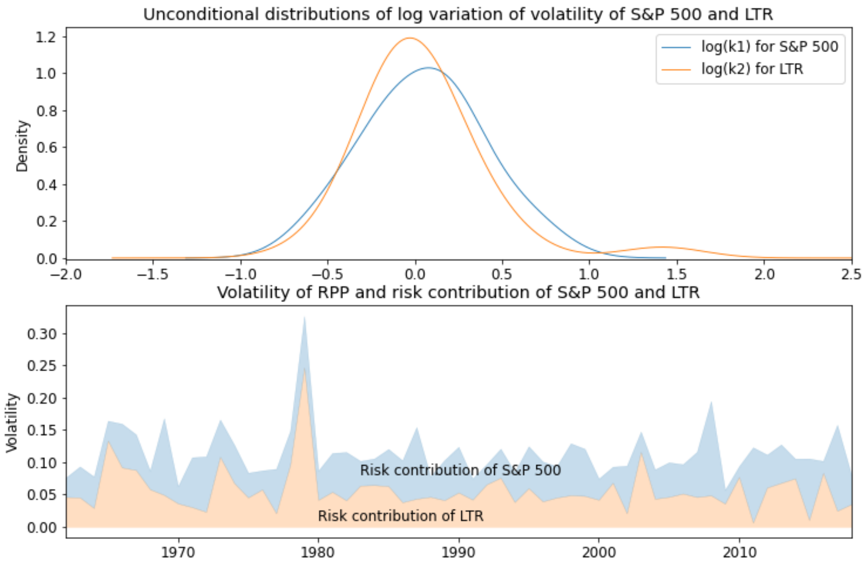

Figure 15: Unconditional distributions of log variation in volatility of S&P 500 and LTR and risk contributions

The top subfigure shows the kernel density estimates of the unconditional log variation in volatility of S&P 500 and LTR, the bottom subfigure shows the RC of LTR and S&P 500, and the sum of the two risk contributions is the volatility of RPP.

7 Conclusion

This paper studies a hypothetical risk parity portfolio(RPP) constructed with only equity and government bond using a longer sample dataset of daily data from 1962 to 2020. RPP equalizes risk contributions from the various components of the portfolio, that is to say, asset allocator distributes the same risk budget to each component so that none is dominating (at least on an ex-ante basis), which is the main difference from traditional equity dominated portfolios. Based on the extended data, our empirical results show that outperformance of RPP to some extent relies on falling interest rate economy. Compared with traditional 60/40 portfolio, RPP has larger VaR, ES and drawdown, in other words, larger tail risk in rising interest rate economy, and the situation gets even worse when faced with recession or positive assets correlation periods. However, in falling interest rate economy, RPP realizes higher excess return and Sharpe ratio and takes less risk in recessions.

Theoretical construction of RPP shows that the actual volatility of RPP with two assets ($\tilde{\sigma}$) could be denoted by target volatility (σ_{target}), the average ratio between actual and predicted volatility of assets (π), the heterogeneity of ratio between actual and predicted volatility of assets (λ), the predicted correlation between assets ($\rho_{1,2}$) and actual correlation between assets ($\tilde{\rho}_{1,2}$), specifically, $\tilde{\sigma} = \sigma_{target} \times \pi \times \sqrt{\frac{\lambda+\tilde{\rho}_{1,2}}{1+\rho_{1,2}}}$. From this equation, we get an interesting finding that RPP tends to have a higher actual volatility than target one ($\tilde{\sigma} > \sigma_{target}$) even if we predict the average volatilities of assets and correlation correctly (π=1 and $\rho_{1,2} = \tilde{\rho}_{1,2}$), because the heterogeneity of ratio between actual and predicted volatility of assets (λ) is greater than 1 in most cases[11], so in most cases, risk contributions of both assets will not be equal, therefore, risk parity funds are hard to equalize risk contribution in practice. This conclusion

[11] $\lambda = \frac{\kappa+\frac{1}{\kappa}}{2} \geq 1$, where $\kappa = \frac{\tilde{\sigma}_1\sigma_2}{\sigma_1\tilde{\sigma}_2}$, $\tilde{\sigma}_i$ is actual volatilities of assets, σ_i is predicted volatilities of assets

is evidenced by our empirical results that portfolio volatility is larger the target one for most of time periods and average portfolio volatility is slightly larger than target one, also, risk contributions of two assets are not equal and S&P 500 on average contributes slightly more. In extreme cases, when volatility shoots up, our usage of historical volatility as predicted ones results in that actual portfolio volatility can be triple the target one, besides, RPP tends to have longer look-back periods and rebalance intervals, making the extreme case even worse. Our empirical results also show that bond have more extreme risk contributions, causing extreme portfolio volatility. In this paper, we use the historical volatilities and correlation as the predicted ones, other prediction methods to construct RPP remains an interesting open question.

APPENDIX

A The treasuries used to construct the LTR

Period	Coupon (%)	Call/Maturity Date
1961-1965	4.250	5/15/1985
1966-1972	4.250	8/15/1992
1973-1974	6.750	2/15/1993
1975-1976	8.500	5/15/1999
1977-1980	7.875	2/15/2000
1981	8.000	8/15/2001
1982	13.375	8/15/2001
1983	10.750	2/15/2003
1984	11.875	11/15/2003
1985	11.750	2/15/2010
1986-1989	10.000	5/15/2010
1990-1992	10.375	11/15/2012
1993-1996	7.250	5/15/2016
1997-1998	8.130	8/15/2019
1999-2001	8.130	8/15/2021
2002	6.250	8/15/2023
2003-2004	7.500	11/15/2024
2005	6.875	8/15/2025
2006	6.750	8/15/2026
2007	6.375	8/15/2027
2008	5.500	8/15/2028

(Continued)

Period	Coupon (%)	Call/Maturity Date
2009	5.250	2/15/2029
2010-2012	5.375	2/15/2031
2013-2016	4.500	2/15/2036
2017	5.000	5/15/2037
2018	4.500	5/15/2038
2019	4.500	8/15/2039
2020	4.625	2/15/2040

B formulations for risk contribution

Because the sum of actual risk contributions of assets should be equal to the actual volatility of portfolio (see equation 3), with equations 18 and 19, the actual risk contribution of assets should be

$$\text{risk contribution of asset 1} = \frac{\sigma_{target}^2 \times \frac{\pi}{1+\rho_{1,2}} \times \frac{(\kappa+\tilde{\rho}_{1,2})}{2}}{\tilde{\sigma}} \tag{B.1}$$

$$\text{risk contribution of asset 2} = \frac{\sigma_{target}^2 \times \frac{\pi}{1+\rho_{1,2}} \times \frac{(\frac{1}{\kappa}+\tilde{\rho}_{1,2})}{2}}{\tilde{\sigma}} \tag{B.2}$$

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