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Investigating the merits of using a Kalman Filter in equity Beta estimation

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"Investigating the merits of using a Kalman Filter in equity Beta estimation"

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1 Abstract

The concept of using alpha and beta to gauge the validity of an investment has been used extensively in both an academic as well as a professional setting. Despite its extensive use, the method of obtaining the alphas and betas estimates is deceptively simple and makes a number of unrealistic assumptions. This report aims to investigate the merits of using a Kalman Filter in equity beta and alpha estimation and thereby circumvent some of the issues of the more traditional approach. Final results show that the error improvements are non-existent or marginal at best. However, while the merits of the Kalman filtering technique is lackluster in this report, it makes a strong case further analysis into the area is warranted.

2 Introduction and Motivation

2.1 Beta and the Capital Asset Pricing Model

The pioneering work of Dr. Harry Max Markowitz in modern portfolio theory started a whole new area of financial innovation. Building on his work, the Capital Asset Pricing Model (CAPM) was developed in the mid 60's by W. F. Sharp, and has since become one of the most iconic models in finance. Relatively simple in nature, the theory elegantly separates investment risk into two key components: systematic and un-systematic risk. The systematic risk, often labeled as "market risk", measures the stocks' correlation to the market. Unsystematic risk is firm specific and, in contrast to systematic risk, is deemed diversifiable as it can be mitigated by combining multiple assets. Since unsystematic risk can be diversified away, it is not expected to yield any return. CAPM can be summarized with the following formula (Francis, Dongheol, 2013):

$$r_i = r_f + \beta_i(r_m - r_f) \text{ (Equation 1)}$$

Where:

- r_i = Return of stock i
- r_f = Risk free return
- r_m = Return of the market
- β_i = Beta of stock i

It is the β in equation 1 that is of particular interest in this report. β represents the systematic risk of the stock and is traditionally calculated using the following formula.

$$\beta_i = \frac{\sigma_{\mathrm{i,M}}}{\sigma_M^2} \qquad \text{(Equation 2)}$$

Where:

- $\sigma_{i,m}$= Covariance between asset i and the market
- σ_M^2 = Variance of the market portfolio return

Despite its almost archaic status, the concept of market risk, represented by beta, is still extensively used in both an industry and academic setting. Equity Betas are at the center of financial theory and are embedded in famous models such as the Market Model and Modern Portfolio Theory (MPT) ; models that are extensively used as investment allocation tools.

Their use does not end with asset allocation strategies however; Betas are also used in the calculation of the Weighted Average Cost of Capital (WACC) within risk management and corporate finance. Corporations use WACC as a discount factor for potential investment projects. With these applications and others, the importance of Beta in the financial markets cannot be understated (Fama, French 1997).

2.2 Shortcomings of Beta and the Capital Asset Pricing Model

The market model, similar in nature to the CAPM, can be summarized with the following equation:

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i \quad \text{(Equation 3)}$$

Where:

- α_i = alpha of stock i
- ε_i = non-systematic risk

Of particular interests is the α term of the equation, which represents the return of stock i not explained by its correlation to the market. For CAPM to hold we combine the two equations and see that:

$$\alpha_i = \beta_i(1 - r_f) \qquad \text{(Equation 4)}$$

However, this equation often does not hold in observed data and many papers have emerged that argue the traditional market Beta fails to fully explain expected stock returns and variance. Many reasons have been given for this, such as (but not limited to): not all investors having equal access to information and equal costs of capital, the model not taking into account taxes and transaction costs and the model not taking the risk of illiquidity into.

Alternative models have been proposed to explain the shortcomings of CAPM, with some gaining widespread academic merit. The famous Fama-French 3-factor model postulates that two additional factors are necessary to fully explain expected stock returns: the outperformance of small versus big companies, and the outperformance of high book /market value versus small book/market value companies. However, each of these models has their own shortcomings. In the case of the Fama-French 3-factor model it is argued that it is empirically inspired and lacks strong theoretical foundations (Fama and French 1992).

The strong academic evidence for the existence of alpha (above market returns) has spearheaded the concept of value investing, where fund managers would use their alleged skills to generate above market returns for their investors. Adherents to the efficient market hypothesis cast doubt on the claims of these fund managers, and empirical analysis of these funds' performance seems to underline this point.

2.2.1 Dynamic Beta

In this paper we make the case for a more fundamental reason behind the shortcomings of the traditional capital asset pricing model: a dynamic Beta. Indeed, the traditional formula for Beta (represented by equation 2) assumes stationarity. However, there is no indication that Beta should not vary over time, and intuition would tell us otherwise. Many factors affecting a stock's correlation to the market change over time, such as leverage levels, changing market conditions and changes to the operations of a company.

Numerous studies have emerged claiming that a time-invariant beta is insufficient to explain the returns and volatility of stocks. As an example, Brook, Faff and Lee (1992), found that there is strong evidence of a dynamic Beta in the Australian equity market and found that a random-coefficient model best describes the variability of Beta.

3 Literature Review

Despite its sophistication and popularity in the engineering world, the difficulty of describing econometric concepts as a state-space representation means that the Kalman filter is relatively uncommon in financial academic literature. However, the latest advances in financial theory have allowed for the Kalman filter to be explored further. As an example, Duan and Simonato (1999) build on the model of Vasicek (1977) and Dothan (1978) and modeled the interest rate term structure as a diffusion process. This allows for the term structure to be represented as a state-space model and for the Kalman filter to be applied with some moderate success.

Kliestik and Spuchlakova (2016) provide the theoretical framework for using a Kalman filter in estimating Beta coefficients and conclude that the filter is optimal for a linear model subject. This theory is explored further in Renzi-Ricci (2016) where a dataset is artificially generated and consists of 1000 noisy data points, with a sudden jump in Beta value from 3 to 6 halfway through the data. An OLS regression on the entire data series, a rolling window OLS regression and a Kalman filter model are all used to explore their ability to accurately predict the sudden jump and its corresponding Beta value. The report demonstrates that the Kalman filter is superior to all other models in accurately finding the correct Beta and is surprisingly accurate despite the noisy data.

The Kalman filter has also demonstrated its merits in empirical data. Choudhry and Wu (2009) compare the Kalman filter to three different Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models for Beta estimation of UK firms. Measures of forecast errors overwhelmingly support the Kalman Fitler approach. A similar experiment is proposed in Lie, Brooks and Faff (2000) where the GARCH and Kalman filter model are compared for Australian financial companies. The conclusions drawn are similar to the UK company case, with the Kalman filter providing the most accurate forecast of equity Betas.

Das and Ghoshal (2010) applies the filter to empirical data from the Indian security market and uses the adaptive Kalman filter in order to estimate the measurement noise covariances which are now assumed to be dynamic instead of static and known beforehand. A RMSE analysis of the results strongly supports the merits of the adaptive Kalman filter in accurately predicting equity Betas. The results had the strongest statistical significance at a window size of 300 time steps for measurement noise covariance estimations.

4 Theory

4.1 The Kalman Filter Model

In 1960 R.E. Kalman published a paper describing a recursive solution of the discrete-data linear filtering problem, later name the Kalman filter. The theory will be briefly explained in this section.

We assume the random process to be estimated can be modeled in the following form (Brown, Hwang 1997):

$$x_{k+1} = \phi_k x_k + w_k \quad \text{(Equation 5)}$$

Where:

- x_k = (n x 1) process state vector at time k
- Φ_k = (n x n) state transition matrix
- w_k = (n x 1) error term (assumed to be white noise with mean 0 and non-zero standard deviation)

The measurements of the process are assumed to occur at discrete points in time in accordance with the following formula:

$$z_k = H_k x_k + v_k \qquad \text{(Equation 6)}$$

Where:

- z_k = (m x 1) measurement vector at time k
- H_k = (m x n) matrix giving the ideal (noiseless) relation between the measurement and the state vector at time k
- v_k = (m x 1) error term (assumed to be white noise with mean 0 and non-zero standard deviation)

Both error terms (w_k and v_k) are assumed to have no autocorrelation and zero cross-correlation. As such:

$$E[w_k w_i^T] = \begin{cases} Q_k, & i = k \\ 0, & i \neq k \end{cases} \qquad \text{(Equation 7)}$$

$$E[v_k v_i^T] = \begin{cases} R_k, & i = k \\ 0, & i \neq k \end{cases} \qquad \text{(Equation 8)}$$

$$E[w_k v_i^T] = 0, for\ all\ k\ and\ i \qquad \text{(Equation 9)}$$

We then assume we have an initial estimate about the process at some time k, which is based on all our knowledge about the process prior to k. This "*a priori"* estimate is denoted as $\hat{x}_k^-$. We then define the estimation error to be:

$$e_k^- = x_k - \hat{x}_k^-$$

With the associated covariance matrix being:

$$P_k^- = E(e_k^- e_k^{-T}) \qquad \text{(Equation 10)}$$

Having a prior estimate of our state vector, we now use the noisy measurement to improve the *a priori* estimate, generating an "*a posteriori*" estimate. In order to find this new estimate we use a linear combination of the data and the *a priori* estimate:

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H_k\hat{x}_k^-) \qquad \text{(Equation 11)}$$

Where:

- K_k is = The Kalman gain at T=k

The Kalman gain is optimized such that the resulting distribution has a minimized mean-square error. As such, we first form an expression for the error covariance matrix associated with the updated (*a posteriori*) estimate.

$$P_k = E[e_k e_k^T] = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \qquad \text{(Equation 12)}$$

We then substitute equations 6 and 11 into equation 12, rearrange, and write the equation out in matrix form, yielding:

$$P_k = (I - K_k H_k)P_k^-(I - K_k H_k)^T + K_k R_k K_k^T \qquad \text{(Equation 13)}$$

We then find K_k that minimizes the above expression by differentiating equation 13 and setting the result to 0, yielding:

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \qquad \text{(Equation 14)}$$

Inserting equation 14 equation 13 and rearranging yields:

$$P_k = (I - K_k H_k)P_k^- \qquad \text{(Equation 15)}$$

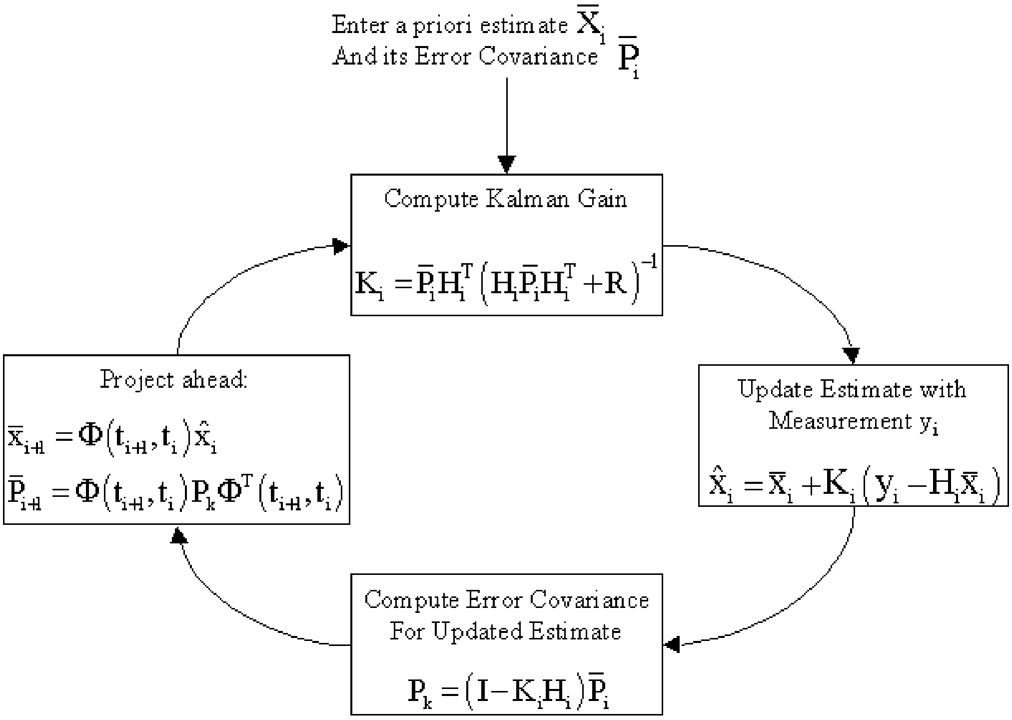

Figure 1 Kalman Filter Loop

The theory described above demonstrates how the Kalman filter can solve some of the issues raised in section 2.2. The feedback loop of the Kalman filter (figure 1) demonstrates the dynamic nature of the model. After updating the *a priori* distribution to the *a posteriori* distribution upon receiving a measurement, the *a posteriori* distribution becomes the new *a priori* distribution. We can then continue updating the distribution when new measurements become available.

The decision to opt for a Kalman filter as opposed to other filtering techniques such as the Wiener filter (which also allows for a dynamic Beta and minimizes the mean-square error of the distribution) is due to the fact that the Kalman filter is recursive. This saves much computational power and allows us to potentially use a much larger dataset.

5 Methodology

5.1 The Data

This report will focus on calculating the dynamic beta values of the 20 largest companies on the S&P500. The S&P500 was chosen over other major benchmarks such as the FTSE250 and EUROSTOXX 50 due to the homogeneity of the US market. Other major benchmarks feature companies headquartered in different parts of the world and therefore introduce unnecessary exogenous factors to the model. The relevant companies are listed in Table 1.

Rank	Company	Weight	Rank	Company	Weight
1	Apple Inc.	3.81	11	Bank of America Corporation	1.26
2	Microsoft Corporation	2.87	12	Wells Fargo & Company	1.18
3	Amazon.com Inc.	2.16	13	Chevron Corporation	1.06
4	Facebook Inc. Class A	1.89	14	Procter & Gamble Company	0.97
5	Berkshire Hathaway Inc. Class B	1.68	15	Home Depot Inc.	0.96
6	Johnson & Johnson	1.65	16	AT&T Inc.	0.95
7	JPMorgan Chase & Co.	1.63	17	UnitedHealth Group Incorporated	0.92
8	Exxon Mobil Corporation	1.56	18	Pfizer Inc.	0.92
9	Alphabet Inc. Class C	1.41	19	Visa Inc. Class A	0.92
10	Alphabet Inc. Class A	1.40	20	Verizon Communications Inc.	0.90

Table 1 List of companies used in the dynamic beta analysis

The share price data itself will be extracted using Bloomberg and exported as a CSV. Just like the S&P500, dividends are assumed to be reinvested instantaneously. All the data will be taken from 03-Jan-2000 onwards. Returns will then be calculated predominantly on a monthly basis and a time series of each asset will be constructed. A smaller subsection of this report will use daily and weekly return data.

The Alphas and Betas of each stock will then be calculated at each point along the time series using both the traditional Market Model, as well as the newly introduced Kalman Filter model. The predictive capacity of each alpha and Beta of then be measured using Root Mean Square Errors (RMSE), Mean Absolute Errors (MAE) and Mean Errors (ME) .

5.2 Calculating the Alphas, Betas and Errors

5.2.1 The Market Model

The default model to which our proposed Kalman Filtering model will be compared is the traditional Market Model. Given a time series of returns of size n, the beta of stock i is calculated using the following formula:

$$\beta_i = \frac{COVAR(R_i,R_m)}{VAR(R_m)} = \frac{\sum_{t=0}^{n}\left(r_{i,t}-E(R_i)\right)*\left(r_{m,t}-E(R_m)\right)}{\sum_{t=0}^{n}\left(r_{m,t}-E(R_m)\right)^2} \qquad \text{(Equation 16)}$$

Where:

- $E(R_i)$ = Expected return of asset i

We can then use beta to calculate alpha for a given time series of size n:

$$\alpha_i = \frac{\sum_{t=0}^{n} r_{i,t} - \sum_{t=0}^{n} \beta_i * r_{m,t}}{n} \qquad \text{(Equation 17)}$$

5.2.2 The Kalman Filtering Model

5.2.2.1 The measurement and state transition equations

As explained in the theory section, the basis for a Kalman Filter model is a state transition equation and a measurement equation. The measurement equation:

$$z_k = H_k x_k + v_k$$

Can be represented using the Market Model:

$$r_{i,t} = \alpha_{i,t} + \beta_{i,t} * r_{m,t} + \varepsilon_i \qquad \text{(Equation 18)}$$

Where:

- $z_k = r_{i,t}$
- $H_k = (1 \quad r_{m,t})$
- $x_k = \begin{pmatrix} \alpha_{i,t} \\ \beta_{i,t} \end{pmatrix}$
- $v_k = \varepsilon_i \sim N(0, \sigma_\varepsilon^2)$

Both the alpha estimate and the beta estimate are assumed to follow a simple random walk model, where the best estimate of alpha and beta at time T+1 is the estimates we calculated at time T. The state transition equation:

$$x_{k+1} = \phi_k x_k + w_k$$

Can be rewritten as two random walk models;

$$\alpha_{i,T+1} = \alpha_{i,T} + u_T \quad \text{(Equation 19)}$$

$$\beta_{i,T+1} = \beta_{i,T} + z_T \quad \text{(Equation 20)}$$

Where:

- $u_T \sim N(0, \sigma_u^2)$
- $Z_T \sim N(0, \sigma_z^2)$
- $\phi_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $w_k = \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix}$

5.2.2.2 The Initial Conditions

5.2.2.2.1 Estimating Q_k and R_k

The Kalman filter model requires prior estimates of both error terms, Q_k and R_k; equations 7, 8 and 9 relate the error terms to the other equations. This report relies on the traditional Market Model to calculate the error terms.

Using equation 6 and 7 we find R_k:

$$R_k = (\sigma_\varepsilon^2)$$

We calculate σ_ε^2 by using the equation for variance:

$$Var(\varepsilon_i) = \sigma_\varepsilon^2 = \frac{\sum_{T=1}^{N} \left(\varepsilon_{i,T} - E(\varepsilon_i)\right)^2}{N-1}$$

Rearranging equation 18 yields:

$$r_{i,t} - \alpha_{i,t} - \beta_{i,t} * r_{m,t} = \varepsilon_i \sim N(0, \sigma_\varepsilon^2)$$

And setting $E(\varepsilon_i) = 0$:

$$\sigma_\varepsilon^2 = \frac{\sum_{T=1}^{N} \left(r_{i,t} - \alpha_{i,t} - \beta_{i,t} * r_{m,t}\right)^2}{N-1} \quad \text{(Equation 21)}$$

Where:

- N represents the amount of data points taken for the Kalman Setup Lookback Period
-

We can therefore calculate σ_ε^2 by calculating the Market Model alphas and betas for N historical time periods and inserting them into equation 21.

Using equation 5 and 8 we find Q_k:

$$Q_k = \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix}$$

We calculate σ_u^2 by using the equation for variance:

$$Var(\varepsilon_i) = \sigma_\varepsilon^2 = \frac{\sum_{T=1}^{N}\left(u_{i,T} - E(u_i)\right)^2}{N-1}$$

Rearranging equation 22 yields:

$$\alpha_{i,T+1} - \alpha_{i,T} = u_T \sim N(0, \sigma_u^2)$$

And setting $E(u_i) = 0$:

$$\sigma_u^2 = \frac{\sum_{T=1}^{N}\left(\alpha_{i,T+1} - \alpha_{i,T}\right)^2}{N-1}$$

Similarly for σ_z^2:

$$\sigma_z^2 = \frac{\sum_{T=1}^{N}\left(\beta_{i,T+1} - \beta_{i,T}\right)^2}{N-1}$$

Once calculated, these error terms are assumed constant throughout the course of the model.

5.2.2.2.2 Prior Estimate and its error Covariance

Similar to Q_k and R_k, we use the Market Model to calculate our initial (T=0) estimates for $\hat{x}_0^-$ and P_0^-. $\hat{x}_0^-$ is estimated by simply taking the Market Model alpha and Beta at time T=0, as the state transition equation assumes a random walk.

P_0^- is estimated by taking the variance and covariance of the alphas and betas over the Kalman intitial conditions time period:

$$P_0^- = \begin{pmatrix} \sigma_\alpha^2 & \sigma_{\alpha,\beta} \\ \sigma_{\alpha,\beta} & \sigma_\beta^2 \end{pmatrix}$$

Where:

- $\sigma_\alpha^2 = \frac{\sum_{T=1}^{N}\left(\alpha_{i,T} - E(\alpha_i)\right)^2}{N-1}$
- $\sigma_\alpha^2 = \frac{\sum_{T=1}^{N}(\beta_{i,T} - E(\beta_i))^2}{N-1}$
- $\sigma_{\alpha,\beta} = \frac{\sum_{T=1}^{N}\left(\alpha_{i,T} - E(\alpha_i)\right) * (\beta_{i,T} - E(\beta_i))}{N-1}$

These initial conditions are then inserted into the Kalman Filter Loop demonstrated in figure 1. As described in the theory section, once these initial conditions have been established, the recursive nature of the Kalman filter allows for the loop to cycle indefinitely without any further additional external inputs.

5.2.3 Error Calculation

Both models will each generate a time series of alphas and betas. These will then be tested by using the observed market return to predict the anticipated security return. The difference between the predicted security return and observed security return will be used as the benchmark for the success of either model.

$$Err_{T,M} = r_{i,T,Pred,M} - r_{i,T,Obs}$$

Where:

- $Err_{T,M}$= Error of model M at time T
- $r_{i,T,Pred,M}$ = Predicted security i return of model M at time T
- $r_{i,T,Obs}$ = Observed security i return at time T

$r_{i,T,Pred}$ is calculated using the alpha and beta of each model at time T-1:

$$r_{i,T,Pred,M} = \alpha_{i,T-1,M} + r_{m,T} * \beta_{i,T-1,M}$$

Where :

- $\alpha_{i,T,M}$ = Alpha estimate for model M at time T-1
- $\beta_{i,T,M}$ = Beta estimate for model M at time T-1

The performance of each model will be determined by observing and comparing the Root Mean Squared Errors (RMSE), Mean Absolute Error (MAE) and Mean Error (ME).

$$RMSE = \sqrt{\frac{\sum_{T=1}^{N} {Err_T}^2}{N}}$$

$$MAE = \frac{\sum_{T=1}^{N} abs(Err_T)}{N}$$

$$ME = \frac{\sum_{T=1}^{N} Err_T}{N}$$

For obvious reasons this report will focus mostly on RMSE and MAE, but ME will also be generated and considered in the analysis.

5.2.4 The Source Code

All relevant calculations in this report have been generated using C++ code. Figure 2 demonstrated the system hierarchy for the source code.

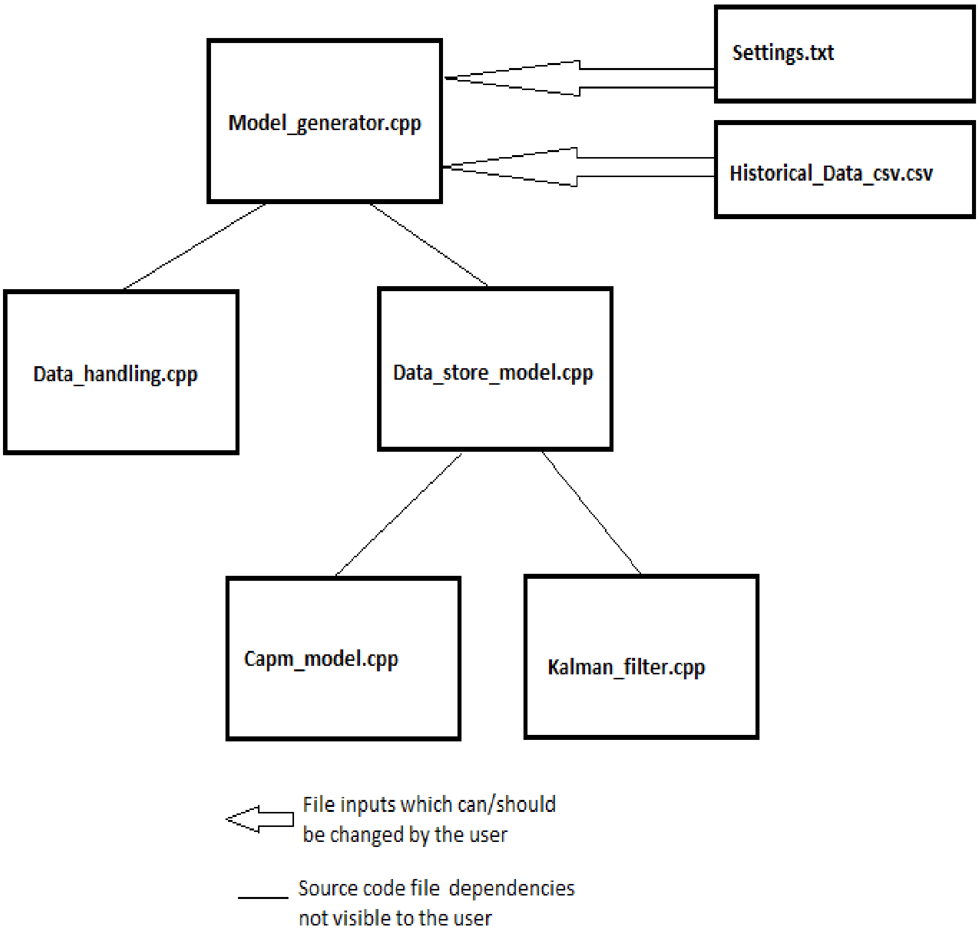

Figure 2 System hierarchy for the source code

A very brief overview of each file will be provided, for further information please consult the attached source code files[1]:

- Model_generator.cpp: The main function of the model generator
- Data_handling.cpp: Responsible for reading the Historical_Data_csv.csv file, extracting the required data and calculating the required returns, list of dates, list of security names…
- Data_store_model.cpp: Responsible for generating the result time series, calls Capm_model.cpp and Kalman_filter.cpp to generate Kalman and Market Model alpha's and beta's. Once the result time series have been generated, the results are exported to a csv file (further information in the next section).

[1] Or email me on Kai.E.Strandmoe@student.bi.no

- Capm_model.cpp: Responsible for generating alpha and beta estimates using the Market Model.
- Kalman_filter.cpp: Responsible for generating alpha and beta estimates using the Kalman Filtering model.

5.2.5 Program Structure

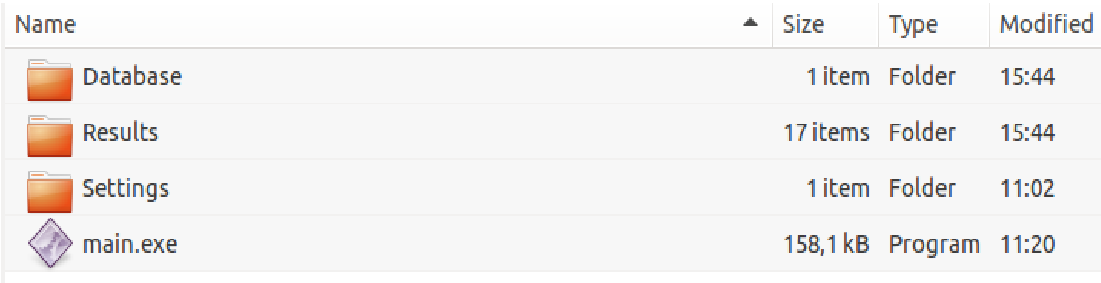

Figure 3 File Layout of Program

The file layout of the program is demonstrated in figure 3. The program is run by simply navigating to the directory where the program was unzipped and launching the .exe file (by typing in "main.exe" if in a Windows OS or by typing "./main.exe" if in a Linux OS).

The settings folders contains a single file labelled "settings.txt". This file sets the required inputs for the model generator and currently features four input parameters:

- CAPM Model Length: Which sets the amount of datapoints to be used when calculating Market Model Alphas and Betas. Note that the units are the same as the units set in the "*Data Frequency*" parameter.
- Kalman Model Estimates Length: Sets the amount of data points to be used when calculating the prior estimates for the Kalman Model
- Alpha Factor: Sets the Alpha factor for the state transition matrix of the Kalman Model. Further information is provided in the results and analysis section
- Data Frequency: Determines whether daily, weekly or monthly data is to be used. 1 stand for daily, 2 stands for weekly and 3 stands for monthly. Default is set to 3.

The Database folder is meant to hold the Bloomberg exported CSV file labelled "*historical_Data_csv.csv*" which holds the raw end of day price data for all the relevant stocks to be analyzed.

The Results folder is where the results of the modelling program are exported to. It will feature a file labelled "*main_results.csv*" which holds the relevant summary statistics. Additionally it will export an "*ASSETNAME results.csv*" file for each of the securities to be analyzed featuring summary statistics of that particular security and a table containing a timeseries of market return, asset return, Market Model alphas and betas, Kalman Filter alphas and betas and their respective errors, absolute errors and squared errors.

Please note that due to time constraints the program is very user unfriendly and any alterations to the database file or settings file need to exactly follow the template provided in order to function correctly[2].

[2] Contact me on Kai.E.Strandmoe@student.bi.no if any problems occur

6 Results and Analysis

6.1 The basic three year historical lookback period

In order to investigate the merits of each model, the residual errors of each model will be compared using RMSE, and occasionally MAE. Instead of comparing each asset individually, the merits of each model as a whole will be tested using the average errors of each of the assets using the following formula:

$$RMSE_{ave} = \frac{\sum_{i=1}^{N} RMSE_i}{N}$$

Where:

- $RMSE_i$ = RMSE of asset i
- Number of assets in the model.

We start of the analysis by calculating the Market Model alphas and betas using 36 data points of monthly returns. This is the standard approach to the Market Model and CAPM and corresponds to three years historical data. Similarly, we will start with 36 data points when calculating the prior estimates for the Kalman Model.

Security	CAPM ME	Kalman ME	CAPM MAE	Kalman MAE	CAPM RMSE	Kalman RMSE
AAPL US Equity	0.38%	0.63%	6.06%	6.02%	8.40%	8.29%
MSFT US Equity	-0.30%	-0.21%	4.38%	4.40%	5.81%	5.86%
AMZN US Equity	-0.94%	-0.67%	7.33%	7.02%	9.84%	9.44%
JPM US Equity	0.20%	0.20%	4.98%	4.91%	7.15%	7.19%
BRK/B US Equity	0.12%	0.13%	3.11%	3.15%	4.51%	4.61%
GOOGL US Equity	-0.14%	-0.01%	4.33%	4.34%	5.94%	6.03%
XOM US Equity	0.00%	0.19%	3.33%	3.28%	4.24%	4.13%
JNJ US Equity	0.06%	0.11%	2.59%	2.58%	3.31%	3.28%
BAC US Equity	0.79%	0.64%	7.86%	7.76%	12.06%	12.02%
INTC US Equity	-0.42%	-0.38%	4.64%	4.68%	5.86%	5.89%
UNH US Equity	0.40%	0.64%	4.98%	5.09%	7.01%	7.11%
V US Equity	0.05%	0.38%	2.86%	3.09%	3.89%	4.00%
WFC US Equity	0.68%	0.49%	4.93%	4.89%	7.71%	7.50%
CVX US Equity	0.00%	0.17%	3.79%	3.70%	4.70%	4.68%
HD US Equity	-0.07%	-0.06%	3.68%	3.67%	5.01%	5.04%
T US Equity	-0.13%	-0.43%	3.71%	3.72%	4.91%	4.94%
PFE US Equity	-0.06%	-0.13%	3.53%	3.54%	4.46%	4.43%
CSCO US Equity	-0.33%	-0.30%	4.49%	4.58%	6.08%	6.12%
Average	***0.02%***	***0.08%***	***4.48%***	***4.47%***	***6.16%***	***6.14%***

Table 2 Error Comparison for Market Model and Kalman Filter Model Alphas and Betas

Table 2 demonstrates that there is virtually no difference in MAE and RMSE between the Market Model (labelled CAPM) and the Kalman model. Note that two securities have been omitted from the table to there not being enough datapoints. These securities are Facebook, Inc Common Stock (FB US Equity) and Alphabet Inc. (GOOG US Equity).

One interesting point to note however is that despite the similar error values, the actual alphas and betas values of both models are strikingly different at times. Figure 4 and 5 demonstrate this effect. Overall both values are highly correlated as expected.

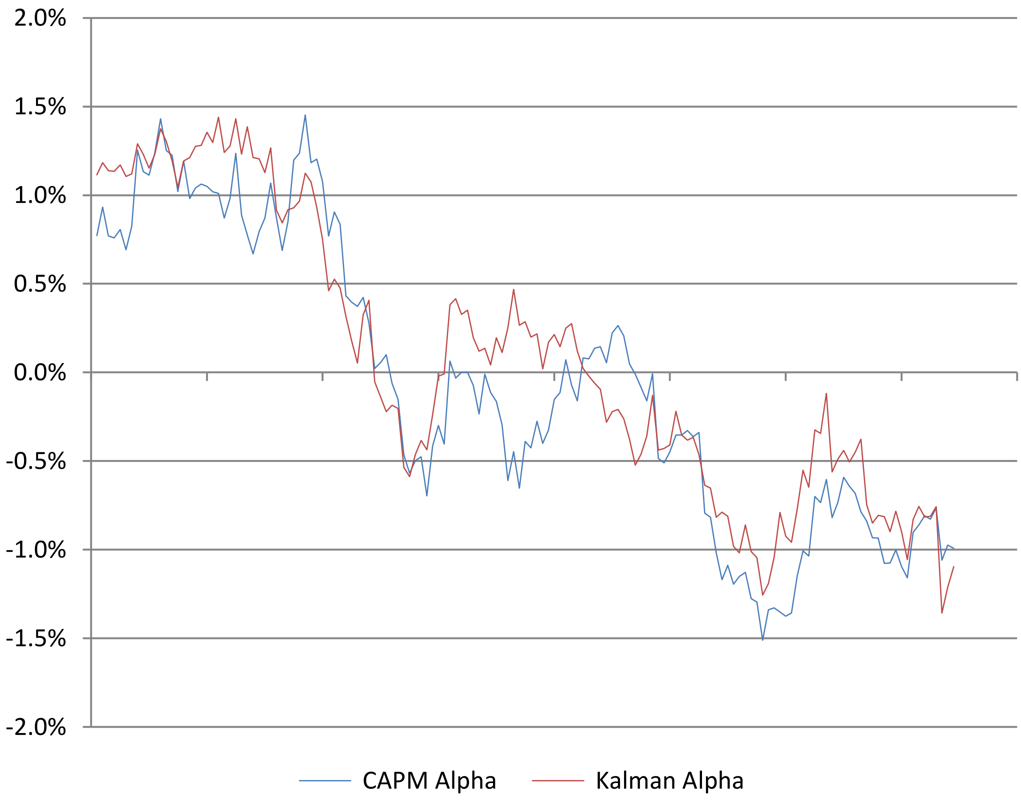

Figure 4 Market Model and Kalman Model Alpha Values for Exxon Mobile Corporation (XOM US Equity) from 2-Feb-2006 to 1-Jun-2018

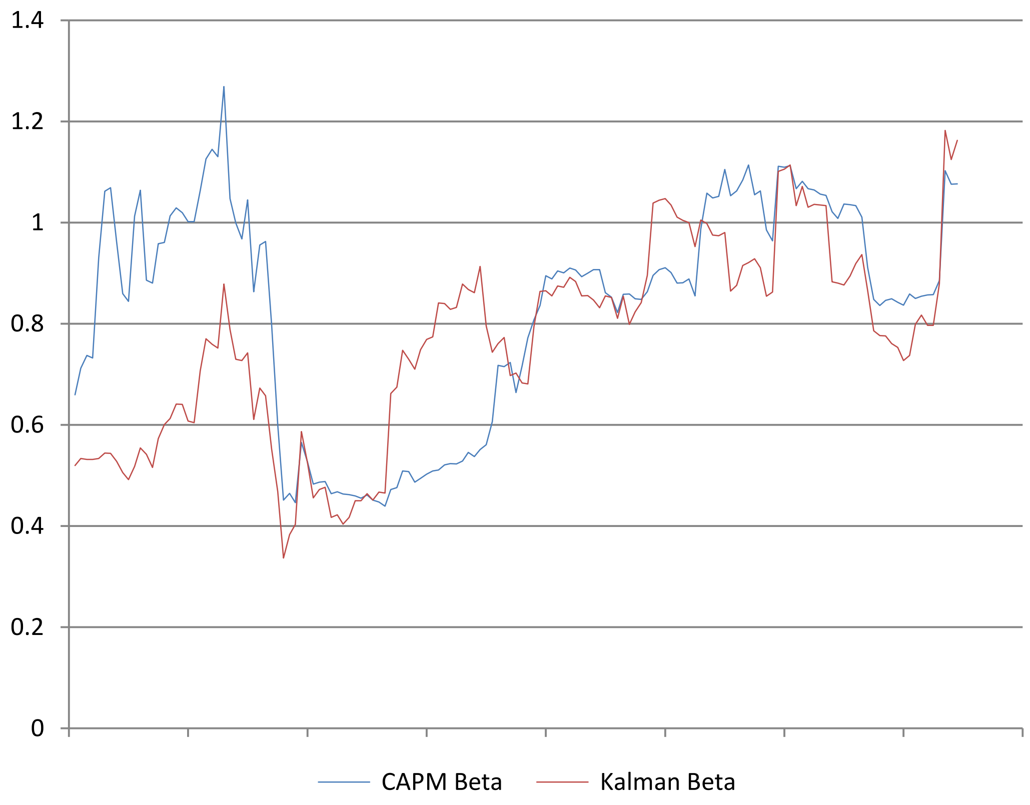

Figure 5 Market Model and Kalman Model Beta Values for Exxon Mobile Corporation (XOM US Equity) from 2-Feb-2006 to 1-Jun-2018

6.2 Lookback period analysis

In this section we will divert from the traditional three year lookback period for CAPM calculations and Kalman prior estimations and observe the effect on model performance.

Figure 7 demonstrates that the Kalman lookback period does not matter much with regards to the RMSE values. This is as expected however, as the Kalman filter is supposed to eventually converge to its equilibrium regardless of the initial prior estimates.

Figure 6 however demonstrates that the CAPM length does seem to matter somewhat. Mainstream literature focuses on 36 data points for CAPM as its values simply become too volatile if too few data points are taken. Decreasing the CAPM lookback period therefore makes the Market Model alphas and betas more volatile which in term enhance the error terms of the Kalman Model. Large error terms decreases the impact of the Kalman loop as as a result decreases the accuracy of the Kalman model. The trend line equations in figure 6 and 7 demonstrate that reducing the CAPM lookback period has an effect that is five times more pronounced than changing the Kalman lookback period.

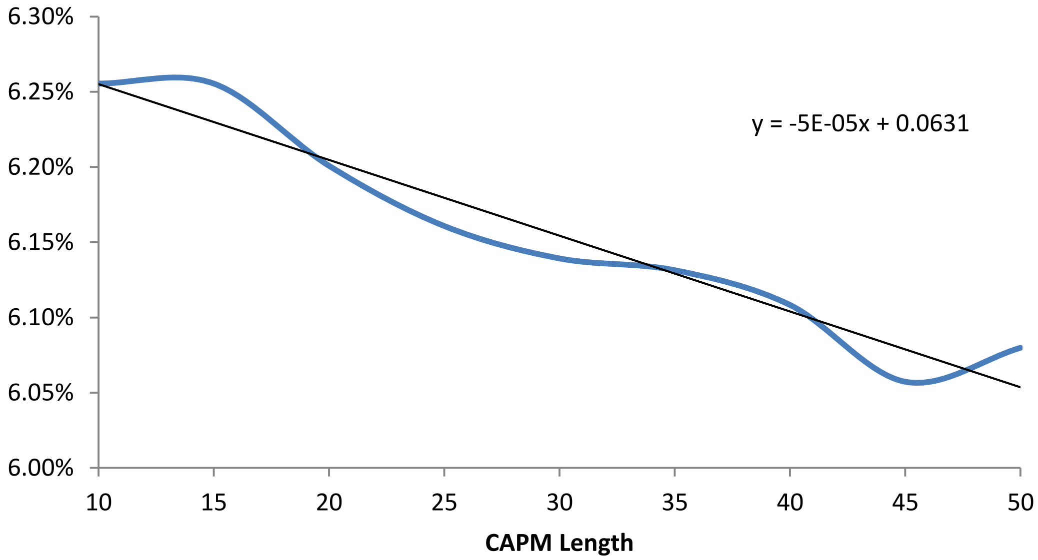

Figure 6 Average RMSE for selected assets keeping Kalman length constant at 35 data points

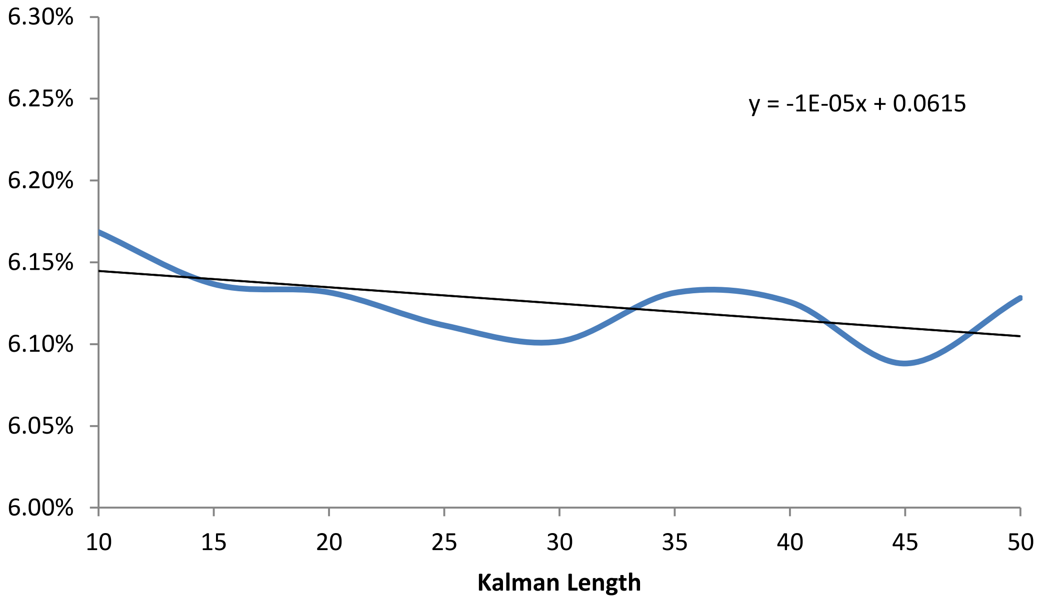

Figure 7 Average RMSE for selected assets keeping CAPM length constant at 35 data points

6.3 State transition analysis

Rewrite equation 20 as new equation.

One very interesting aspect of the Kalman filter not yet explored is that it features a state transition matrix. The vast majority of financial theory stipulates that non-zero alpha should be a temporary phenomenon as the market is expected to smoothen out any outperforming stocks. We could therefor try to improve upon the existing model by replacing the random walk of the alpha state transition matrix (Equation 19) with an AR(1) model of varying factor loading:

$$\alpha_{i,T+1} = FL_{\alpha} * \alpha_{i,T} + u_T$$

Where:

- FL_{α} = Alpha factor loading of the state transition equation.
  Conventional market hypothesis expects this value to between 0 and 1

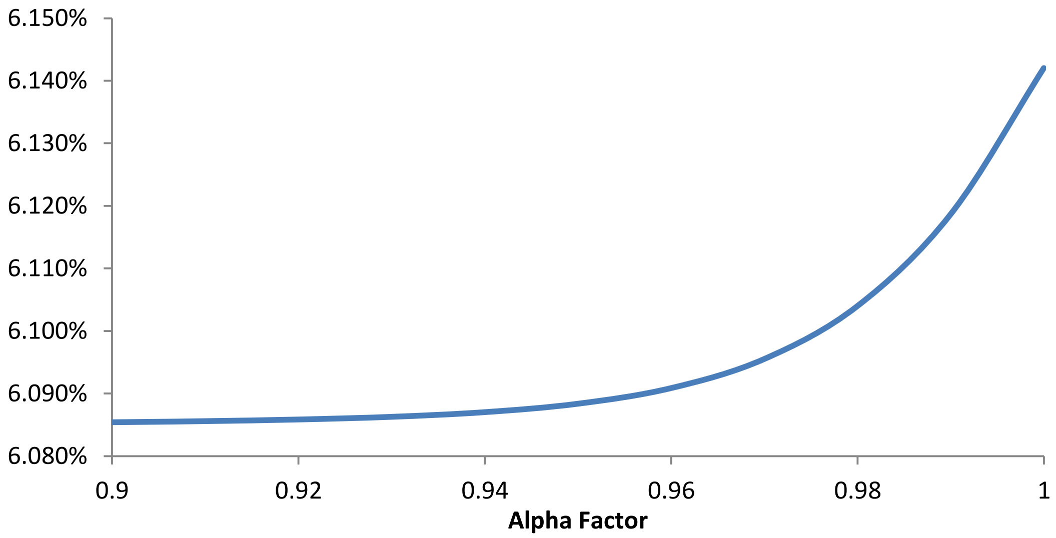

Figure 8 Average RMSE for selected assets with varying Alpha Factor

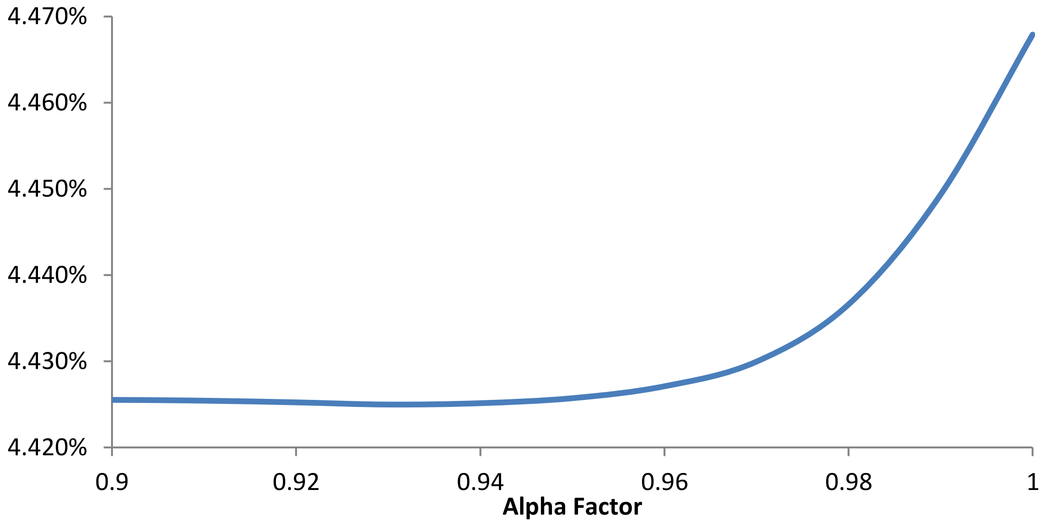

Figure 9 Average MAE for selected assets with varying Alpha Factor

Observing figure 8 we can observe see a decreasing RMSE trend for a decreasing alpha factor loading. The effect seems to plateau at around 0.93. Observing figure 4, which demonstrates a similar effect using MAE shows a very small local minima at an alpha factor loading of 0.93. Figures 8 and 9 demonstrate the benefits of investigating the state transition matrix of the Kalman Model.

6.4 Data Frequency Analysis

The CAPM model traditionally uses monthly return data due to the fact that any shorter time period produces values that are too noisy. One of the predominant reasons for using the Kalman filter, both in financial as well as in other scientific and engineering applications, is that it is meant to filter out noise more effectively. The Kalman filter could therefore potentially allow for daily or weekly data to be analyzed.

The model will use a CAPM lookback period of 150 data points for the daily returns, roughly equating to half a year worth of data. The prior alpha, beta and error estimates of the Kalman filter are then calculated using 50 Market model data points.

The weekly data CAPM lookback period will in turn use 52 data points, roughly equating to a year worth of data. The prior alpha, beta and error estimates of the Kalman filter are then calculated using 26 Market model data points, roughly equating to half a year worth of data.

Frequency	CAPM MEA	Kalman MEA	CAPM RMSE	Kalman RMSE
Daily	*0.93%*	*0.92%*	*1.50%*	*1.49%*
Weekly	*2.08%*	*2.08%*	*3.10%*	*3.10%*
Monthly	*4.48%*	*4.47%*	*6.16%*	*6.14%*

Table 3 Average Errors for selected assets for daily, weekly and monthly data frequency

Unfortunately, table 1 demonstrates that there is virtually no difference between the error terms of both models for daily and weekly data. We can conclude that the daily and weekly data points are simply too noisy to yield any meaningful difference for the Kalman model in its current state.

6.5 Final Observations

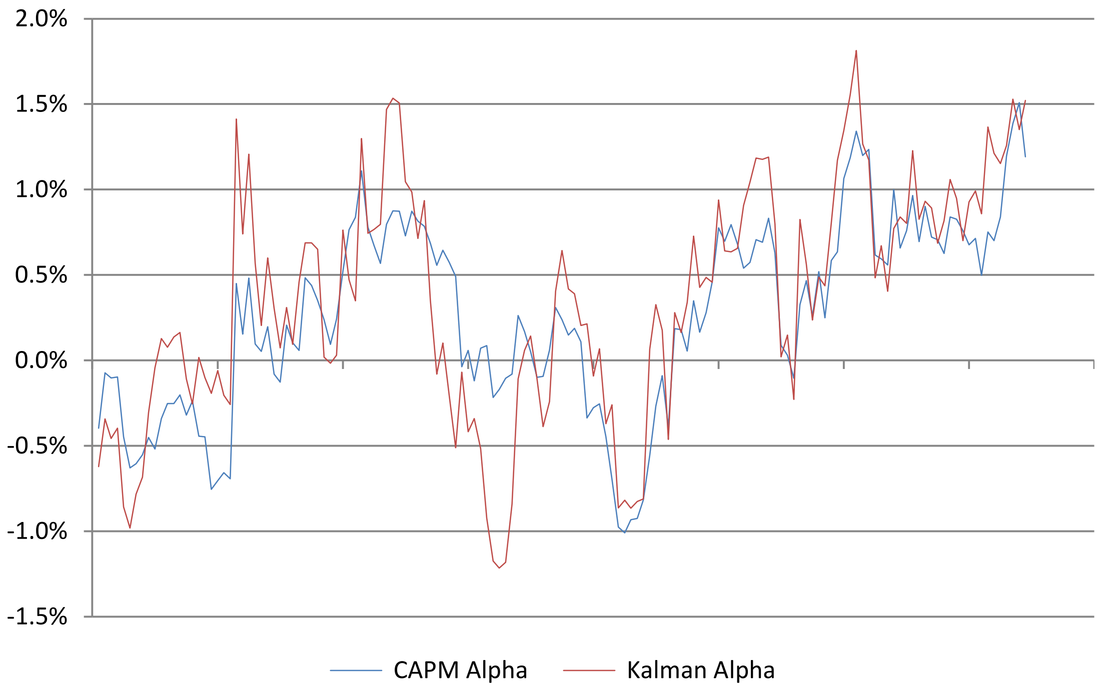

Figure 10 Market Model and Kalman Model Alpha Values for Microsoft Corporation (MSFT US Equity) from 2-Feb-2006 to 1-Jun-2018

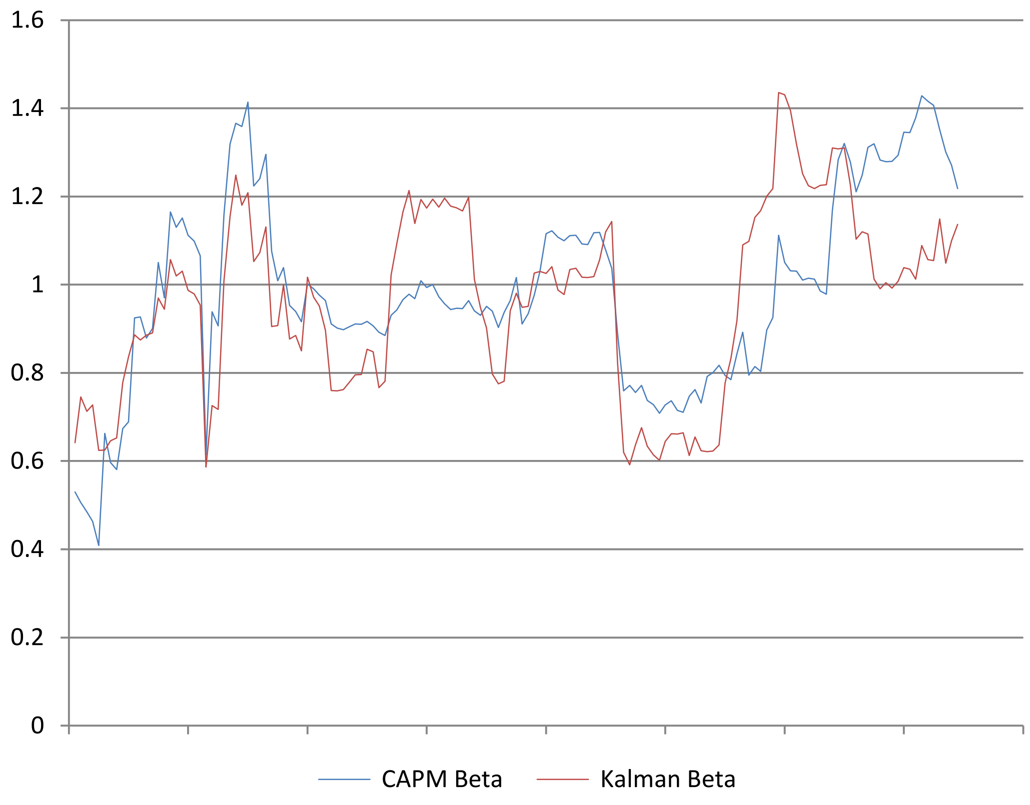

Figure 11 Market Model and Kalman Model Beta Values for Microsoft Corporation (MSFT US Equity) from 2-Feb-2006 to 1-Jun-2018

A point briefly touched in the first part of this section is that despite the relatively similar error values, the Market model and Kalman model yield strikingly different alphas and betas at certain time periods. This observation holds true for all securities analyzed in this report and another example is demonstrated in figures 10 and 11. This effect is more pronounced during times of crisis, as all figures 4, 5, 10, 11 show the biggest divergence during the flash crash of 2011.

7 Conclusion and Potential Improvements

Investigating the merits of using a Kalman Filter in equity Beta and Alpha estimation across four areas has on one occasion yielded a noticeable error improvement but overall the performance of the model when compared to the traditional CAPM model was modest at best. This report has therefore not conclusively demonstrated the merits of the Kalman model in alpha and beta estimation, however it has demonstrated to be a promising area of further research. The potential of varying the state transition matrix is an area that proved particularly promising and a more in-depth analysis of alpha and beta auto-regressive models (asset or sector-specific) has the potential to significantly improve the Kalman model. For example, the beta state transition equation could be replaced with a different mean-reverting equation:

$$\beta_{i,T+1} = \mu + FL_{\alpha} * \beta_{i,T} + z_T$$

Where:

- μ = the long term beta equilibrium value of asset i.

Overall, all areas investigated in this report could benefit from further investigation, including the optimal lookback periods for the CAPM model. The CAPM lookback period changes the performance of the Kalman model because it affects prior error estimates (Q_k and R_k). Alternative ways of estimating these errors could therefore improve the overall model. One particularly interesting area in this regard is to allow for the error terms to periodically update. In the current Kalman model the error estimates are estimated at the beginning of the model, and then assumed to be constant throughout the entire forecast period (usually running between 11 and 12 years). These errors are however likely to change over that long time horizon and periodically updating the estimates might make the model more accurate.

The report has not only demonstrated the case for further academic investigation, the different alpha and beta values during times of crisis indicates that the model would be an interesting area of research for alternative portfolio strategies.

8 References

[1] Jack Clark Francis and Dongheol Kim "*Modern Portfolio Theory : Foundations, Analysis, and New Developments*", (Jan 2013)

[2] Eugene F. Fama and Kenneth R. French, "Industry cost of Equity", *Journal of Financial Economics* 43, (1997), p153-193

[3] Eugene F. Fama and Kenneth R. French, "The cross-section of expected stock returns", *Journal of Finance* 47, (1992), p427-465

[4] Brooks R, Faff R, Lee J "The form of time variation of systematic risk: Some Australian evidence." *Applied Financial Economics 2*, (1992), p191–198

[5] Robert G. Brown and Patrick Y.C. Hwang, "*Introduction to Random Signals and Applied Kalman Filtering, with Matlab Exercices and Solutions"*, (1997), p-215-232

[6] Jin-Chuan Duane, Jean-Guy Simonato, "Estimating and Testing Exponential-Affine Term Structure Models by Kalman Filter", *Review of Quantitative Finance and Accounting.*" New York Vol. 13, Iss. 2, (Sep 1999): p111-135.

[7] T. Kliestik and E. Spuchlakova, "Alternative Methods of the Beta Coefficient Estimation", *International Journal of Trade, Economics and Finance*, Vol. 7, No. 5, (October 2016)

[8] Giulio Renzi-Ricci, "Estimating Equity Betas: What Can a Time-Varying Approach Add? A Comparison of Ordinary Least Squares and the Kalman Filter", *Nera Economic Consulting*, (July 2016)

[9] Taufiq Choudhry and Hao Wu, "Forecasting the weekly time-varying beta of UK firms: GARCH models vs. Kalman filter method", *The European Journal of Finance*, volume 15 Issue 4, (2009)

[10] F. Lie, R. Brooks and R. Faff, "Modelling the equity beta risk of Australian financial sector companies", *Australian economic papers*, (2000)

[11] Atanu Das and Tapan Kumar Ghoshal, “market Risk Beta Estimation using Adaptive Kalman Filter”, *Atanu.Das et. al. / International Journal of Engineering Science and Technology Vol. 2(6)*, (2010), p1923-1934

[12] Mohinder S. Grewal and Angus P. Andrews, “*Kalman Filtering: Theory and Practice, Using Matlab*”, 2nd Edition, A Wiley-Interscience Publication, (2001)

[13] R.E. Kalman, “A new approach to linear filtering and prediction problems”, *Journal of basic Engineering*, (1960)

[14] Helmut Lutkepohl (1993). *Introduction to Multiple Time Series Analysis*. Germany: Springer-Verlag.

[15] Peyton Z. Peebles, Jr. (1993). *Probability, Random Variables and Random Signal Principles*. 3rd ed. United States of America: McGraw-Hill, Inc.

[16] Andrew C. Harvey (1991). *Forecasting, structural time series models and the Kalman filter*. 3rd ed. Great Britain: Cambridge University Press.