BI NORWEGIAN BUSINESS SCHOOL

-GRA 1952 Preliminary Master Thesis-

-Option Pricing using fraction Brownian motion-

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Contents

1. Introduction

There are several ways to manage risk. Operations in the financial markets follow the oldest and simplest Fundamental analysis. If a stock rises, one seek out the cause by studying the company behind it, or the economy in general. Further, one would study the company's expectations and its corresponding market and try to predict whether the stock should go up or down. Prices adjust to events or facts that change the outlook of the company. Being able to know the cause of an event could make you able to forecast the effects and manage the risk they imply.

Information is however difficult to interpret, and the cause effect may be obscure. Information can be concealed, it can be impossible to come by or easily misrepresented. There could be information implying events are about to happen, but still they may not. The cause to effect relationship might not be consistent. How does one make a decision in such cases of uncertainty?

Investors who have invested in a risky asset will deal with this uncertainty by hedging their position to reduce the possible downside. One way to achieve this is through buying derivatives in the market. The derivatives market has been rapidly growing over the last two decades in both advanced and emerging markets; both exchange traded and OTC. This is true across all underlying classes, including equity, currency, interest-rate etc.

Because of this increasing popularity, pricing derivatives correctly has become even more important, as mispricing can have grave consequences. In 1998, Long Term Capital Management suffered a huge loss using the Black-Scholes model to hedge their investments. Following this strategy, they occurred a loss which amounted to 4 billion dollars leading to a government bailout, until the fund declared bankruptcy in 2000. Alas, this made it evident that the current models are elegant, but flawed. To understand how the price modelling works and how it could fail in such a way, one must start with the history and development of the pricing theory.

Development of the options modelling:

Current models are based on the theories first explained by Louis Jean-Baptiste Alphonse Bachelier in his PhD paper *Théorie de la spéculation* published and defended in 1900. Samuelson (1964) further developed the model to circumvent the problem that prices could become negative in the Arithmetic Brownian Motion model, which was a problem as values below zero is impossible for a stock. Osborne (1959) concluded that one could use independent and normally distributed random variables to model the logarithm of stock returns. However, it has been shown that the normality assumption is not satisfied by most returns.

The basis of the model used in this thesis was created by Mandelbrot (1963) and Mandelbrot and Taylor (1967) who considers non-normality in stock returns. They observed a fractal behaviour in stock prices. These series are characterized by distinct but non-periodic cyclical patterns. Their research showed that the autocorrelations decay with a slow mean-reverting hyperbolic rate. This means that stock prices seem to have the properties of long memory or long-range dependence. Lo & Mackinlay (1988) further confirmed dependence by rejecting the random walk model, while Ding, Engle & Granger (1993) also found empirical evidence of long memory in volatility for stocks in S&P 500.

Motivated by this, our thesis tries to deal with non-normality and long-term dependence by implementing and testing a model based on fractional Brownian motion. Influenced by the works of Mandelbrot, Hu & Øksendal (2000) and Necula (2002) we compare the results of fBm to conventional models and see if this model could empirically perform better in pricing derivatives, specifically options in the Norwegian and American market.

This thesis will be organized as follows. Chapter 1, introduction. 2. we review current literature. 3. we introduce the assumptions of the Black-Scholes model and a theoretical model for fractional Brownian motion. 4. we explain the data used in our analysis. 5. we conduct an empirical analysis and interpret the results. 6. concludes the thesis.

2. Literature review

Long term dependency

Long-term dependency can be defined as the existence of a statistical dependence between two points in a time series. There is a broad amount of literature regarding so called long-term dependency or long-term memory in financial data. Lo and Mackinlay (1988) obtained empirical evidence against the random walk model. However, Lo (1991) conducts a study of long term dependence in the stock market prices. He uses an extension of the "range over standard" deviation or R/S statistic developed by Hurst. The test is performed on daily and monthly returns of indices over several time periods. He concludes that statistically there is little evidence that there is any long-term memory in the US stock market. He states that there could be a long-term dependence, but the current statistical tools are not able to catch it.

Barkoulas and Baum (2006) also tested for long term dependence in the stock market. They applied a spectral regression method to test for fractal structure in aggregate stock returns, sectoral stock returns, and stock returns for the companies included in the Dow Jones Industrials index. They find no fractal structure in the stock indices. They did however find some long-term memory in individual stocks. Barkoulas and Baum believe that the long-term memory in the indices is masked due to aggregation, but they think that the overall finding does not offer convincing evidence for long-term memory.

Ding, Engle and Granger (1993) investigated the long memory property of stock market returns and came up with a different conclusion. They focused on the absolute returns rather than the returns themselves, additionally they looked at the power transformation of absolute returns. Their findings show that there is a significantly higher correlation between absolute returns than returns themselves. They also find that the power transformation of returns has a strong autocorrelation for long lags. Thus, they conclude there exists long-term memory effect in the stock market.

The literature on long term dependency is ambiguous and there is no definite answer if long term dependencies in stock returns truly exist. There are also some doubts if the methods used today are adequate to capture these long-term dependencies. As the objective of this thesis is to price options, assuming long term dependency, we will take considerations to this problem when the analytical results are interpreted.

Earlier practitioners of similar model(s)

There have been empirical studies on pricing options with fBm. Literature is however scarce and few have been published in recent years. Cajuerio and Fajardo (2005) estimate volatility and price European options using a modified Black-Scholes model with fractional Brownian motion(fBm). Their study applies to the Brazilian market. Cajurio and Fajardo are motivated by the fact that the Brazilian market returns exhibit fat tails and have a significant long-term dependency. They believe that fBm is good fit to deal with these problems.

The data used is a collection of intraday prices of TELMAR PN from February to April. The objective is to price call options by using regular Black-Scholes and fBm Black-Scholes and then check them up against the real prices. Cajurio and Fajardo find that the Black-Scholes prices are closer to the real prices. They conclude that the fBm model does not deal with the fact of fat tails. They recommend that the model is extended to incorporate better for dependencies and fat tails.

On the other hand, Shokrollahi and Kilicman (2006) price currency options by using a modified Black-Scholes model. The authors find that the fBm Black-Scholes model is better suited to capture the behaviour of the Forex market. They show that it outperforms the two other models with respect to option pricing. Moreover, they run tests to check for value discrepancies and find out the fBm is better matched with the Garman Kolhagen model. Shokrollahi and Kilicman (2016) conclude that the model is reasonable to use.

Takayuki Morimoto (2015) investigates European option pricing under fBm and applies it to realized volatility (RV). The RV measure is selected because it uniquely exhibits simultaneous stationarity and long-range dependency properties in financial time series, as shown in our empirical study. Meanwhile, the Black-Scholes differential equation is not well defined when the underlying assets follow fBm with the Hurst exponent H = 1/2 because fBm is not a semimartingale.

Conducting an empirical study by computing the European option prices using a previously proposed fractional Black-Scholes formula, Morimoto (2015) finds that the real volatility expresses both stationary and long term dependencies in the financial times series. The author also find out that the European call options are very sensitive to the Hurst parameter. Furthermore, there were differences between the simulation and the empirical data such which they conclude is due to different volatility parameters.

Cajuerio and Fajardo (2005) found that the model underperformed when applied to the Brazilian market. However, Shokrollahi and Kilicman (2006) concluded that the model outperformed in the currency market. There is no consensus if the model has all the desired qualities. Further empirical research and testing of fBm could therefore enrich option pricing theory.

We would therefore test the model versus the Norwegian and the American market. This to see how the model could perform in a small and big market. There are few available empirical papers that tests the model directly. As mentioned, trade of options has been rapidly increasing, we believe that to find the model that performs best in pricing them correctly is therefore a highly relevant subject in finance. We would like to add an empirical study to further test the theory in practice.

3. Theory/Methodology

Traditional financial modelling has for a long time been based on a semimartingale processes with stationary and independent increments. However, some empirical studies have shown that these assumptions do not hold. It is a well-known fact that financial data is leptokurtic, having a high peak and fat tails. The normality assumption of returns could therefore lead to wrong estimations. This means that there is a need for a new model; one that takes into account for non-normality and the possible long-term dependence in returns. We believe fBm can help solve these problems.

The most commonly used option pricing models are Black-Scholes and the binomial model. We have chosen to focus on the assumptions that make up a Black-Scholes market as this model is most prominent. The assumptions in Black-Scholes are categorized in four groups. The risky asset, riskless asset, option and market.

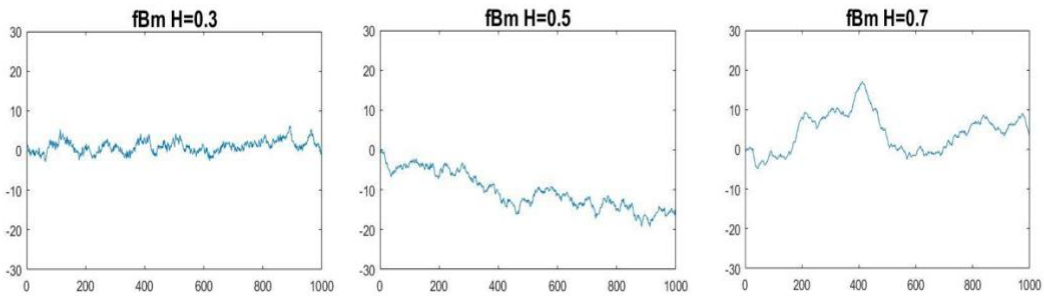
The risky asset has the following attributes: it follows a random walk, has a constant volatility, normally distributed returns and no dividends. The riskless asset can be borrowed or invested in at a constant risk free interest rate. The model requires the option to be priced to have the characteristics of a European option. Black-Scholes imposes the following restrictions in the market; no transaction cost, perfect liquidity, no restrictions for short selling and no arbitrage possibilities.

These assumptions do not always mimic the real world. Discrepancies between the model´s estimated and the market´s prevailing prices can be significant. Criticism has therefore been raised against the model and how it performs in practice. Researchers have therefore been dedicated to find a better model or make improvements by introducing elements that take into account non-normalities, long term dependencies and discontinuous jumps.

Fractal theory

Fractal theory was introduced in The Fractal Geometry of Nature (1982) by Benoit Mandelbrot. He introduced the model to describe objects "whose portions have the same structure as the whole". Mandelbrot found out that stochastic process could be regarded as fractals. He explains this with the comparison of a fern; "If you look closely at the frond of a fern, for instance, you see it is made up of smaller fronds that, in turn, consists of even-smaller leaf clusters"[1]. Edwin Hurst was influential in development of the fBm. His studies showed that the Nile and its floods exhibited a special behaviour that suggested a different covariance structure than Brownian motion. Mandelbrot was inspired by these studies and tried to apply this to financial data as financial data had shown signs of fractal behaviour. Later he used these insights to define fBm including the Hurst parameter.

In the fBm, the Hurst parameter measures the degree of self-similarity in the data. When the value of H is above ½, fBm exhibits positive correlation between increments and it is said to be persistent. When the value is below ½, it has a negative correlation between increments and its said to be anti-persistent. If H is equal to ½, the fBm becomes a regular Brownian motion and increments do not depend on each other. In the persistent case the fBm is said to show long term dependence. This means that if there is an increase in the historical trend there will most likely be an increase in the future. If the process is anti-persistent then a decrease in the trend will most likely lead to an increase.



Using the fBm is more complicated than implementing standard Brownian motion. This is especially true for finance applications as H values that differs from ½ means it is no longer a semimartingale and it generates opportunities for

[1] The (Mis)behaviour of markets, Mandelbrot B., Hudson R.

arbitrage. Hence it becomes unusable for the financial market. In order to apply fBm to finance, one has to develop an Ito formula and a risk neutral measure similar to the one in the geometric Brownian market. Hu and Øksendal developed a model that could be applied, defining a fractional Ito formula.

Market model

Similar to the regular Black and Scholes, we need to define a market to price the options. In our model, the market is defined as a fractional Black-Scholes market. The investors face two investment assets, one risky and one riskless.

The price of the riskless asset is given by:

$$dB_H(t) = \rho B(t)dt \tag{1}$$

The price of the risky asset is given by:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t) \tag{2}$$

where ρ, μ and σ are given constants and B_H is the respective fractional Brownian motion with a Hurst exponent H, (H/=½).

Later we follow the work of Hu and Øksendal (2003) where they show that the Ito type of stochastic processes with respect to B_H, does not present an arbitrage opportunity for the fractional Black-Scholes market.

Proposition 1. (Geometric Fractional Brownian Motion) The solution of the fractional differential equation.

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t) \qquad S(0) = s_0$$

given by

$$S(t) = s_0 exp(\sigma S(t)dB_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H})$$

Hu and Øksendal (2000)

The stochastic differential equation above consists of a deterministic part $\mu S(t)dt$ and a stochastic part $S(t)dB_H(t)$. The B_H term is the one representing the fractional Brownian motion. It is a random process which is normally distributed with a mean equal to zero and a variance equal to t. The assumption of log-normal prices, log-prices are normal, implies that returns of the underlying asset will be normally distributed.

Option Pricing

The results provided by Necula (2002) makes a framework for pricing a European option based on the Black-Scholes.

Necula states a Theorem that: the price at every time $t \in [0,T]$ of an European option with a price S, strike price K and maturity T is priced by:

$$C\big(t,S(t)\big) = S(t)N(d_1) - K^{e-r(T-t)}N(d_2) \tag{3}$$

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right)+r(T-t)+\frac{\sigma^2}{2}(T^{2H}-t^{2H})}{\sigma\sqrt{T^{2H}-t^{2H}}} \tag{4}$$

$$d_2 = \frac{ln\left(\frac{S(t)}{K}\right)+r(T-t)-\frac{\sigma^2}{2}(T^{2H}-t^{2H})}{\sigma\sqrt{T^{2H}-t^{2H}}} \tag{5}$$

The $N(d_s)$ is the cumulative probability of the standard normal distribution. $N(d_2)$ is the probability that the option will be exercised in a risk neutral world. Therefore, $K^{e-r(T-t)}N(d_2)$ is the strike price multiplied with the probability that the strike price will be paid. $S(t)N(d_1)$ is the expected value of the stock in a risk neutral world given that the $S_t > K$, while it is zero otherwise.

An important note is that using Black & Scholes to price of two options with a maturity at time T written in t(1)and t(2) would be the same. In fBm however, the assumption of long dependency would price the options differently. The long memory property makes the price of the option written in time t1 different as it contains information from period [t1,t2]. This influence is reflected by the Hurst parameter.

This framework will be used to price European call options in the US stock market and the Norwegian stock market. These prices will be compared to the prevailing prices at the time and the accuracy of the fBm Black-Scholes will be compared to other models too.

4. Plan for data collection and the road ahead

To conduct the empirical analysis and pricing, we need to collect historical option data from both the US and Norwegian stock market respectively. The data should contain at least stock prices, option prices, strike prices and times to maturity. The plan to gather the Norwegian data from the Norwegian stock exchange primarily the OBX, concentrating on the most liquid traded options. The same process will be applied for the US data, where we will collect data from the S&P 500. This data should be relatively easy to obtain as it is recorded frequently.

The next upcoming months we will first begin by implementing the framework and model for the fBm Black-Scholes in Matlab. Later we will also implement the regular Black-Scholes and the Heston model. We will describe all the decisions made while collecting the output and keep a record of all data handling such that triangulation and replication of the results will be possible in the future.

We are going to conduct an empirical comparison of the option pricing models based either on the research methods by An&Suo (2009), Bakashi & Chen (1997) or Bates(2003). How to conduct this empirical analysis and comparison will be discussed with the supervisor.

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