BI Norwegian Business School - campus Oslo

# GRA 19502

Master Thesis

Component of continuous assessment: Thesis Master of Science
Final master thesis – Counts 80% of total grade

An empirical application of Black and Scholes option pricing with fractional Brownian motion

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Start: 02.03.2018 09.00

Finish: 03.09.2018 12.00

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*An empirical application of Black and Scholes option pricing with fractional Brownian motion*

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# Aknowledgements

We would like to thank Costas Xiouros for all the helpfull feedbacks and advice in the process of writing this thesis.

We would also like to thank our friends for making our time at BI such a great experience. A special thanks to Jaran for these last 5 years, being our study partner for exams and for all the fun we have had on the way to finishing our masters.

# Abstract

This thesis examines the empirical properties of a fractional Black and Scholes model developed by Röstek and Schobel. The model is tested and compared to the standard Black and Scholes for Standard and Poor's 500 call options in the period 10th May to 10th of July 2018. We first go through the theoretical differences of using a geometrical and a fractional Brownian motion. We test the models using three different empirical tests, following the methods of Bakshi, Cao & Chen (1997). The performance is measured using an in-sample test, an out of sample test and running a dynamic delta hedging strategy over a period of 41 trading days. While testing the models, we highlight the importance of estimating the correct Hurst value (long-term dependence) as the model becomes time dependent. We find that the fractional Black and Scholes model is misspecified, but performs slightly better in the out of sample test. In total, we rank the fractional equal to the geometric model.

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# 1. Introduction

## *1.1 Motivation*

In this thesis, we are studying option pricing under the assumption that stock returns are persistent, meaning that the market exhibits long-range dependence. The success of the Black & Scholes model (*B&S*) within option pricing, but with its limitations and criticism, researchers have aspired to extend this model within a fractional context. There is a dispute on the proper application of fractional Brownian motion (*fBm*) in valuation theory. There has been raised concerns that the properties of a fBm prevent no arbitrage pricing, but the consequences of applying the model on real market data is not clear.

We will present a market driven by fractional randomness and test the fBm B&S model's performance. Comparing its ability to price options with that of the original B&S we hope to show the fragility of prevailing option prices and test empirically if the fractional model is a viable extension.

## *1.2 Overview and findings of our study*

The model chosen for our empirical tests price options using an underlying asset driven by fBm (Röstek 2009). We impose restrictions on time increments between transactions for any single investor based on the idea that other investors ensure transactions in between. Thereby creating a continuous market while removing arbitrage opportunities within the fractional market. For mathematical proof, interested readers are asked to see Cheridito (2003). The tests are performed on European options, more specifically call options which gives the buyer the right, but not the obligation to buy an asset at a certain time in the future at a determined value, known as the strike price.

Our study extends the B&S model, incorporating a Hurst parameter, which is an estimate of long-range dependence. We study both models using a simulation to show how the fBm performs in a non-restrictive market. This is done to show the problems of traditional non-arbitrage pricing when applying the fB&S.

The models are then run to calculate option prices for S&P500. Their performance is measured and ranked for their ability to reduce the errors between the estimated and prevailing option price. The end results show that the models performance equally, though the fractional model is slightly better in the static out of sample test. We show how the fractional model is dependent on the correct estimation of the Hurst parameter and discuss how calibration could improve performance.

## *1.3 Literature review*

Current models, the most common being Black-Scholes, are based on the stochastic process Brownian motion (*Bm*), first modelled by Bachelier in his PhD paper *Théorie de la spéculation* (1900). To model stock returns, Osborne (1959) concluded that one could use independent and normally distributed random variables by using their logarithmic functions. Samuelson (1964) developed the model to only include stock prices greater than zero in the Arithmetic *Bm* model. Motivated by these results, Myron Scholes and Fischer Black developed the Black-Scholes framework and model. The returns in this model are assumed independent, meaning there are no autocorrelation in price changes.

Black and Scholes (1973) stated that if options are correctly priced, there should be no strategy that can replicate a long or short positions in options and their underlying stocks to make a sure profit. They were the first to derive the theoretical valuation formula that could be applied for bonds, stocks and warrants. Now the fundamental basis for option pricing theory in current academia.

Cox & Ross (1976) find that the B&S overprice options with low volatility, and tend to underpriced options with high volatility. They see it as a problem that volatility is held constant. They argue that the most important tests for option pricing theory is the empirical ones, as these show the true abilities of the models.

Bakshi, G., Cao, C., & Chen, Z. (1997) find that B&S tend to underhedge for all securities, while other option pricing models can go either way. They believe B&S shows a systematic hedging bias, that could be improved using stochastic volatility to capture long-observed features of implied volatility.

Bates (2003) conducts an overview of empirical option modelling. Analyzing tests of model performance, alternative models and future research. He states that research has rejected many of the assumptions of traditional option pricing, such as independent volatility, identical volatility across all strike prices and the concept of no arbitrage. He suggests that new alternative option models should investigate relaxing the geometric Brownian motion assumptions and consider equilibrium pricing instead of the no arbitrage pricing.

### *1.3.1 Long-range dependence in stocks, option pricing utilizing the fBm and equilibrium pricing.*

Researchers within option pricing theory has looked to incorporate fractional theory to improve pricing by changing how we look at the underlying behaviour. Mandelbrot (1967) showed that autocorrelations exists in returns. They exhibit long-range dependence and decay with a slow mean-reverting hyperbolic rate. These findings have motivated researchers to conduct studies about long-range dependence in asset prices.
Long-range dependence can be defined as statistical dependence between two points in time. There is a broad amount of literature regarding long-range dependency or long-range memory in financial data. Lo and Mackinlay (1988) obtained empirical evidence against the random walk model by finding persistence in stock returns, and argue this behaviour cannot be explained by infrequent trading or changing volatility. This implies a misspecification of the B&S model. However, Lo (1991) conducts a study of long-range dependence in the US stock market prices by using an extension of the "Rescaled Range" (R/S) statistic and finds little evidence for any persistence contrary to previous results, but states he cannot exclude this to be caused by inefficiency in current statistical tools. Therefore, one cannot conclude that there is no persistence.

Analyzing absolute returns, Ding, Engle and Granger (1993) show that there is a significant correlation between returns and conclude there exists a long-range memory effect in the S&P500. Other researchers have found different results of persistence in foreign exchange markets. For example, Cheung and Lai (1995) and Grau-Charles (2000) supports Lo (1991) in that there are no clear signs of any persistence in stock returns, but they also argue that Ding, Engle and Granger (1993) do find some long-range dependence, and that the models require further robustness checks. They cannot conclude with certainty.

Sadique and Silvapulle (2001) find evidence for long-range properties in small economies such as Korea, Singapore, Malaysia and New Zealand, but no significance for the US market. Implying smaller markets are inefficient. Cajueiro and Tabak (2008) conduct research on a sample of 41 indices, including both developed and emerging countries. Applying the R/S- and the V/S statistic, which they propose is superior, they find little evidence in equity returns and that results differs between countries indices and is dependent on the method used. However, they find that volatility exhibits quite strong long-range dependence in general.

Hu and Øksendal (2003) developed a fractional market model. Based on this model Necula (2002) developed a closed form *B&S* option model generalized for the fBm. The model has become quite popular as a base for conducting empirical studies on option pricing.

Cajueiro and Fajardo (2005) price European options using Necula's fractional pricing model. They argue that this model is a good fit when the market returns exhibit fat tails and has a significant long-range dependency. They use a collection of intraday prices of TELMAR PN from February to April. Their findings show that the regular B&S is closer to the real prices. They conclude that the model cannot capture the correct price when data has the property of fat tails. They recommend that the model needs further development.

On the other hand, Shokrollahi and Kilicman (2016) apply Necula's fB&S to price currency options. The fB&S is shown to be better suited to capture the behaviour of the forex market. Moreover, they run tests to check for value discrepancies and find the fB&S to perform better than the B&S with results similar to a Garman Kolhagen model. They conclude that the model performs well in practice.

Takayuki Morimoto (2015) investigates European option pricing under fBm with historical volatility (HV). The HV measure is selected because it exhibits simultaneous stationarity and long-range dependency properties in financial time series. During his empirical study, Morimoto finds that the real volatility expresses both these properties in his series. Further, he finds that the European call options are very sensitive to the Hurst parameter.

# 2. Option pricing

## *2.1 Arbitrage*

Option pricing is based on the assumption of no arbitrage. Arbitrage can be defined as weak and strong. Weak arbitrage states that highly liquid assets should not exhibit any arbitrage, but illiquid assets could have some discrepancies in prices. Strong arbitrage is defined as a market with no mispricing of any kind. Imagine an investor being able to buy two assets with different payoffs, but to the same price. An investor would then be able to make a portfolio by selling and buying these assets, making a risk-free profit with zero investment. This information would spread and the market demand for this portfolio would increase, stabilizing and setting a price level that prevents arbitrageurs from replicating the strategy. Option pricing theory is dependent on strong arbitrage where markets are efficient.

## *2.2 Geometric B&S model*

### *2.2.1 Geometric Brownian motion*

The geometric Brownian motion (gBm) is a logarithmic extension of the stochastic continuous-time process Brownian motion ( $B_t$) with drift. $B_t$ the description of the physical phenomenon a small particle exhibits a zigzag motion through a liquid or gas. The motion was discovered by Robert Brown in 1827 and later in 1918 Norbert Weiner was able to create a mathematical description of the movement.

GBm must satisfy the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \quad (2.1)$$

The drift ( $\mu$ )and volatility ( $\sigma$ )are assumed to be constants.

Equation (2.1) consists of two parts; one deterministic and one stochastic part. $\mu S_t dt$ is used to model deterministic trends, while the $\sigma S_t dW_t$ models the stochastic movement[1]

GBm has properties such that it is a log-normally distributed random variable for each value of time $t$. With mean and variance:

[1] The Brownian motion has the following properties:

(1) $B_t - B_s \sim N(0, t-s)$ .

(2) For every pair of disjoint time intervals, the increments are independent, normal distributed random variables.

(3) $B_0 = 0$

(4) $B_t$ is continuous for all t.

$$E(S_t) = S_0 e^{\mu t} \qquad (2.2)$$

$$Var(S_t) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1) \qquad (2.3)$$

The logarithmic properties of the gBm make it quite preferable to model market prices in finance as it is positive for each point in time $t$. This coincides with the value restriction of stocks always being equal or greater than zero.

### 2.2.2 Black and Scholes

Black and Scholes (1973) incorporate the insight of no arbitrage pricing to develop a market model where the law of one price holds and the underlying is driven by a gBm. They assume volatility to be constant and the mean to be equal the risk-free rate. Returns to be independent and follow a log-normal distribution.

These assumptions have shown to come with a set of problems. Volatility has shown to follow a convex shape known as the volatility smile, and has shown to vary over time. Returns have been found to be non-log-normal and there are indications of long-range dependence (Lo and MacKinlay 1999). There are also exogenous factors affecting prices that a model of Gaussian character won't be able to capture.

The famous B&S option pricing model is built up around a delta hedging technique, creating a replicating portfolio with positions in the underlying and risk-free instruments to create a payoff equal to a call option. The cost of the option then equals the cost of the portfolio.

*The B&S model and its following assumptions.*

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \qquad (2.4)$$

$$d_1 = \frac{\ln(\frac{S(t)}{K}) + r(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \qquad (2.5)$$

$$d_2 = \frac{\ln(\frac{S(t)}{K}) + r(T-t) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \qquad (2.6)$$

Proof. Black and Scholes (1973)

$N(.)$ is the cumulative probability of the standard normal distribution.

*Black and Scholes assumptions*

1. Stochastic differential equation has an Ito Skorohod interpretation.
2. The stock prices are log-normally distributed.
3. The underlying follows an gBm.
4. No arbitrage possibilities.
5. Complete and efficient market.
6. Constant volatility and drift.
7. Risk-free rate is constant and known.
8. Short selling is allowed.
9. There are no transactions costs or taxes.
10. The underlying has no dividends or other payoffs in its lifetime.
11. A single investor can trade continuously.
12. Securities are perfectly divisible.
13. The option is European, exercisable at time T.

## *2.3 Fractional B&S model*

In 1967 Mandelbrot and Van Ness developed the fractional Brownian motion, influenced by seminal work of Edwin Hurst (1951). They wanted to develop a stochastic process that could capture fractal properties such as long-range dependence and self-similarity.

The stochastic integral representation is given as a moving average of Brownian increments. The term dB(s) is a two-sided Brownian motion added to incorporate covariance between increments. This is the main distinction between fractional Brownian motion and Brownian motion.

$$B^H(t) = C_H \left( \int_{-\infty}^{0} \left[ (t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dB(s) + \int_{0}^{t} (t-s)^{H-\frac{1}{2}} dB(s) \right) \qquad (2.7)$$

*$C_H$ is a normalizing constant.*[2]

*The motion has the following properties:*

1. $B^H(0) = 0$.

2. The expectation of the fBm $E[B^H(t)] = 0$ for all $t > 0$.

3. The variance function of the fBm is given: $Var[B^H(t)] = t^{2H}$.

4. At $H = \frac{1}{2}$ the fBm becomes a gBm.

The fBm has stationary increments. Meaning the distribution of increments depend only on the interval and not the time in which they occur. Another property of the fBm is its self-similarity. In short, the process looks similar no matter how you scale it. The Hurst parameter is the determined degree of long-range dependence.

### *2.3.1 The Hurst parameter*

In 1951 hydrologist Edwin Hurst worked on building dams in the river Nile. He planned how to store water from lake inflows in good years, which in turn could be used in bad years. This meant that the dam had to be built in such a way that it could hold unexpected amount of water. By this time, the consensus was that water inflows where random processes. Hurst found out that the data showed Gaussian features. However, it showed more clusters than a random process. Using this insight, he developed the Rescaled Range Analysis to measure the degree of clustering. Mandelbrot used this as a basis for the fBm, naming the measured value of long-range dependence and degree of randomness the Hurst exponent (*H*).

A time series is described to exhibit long-range dependence if it has correlation that persist over all time scales. The H parameter takes a value between 0 and 1, where values >0.5 indicates

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[2] $C_H = \frac{1}{\Gamma(H + 1/2)}$

that the series exhibits persistence. The opposite is true for < 0.5 and the series is anti-persistent. Peters (1991) describes that one can see (anti)persistence in the jagged lines of a time series chart *(Figure 1)*. H=0.3 is categorized as anti-persistent, it displays jagged lines as it is subjected to mean reversion. This means that if the price goes up there is a higher probability it will move downwards on the next step. H=0.7 shows a persistent chart. If the price goes up it will most likely be followed by another increase. One can think of the Hurst as the probability of the next movement. When H=0.7 and the previous movement was in the upwards direction, there is a 70% change that the next movement will be up again. A normal geometric Brownian motion has a Hurst=0.5 where increments are independent, thus there is a probability of 0,5 that the value moves either up or down.

*The figure displays the increments of the fractional Brownian motion with Hurst parameters set at 0.15, 0.5 and 0.95. As the Hurst value increases one can see that the lines smoothen and becomes almost perfectly flat as the motion increases in persistence.*

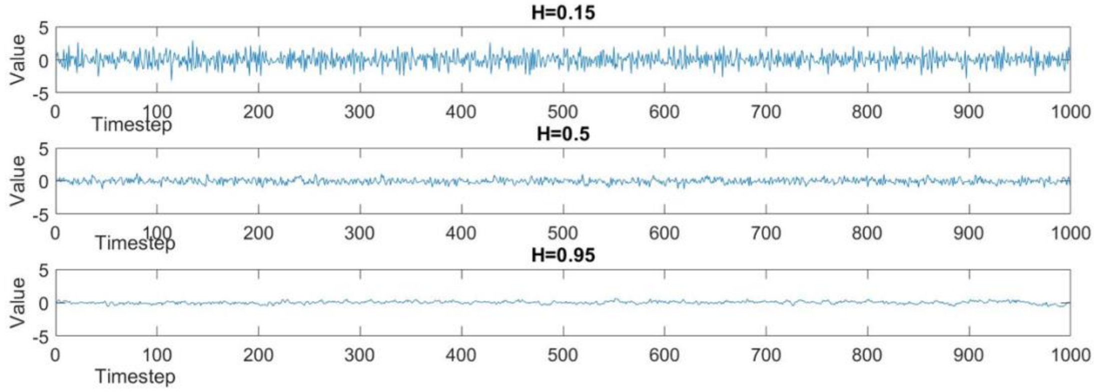



***Figure (1)***

*Comparison of the Hurst parameters with values 0.15, 0.5 and 0.95. The lines are simulated as fractional Brownian motion with different degrees of persistence.*

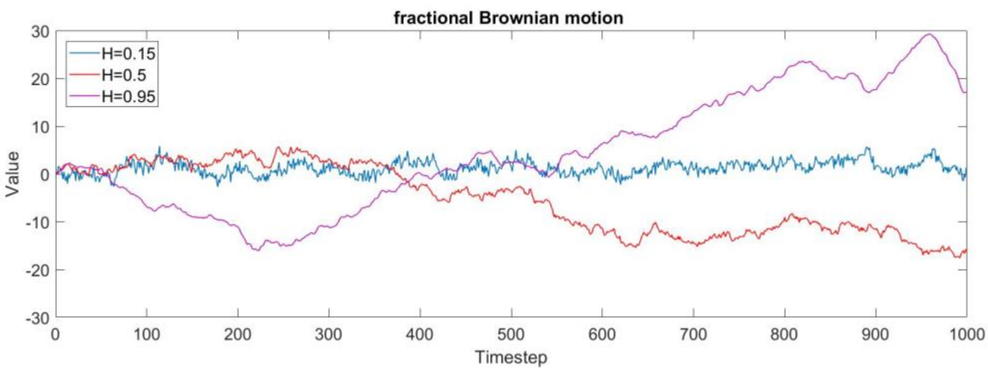



***Figure (2)***

The Hurst estimation measures to what degree the time series exhibits a slow decay in time, through the autocorrelation function. Easier said, the estimation methods display a numeric value of the degree of long-range dependence. The Hurst parameter can be defined mathematically, but to measure it has been shown to be problematic, and there are several suggested ways to estimate the Hurst. No method has however shown to outperform the others empirically (Rea 2009). Most methods have shown to converge slowly to the correct value with increasing data, but would then likely be affected by trends and periodicity when used in economics.

For financial data, a common Rescaled range (R/S) method has been a popular measure, and has been chosen for our analysis as it has shown to perform quite well when the series length is greater than 1000 independent of time length, which is the case for our data.

The rescaled range method is based on the following algorithm:

Range series formula:

$$R(t,k) = \max_{0\le i\le k}\left\{Y(t+i)-Y(t)-\frac{i}{k}(Y(t+k)-Y(t))\right\} - \min_{0\le i\le k}\left\{Y(t+i)-Y(t)-\frac{i}{k}(Y(t+k)-Y(t))\right\} \quad (2.8)$$

Standard deviation of series:

$$S(t,k) = \sqrt{\frac{1}{k}\sum_{i=t+1}^{t+k}\left(X(i)-\frac{1}{k}\sum_{i=t+1}^{t+k}X(i)\right)^2} \quad (2.9)$$

Rescale range computation:

$$RS(k) = \frac{R(t,k)}{S(t,k)} \quad (2.10)$$

Rescaled range is the range of values in a portion of a time series divided by the standard deviation. The range series is the difference between the max and minimum of a cumulative function in a mean adjusted series R(t,k). Combined with a standard deviation series we are able to create the R/S statistic RS(k). The Hurst parameter is then created as a the slope between the logarithmic function of R/S vs the logarithmic function of *n* observations.

Proof: Biagini, Hu, Øksendal and Zhang, 2008

### *2.3.1 Building the fractional market model*

Delbean and Schachermayer(1994) proved that for continuous market models, the assumption of no arbitrage is broken as there is a weak form of arbitrage with diminishing risk. For an fBm with H ≠ 0.5, the fB&S exhibits weak arbitrage when applied in a continuous time setting (Peter 1997).

To circumvent the arbitrage problem of continuous trading, the fBm model is built on a different framework. A trader can move arbitrarily fast, but cannot react immediately to new market movements. We exclude continuous trading and assume there is an arbitrarily small time between each trade to prevent arbitrage possibilities (Cheridito 2003). Traditional non-arbitrage option pricing approaches are no longer viable, and dynamical hedging and replication methods cannot be used to derive prices. This is solved using equilibrium theory. Replacing preference free pricing with risk preference pricing (Röstek and Schobel 2009). Risk preference states that the market must satisfy a basic equilibrium condition. For the risk-neutral investors the option price is set by calculating the discounted conditional mean of the relevant payoffs.

The fractional Brownian market is built up with the following two assets:

A riskless bond $$A_t = A_0 e^{rt} \qquad (2.11)$$

and a risky stock $$S_t = S_0 e^{\mu t - \sigma^2 t^{2H} + \sigma B_t^H} \qquad (2.12)$$

The riskless rate ($r$), drift ( $\mu$ )and volatility ( $\sigma$ )are assumed to be constants.

The assets can be described with differential equations and a format more akin to Black and Scholes.

Riskless asset satisfies

$$dA_t = rA_t dt \qquad (2.13)$$

and the risky asset follow

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H \qquad (2.14)$$

The stochastic differential equation (2.14) above consists of a deterministic part and a stochastic part. The term $B_t^H$ is the representation of the fractional Brownian motion. It is a random process which is normally distributed with a mean equal to zero and a variance equal to $t^{2H}$. The assumption of log-normal prices implies that returns of the underlying asset will be normally distributed.

*Rostek and Schobel Fractional Conditional Black-Scholes option pricing model:*

$$C_{T,H}(t,S(t)) = S_t N(d_1^H) - Ke^{-r(T-t)} N(d_2^H) \tag{2.15}$$

$$d_1^H = \frac{\ln\left(\frac{S_t}{K}\right) + r(T-t) + \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H}}{\sqrt{\rho_H}\,\sigma(T-t)^H} \tag{2.16}$$

$$d_2^H = \frac{\ln\left(\frac{S_t}{K}\right) + r(T-t) - \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H}}{\sqrt{\rho_H}\,\sigma(T-t)^H} \tag{2.17}$$

Narrowing factor:

$$\rho_H = \frac{\sin\left(\pi\left(H-\frac{1}{2}\right)\right)}{\pi\left(H-\frac{1}{2}\right)} \frac{\Gamma\left(\frac{3}{2}-H\right)^2}{\Gamma\left(2-2H\right)} \tag{2.18}$$

Worth to note that equation (2.15-2.17) is almost identical to (2.4-2.6).

*N(.)* is the cumulative probability of the standard normal distribution.

*The fractional Black-Scholes assumptions:*

1. The Hurst parameter 0 < H < 1.
2. The stock prices are log-normally distributed.
3. The underlying follows an fBm.
4. Market participants have a constant relative risk aversion.
5. The investors maximize their utility.

6. No arbitrage possibilities.
7. Complete and efficient market.
8. Constant volatility and drift.
9. Risk-free rate is constant and known.
10. Short selling is allowed.
11. There are no transactions costs or taxes.
12. The underlying has no dividends or other payoffs in its lifetime.
13. Investors can only trade in discrete time.
14. Securities are perfectly divisible.
15. The option is European, exercisable at time T.

We see that if H=½, p converges to 1 and the formula becomes the well-known B&S option formula, yielding familiar results as the classic model.

# 3. Theoretical modelling

## *3.1 Simulation and delta hedging framework*

We simulate the effect of extending the B&S model with the Hurst parameter. By applying a continuous delta hedging strategy we can compare how the models perform if the models are applied in a market with no restrictions for trading.

Stockpaths are modelled using Monte Carlo simulation for both *Bm* and *fBm*. The simulation is used to see how the models differ in results when creating a replicating portfolio for either a short or long position to price a call option. The simulation is run with a starting price of 100 for the underlying and a time to maturity (*T)* of 0.25 years. Time increments are set to $dt = T/n$, $n$ being the number of changes in the stock price until maturity. The strategy to replicate the call option payoff will be rebalanced at each timestep. When hedged correctly, errors as a function of profit and loss should converge to zero as the time increment becomes smaller and $dt \to 0$, creating a risk-free investment with return $r$.

**Simulation variables:**

$S_0$ : Initial stock price

$K$ : Strike Price at time t

$\sigma$ : Volatility for the stock

$r$ : Risk free rate

$T$ : Time to maturity

$n$ : Number of time steps

$dt = T / n$ : Time interval increment

$C$ : Call option price

$S_t$ : Stock price at time t

$\Delta_t$: Delta for the option

$b_t$ : Bank balance at the end of time t

The stock paths following the stochastic differential equations:

gBm: $dS_t = rS_t dt + \sigma S_t dB_t$ (3.1)

fBm: $dS_t = rS_t dt + \sigma S_t dB^H$ (3.2)

The call option price is calculated for B&S and fB&S with equation (2.1) and (2.14) and is subsequently used to calculate $\Delta_t = N(d_1)$ .

At *t = 0* sell the option C and purchase $\Delta_0$stocks. Thus obtaining the bank balance:

$$b_0 = C - \Delta_0 S_0 . \qquad (3.3)$$

At intermediate times *t* where $t \neq 0$ and $t \neq n$ :

The bank balance earns $r$ interest. We reposition by buying ($\Delta_t$-$\Delta_{t-1}$) stocks. The new bank balance is then at time t:

$$b_t = e^{rdt} b_{t-1} - (\Delta_t - \Delta_{t-1}) S_t \qquad (3.4)$$

This process is repeated for each *dt* until maturity.

At time T, we sell $\Delta_{n-1}$ shares ; and cover our position:

$$\max\{S_n - K, 0\} \tag{3.5}$$

The bank balance at maturity becomes:

$$b_n = e^{rdt} b_{n-1} + (\Delta_{n-1}) S_n - \max\{S_n - K, 0\} \tag{3.6}$$

At each timestep the portfolio is rebalanced and the option price is recalculated as it moves towards maturity. As we increase the number of timesteps we reduce the time increment between each rebalance. This should give the mentioned results; increasing the accuracy and profit/loss should converge to zero at each timestep.

### 3.1.1 Simulated scenarios

The simulation results are given in (*Figure 3 & 4*). The hedging error is plotted as a function of *dt*. The observed profit at time to maturity converges to zero as we increase timesteps. This is in line with the theory of the classical B&S.

*B&S hedging error from simulation as a function of dt*

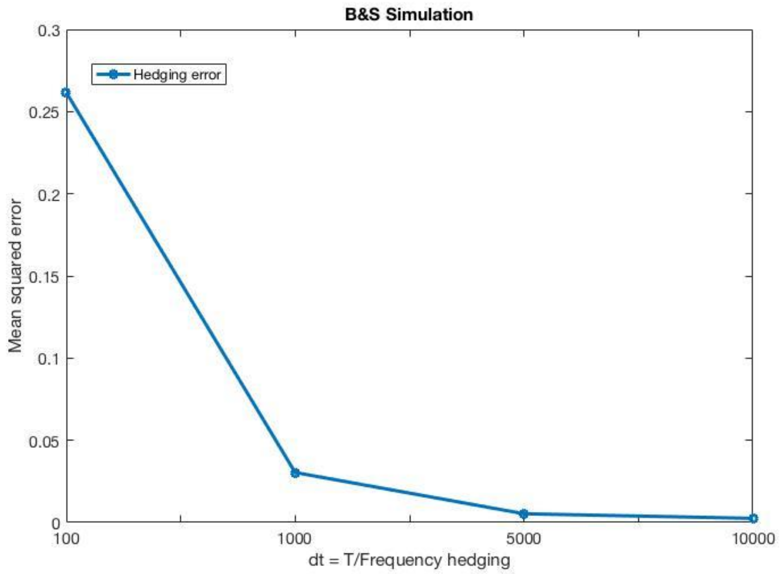



***Figure (3)***

We followed the same procedure for fBm with a constant Hurst of 0.6 (*Figure 4*). The results were quite different from the classical B&S theory. The hedging strategy is tested on the same time to maturity *T* and time increments *dt*. We observed that the hedging errors increased with the hedging frequency. Röstek (2009) states that for this model traditional continuous delta hedging is not applicable and the model should be used under the assumption of discrete trading. The results of the simulation shows what happens in the continuous market.

*FB&S profit and loss from simulation as a function of dt*

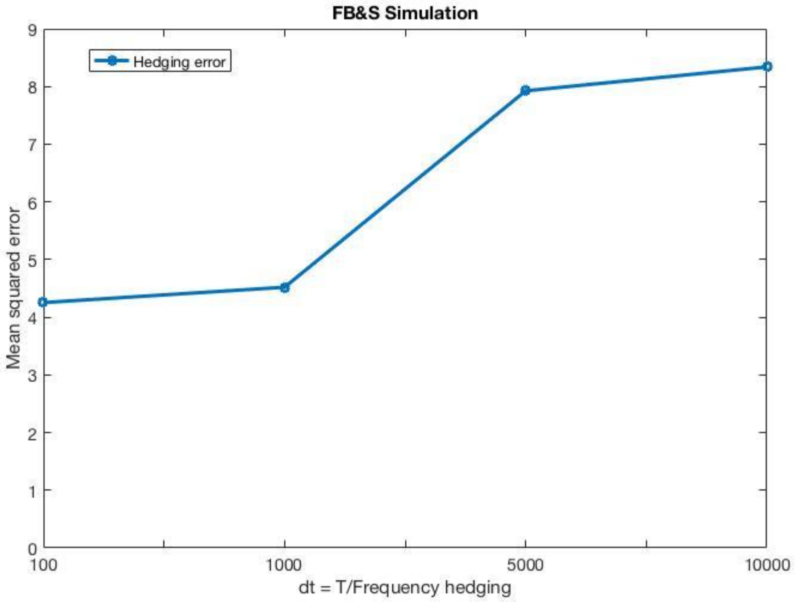



***Figure (4)***

The fact that delta hedging in continuous time is not applicable is unfortunate. However, the Rostek & Schobel proves the model does not exhibit arbitrage when discrete trading time is assumed. We believe this to be a reasonable assumption when one considers that continuous trading is not easily applicable in the real world.

# 4. Data

The data set is comprised of call options with contracts on the Standard & Poor's 500. These were chosen because of the high trading volume, making them highly liquid. Each option contract is traded at a quantity of minimum one hundred at the date it is collected. That the S&P 500 options is followed closely by the market, should prevent or minimize skewness from mispricing.

The data is gathered from Bloomberg and consists of S&P500 index data in the period 1st February 2013 to 10th July 2018. The series sample contains 1370 trading days. The option data ranges from 10th of May 2018 to 10th of July 2018 and contains a sample of 252 call options, divided between different maturities and strike prices. Both the index and option prices are quoted as closing prices. This is done to best match the price of the option to its corresponding spot price. *Table 1* presents the sample properties of the options used to conduct the empirical tests. Summary statistics are reported for the mean, standard deviation and total number of observations for each maturity-moneyness category.

**Sample properties of S&P 500 Index Options**

The table shows average and standard deviation of closing prices for the S&P 500 collected from Bloomberg. The prices are collected over two month period from May to July.

| | Days to Experation | | | |
|---|---|---|---|---|
| Moneyness | <30 | 30-60 | 60< | Subtotal |
| OTM | 38,41 | 38,20 | 46,56 | 84 |
| | (19,32) | (11,44) | (13,62) | |
| ATM | 25,13 | 41,18 | 57,23 | 84 |
| | (9,677) | (7,064) | (9,531) | |
| ITM | 25,50 | 41,50 | 57,53 | 84 |
| | (9,694) | (7,2781) | (9,499) | |
| Total | 25,10 | 41,40 | 57,24 | 252 |
| | (12,9) | (8,5948) | (10,8864) | |

***Table (1)***

Interest rates are collected from the US Treasury Department. The rates are quoted monthly and adjusted to daily rates.

## *4.1 Normality and Long-range dependence*

The classic Black-Scholes assumes log-normality and independent returns. The return distribution of S&P500 can be seen compared to a normal fitted distribution (Figure 5). The returns exhibit skewness and kurtosis. The high peaks indicate that there is a larger frequency of values close to the mean than that of a normal distributed series, while the tails imply period returns outside the normal probabilistic theory. A Jarque-Bera test confirms these beliefs, as such we can reject normality for S&P 500 (Appendix 1). This reinforces the conviction that using B&S under its basic assumptions may lead to erroneous results.

*The S&P 500 returns for each timestep compared to a normal distributed set*

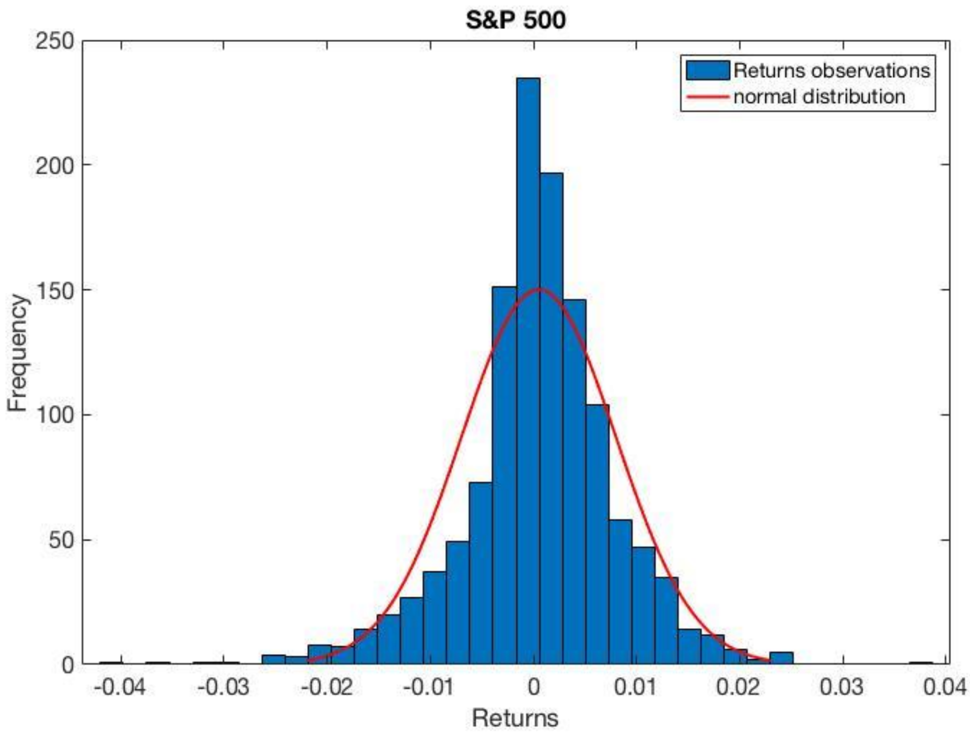



***Figure (5)***

We run a Ljung-Box test for autocorrelation in the return series (*Table 2*), testing if the hypothesis for independence holds.

**Ljung-Box test**

*The table represents the results of a Ljung-Box test for each time series. The test tells if a series shows sign of autocorrelation in lags. The test is run for the three first lags to test the independence hypothesis. The numbers represent the p-values for each lag.*
*H0: The data are independently distributed*

*H1: The data are not independently distributed; they exhibit serial correlation*

| Lags | S&P500 |
|---|---|
| 1 | 0,3848 |
| 2 | 0,4948 |
| 3 | 0,6476 |

***Table (2)***

We cannot reject the null hypothesis of independence, supporting the use of a fractional model.

As we assume a constant Hurst parameter, we run an augmented Dickey-Fuller test (*ADF*) to test for stationarity in the incremental series (*Table 3*). If the increments are non-stationary we would most likely observe a time varying Hurst parameter, which would skew our results for the fB&S. The *ADF* is significant at a 1% level, meaning we can reject the null hypothesis of a present unit root.

**Augmented Dickey-Fuller test**

*The table represents the results of augmented Dickey-Fuller (ADF) test for the S&P time series. If there is an unit root present in a series, the series is explosive. A shock to the series will have a permanent impact on the mean, which is problematic for statistical inference.*
*H0: An unit root is present*
*H1: The time series is stationary*

| | S&P500 |
|---|---|
| P-value | 0,001 |

***Table (3)***

The long-range dependence (Hurst) is measured using the R/S model to get an approximation of whether the series is persistent or antipersistent (*Table 4*).

***Hurst estimation***

*The table represents the Hurst estimates for each time series. R/S is the estimated Hurst value using the Range over standard statistic.*
*H<0.5 antipersistence*
*H=0.5 Brownian motion*
*H>0.5 persistence*

| Method | *S&P500* |
|---|---|
| *R/S* | 0,5234 |

***Table (4)***

The Hurst measure indicate that the index weak persistent, with a Hurst value close to 0,5. We believe that the index is close to an gBm as it is comprised of several stocks. Thus preventing a manifestation of a strong persistent or antipersistent pattern. This is in line with previous literature about the long-term dependence in S&P500. Even if the index shows weak persistence, we still believe that it could contain information to improve option pricing.

## *4.2 Volatility estimation*

We relax the assumption of constant volatility. The tests are performed with the estimated implied volatility of options at time *t-1*. The reasoning is simple. If the models are correctly specified, the models implied volatility should be consistent with the volatility implicit by the observed time series. Furthermore, the implied volatility withdrawn from the options should reflect future instead past information recorded in historic data.

The implied volatility estimation method has the objective to infer the volatility parameter by minimizing the distance between the model price and the quoted market price. We collect the necessary historical parameters needed to price an option, such as the index price, Hurst, strike price and interest rate for each time *t=0 to t=T* as they are quoted in Bloomberg. Holding the Hurst constant throughout the period, we are able to estimate the implied volatility of the index using an error minimization function.

The fractional option price is denoted $C_n^H(C,S,K,t_n,T_n,\sigma_{H,n},H)$ while the market price is $C_n^M$ . The error function is the difference between the two prices.

$$\varepsilon_n\left[\sigma_{H,n}\right] = C_n^H(C,S,K,t_n,T_n,\sigma_{H,n},H) - C_n^M(t_n,T_n) \qquad (4.1)$$

The objective is to minimize the function SSE which is the sum of absolute squared errors with respect to implied volatility. The same method is applied for both models.

$$\min_{\sigma_{H,n}} SSE(\sigma_{H,n},t) = \min_{\sigma_{H,n}} \sum_{n=1}^{N}\left(\left|\varepsilon_n\left[\sigma_{H,n}\right]\right|^2\right) \qquad (4.2)$$

### *4.3 Out of sample framework.*

For this test, we assume that we can observe the stock and strike price directly from the market, while the Hurst is collected from historic observations. To price the options, we rely on the previous day's option prices to back out the implied volatility. At time *t*, we compute the call price for *t+1* and calculate a mean absolute percentage error *(equation 4.3)* between the forecasted price and the quoted market price (Bakshi, G., Cao, C., & Chen, Z. 1997).

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{A_t - F_t}{A_t}\right| \qquad (4.3)$$

*Mape is measured as the cumulative sum of the absolute value of the difference between the actual (A) and predicted value (F).*

This procedure is repeated from $10^{th}$ of May to $10^{th}$ of July 2018, for call options with different maturities and strike prices equal to the spot and $\pm 5$ the spot price.

# 5. Model Performance

The empirical testing is built up the following way. We run an in-sample test to see how the add Hurst parameter affects the option price and implied volatility. Furthermore, we follow the out of sample framework to directly measure model misspecification. The reasoning is as follows; adding parameters could potentially improve the performance in an in-sample test. However, this is not necessarily true for the out of sample as overfitting may be penalized. Our last test, the dynamic hedging performance, shows how well the models are able to capture the dynamic properties of the option and the underlying security.

## *5.1 In-sample*

The Hurst parameter influences the fractional option prices in two ways. The narrowing effect and the power effect. The narrowing effect concentrates the distribution of the call option. While the power effect is the effect captured by the variance of the call as the Hurst changes. The power effect shows that when the Hurst is large, the variance will decrease in the short term. However, it will also increase the cyclical motion from the mean. Vice versa for a lower Hurst.

In figures *6-8* we show a simulated scenario of how the value of the Hurst parameter affects the option price and how this is dependent on the options time to maturity.
The simulation is run on an option where $S = 100$, $K = 100$, $r = 2\%$ and the $\sigma = 0,2$

Running the simulation with a time to maturity $T=0,25$, the power effect strictly dominates the narrowing effect. When the Hurst has a higher value, the variance and the option price will be lower (*Figure 6*).

*The option price dependency of the Hurst value with a constant maturity of T=0,25*

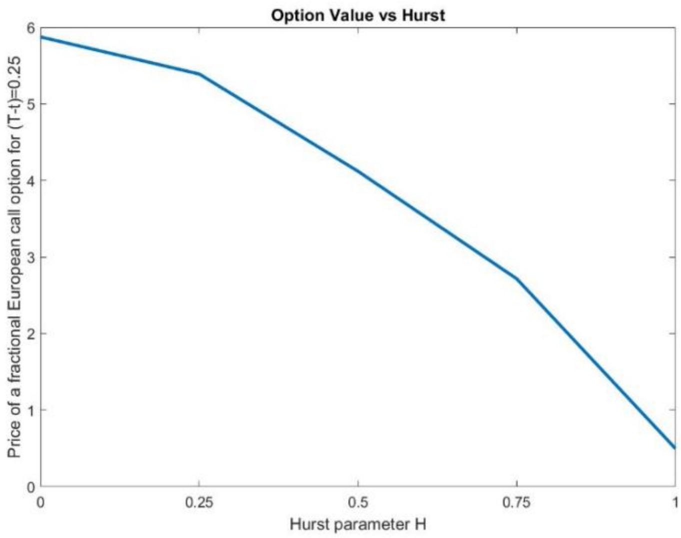



***Figure (6)***

If we increase the maturity $T$ from *0,25* to *0,75*, one can see both effects *(Figure 7)*. With a Hurst below 0,5 the positive narrowing effect dominates the negative power effect. This results in a concave curve for the option price relative to an increasing Hurst. For a series with a Hurst close to 0,5 and over, the power effect takes precedence and drives the variance and price down.

*The option price dependency of the Hurst value with a constant maturity of T=0,75*

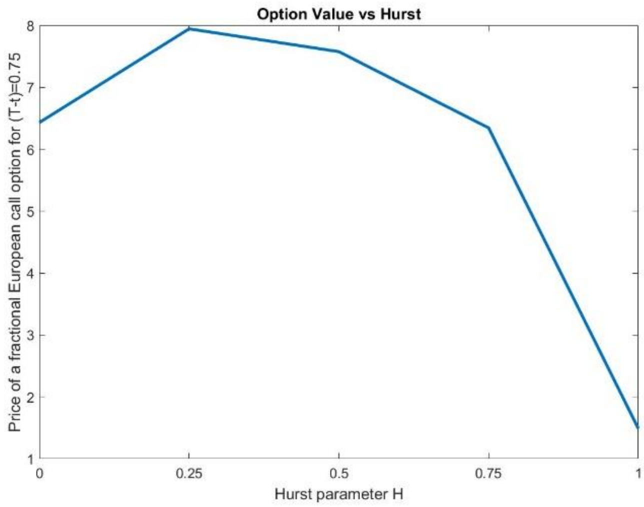



***Figure (7)***

If we increase the maturity to an even longer period *T=5*, the option price exhibit a quite peculiar pattern *(Figure 8).* As the Hurst value increases, the option price becomes higher, until the power effect takes over and the price quickly declines.

*The option price dependency of the Hurst value with a constant maturity of T=5*

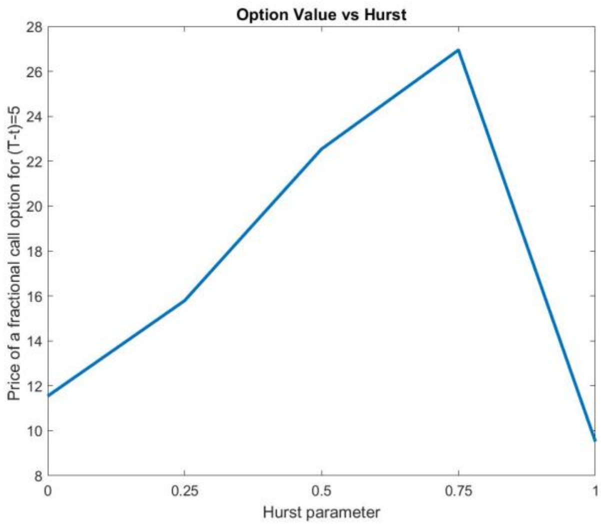



***Figure (8)***

The Hurst parameter affects option prices when maturity is changed dynamically through the unconditional variance $(T-t)^{2H}$. This gives the option prices the following properties depending on the Hurst. For the antipersistent case H<0,5 the relation between the price and time becomes concave. For the persistent case H>0,5, the shape is now convex.
This means that for a lower Hurst <0,5, by moving closer the maturity, the negative effect on option value is greater than for a series with a Hurst >0,5.[3] (*Appendix 2-4)*

*The implied volatility of fB&S, B&S and as quoted in Bloomberg*

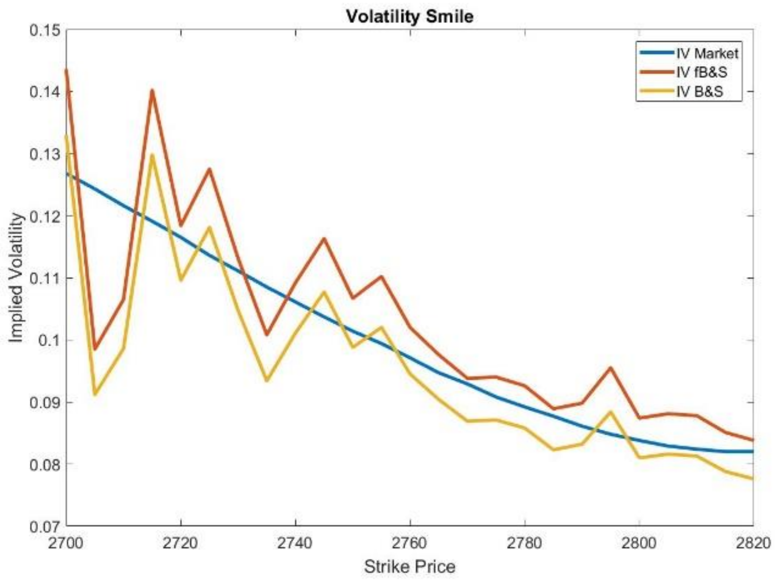



***Figure (9)***

Figure 9 show how the fB&S and B&S implied volatility measures compared to the market. One can see that both models model does in fact exhibit the properties of volatility smile. The advantage of the fB&S, is that the extension of the model gives us the ability to derive the implied volatility analytically from the B&S implied volatility.

$$\sigma_{H,n}^2 = \sigma^2 \frac{(T-t)}{p_h (T-t)^{2H}} \qquad (5.1)$$

FB&S replaces the classical variance with the conditional fractional variance. This means the implied volatility should not be affected by moneyness and should create a flat volatility line. This is however not the case as shown from figure 9. Backing out the implied volatility, we

3 Rostek and Schobel 2009

can see it changes with moneyness. FB&S is not able to correct the tendencies of a u-shaped smile, indicating that the model is misspecified.

*Implied volatility for different (anti)persistence in the series compared to a constant B&S*

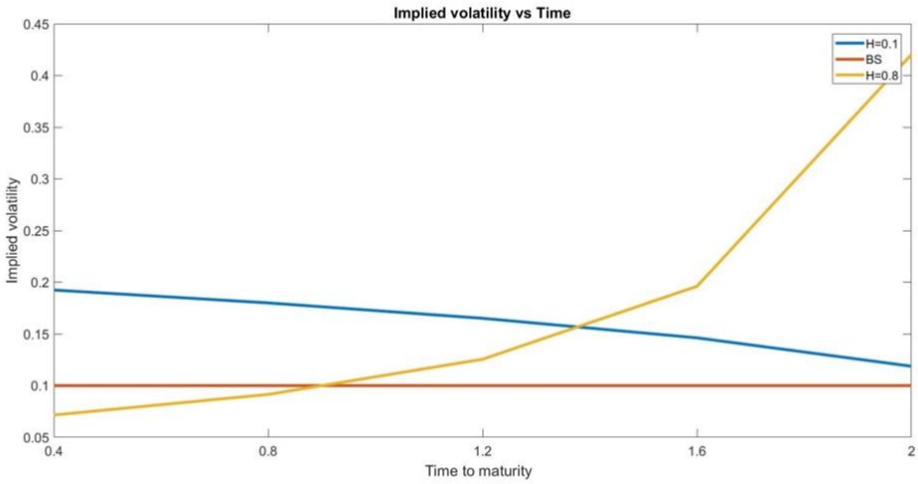



***Figure (10)***

Figure 10 shows how the implied volatility is dependent on the Hurst value and time to maturity. Holding the B&S volatility constant, we infer the fB&S implied volatility for different maturities. We see that in the case of strong persistence H=0,8 the volatility has a convex curve, that explodes when we increase the maturity. For the antipersistent case H=0,1, the volatility is high in the short run as the fractional Brownian motion induces a high level of uncertainty. The curve is however concave, and the volatility falls as we increase time to maturity.

We backed out the implied with different our estimated Hurst to see how this effect on implied volatility compares for the S&P500 data. We see that when the Hurst is close to 0,5 the volatility is quite stable, and similar to the regular B&S. However, the same pattern emerges as the time to maturity increases. Setting a higher Hurst, we see how the persistent cases estimate a different implied volatility. This shows that estimating the correct Hurst value is of great importance when applying the model for asset pricing.

*Comparison of implied volatility from real market data for different persistence in the series*

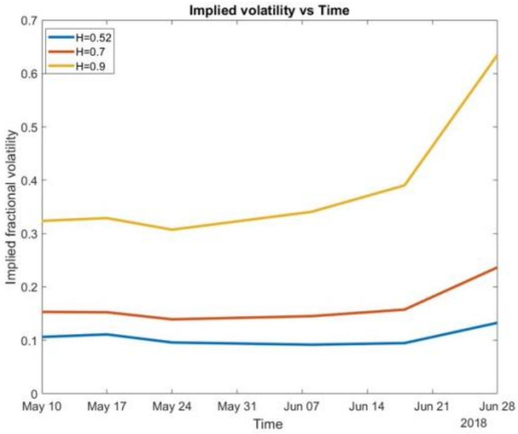



***Figure (11)***

## *5.2 Out of Sample*

*Table (7)* shows the results of the out of sample test for each of the respective models. The results are categorized in moneyness to see how the models perform when the options are in, at and out of the money.

***Pricing errors S&P B&S and FB&S***

*For each given model, we compute the price of each option at time t using the implied volatility at t-1. Each option is priced using the implied volatility of the option with the closes time to maturity. The results reported show the mean absolute pricing error and the actual dollar error between the prevailing market price at each date and the models forecasted prices in percentage numbers. The table show the percentage errors for options at the money (ATM), in the money (ITM) and out of the money (OTM).*
TOTAL shows the total MAPE and Dollar error for the whole sample.
*of S&P Call options which consists of 82 call options of each category.*

***MAPE***

| | B&S | | | | FB&S | | | |
|---|---|---|---|---|---|---|---|---|
| *Maturity* | *<30* | 30-60 | 60> | *Total average* | *<30* | 30-60 | 60> | *Total average* |
| *OTM* | 16,33 % | 9,74 % | 9,34 % | 11,86 % | 16,38 % | 10,10 % | 8,26 % | 11,84 % |
| *ATM* | 14,64 % | 14,41 % | 9,47 % | 13,11 % | 14,41 % | 12,53 % | 6,43 % | 11,51 % |
| *ITM* | 15,02 % | 11,12 % | 6,29 % | 11,03 % | 15,28 % | 11,82 % | 5,96 % | 11,44 % |
| *Total sample* | | | | 12,00 % | | | | 11,60 % |

***Errors in actual dollar value***

| | B&S | | | | FB&S | | | |
|---|---|---|---|---|---|---|---|---|
| *Maturity* | *<30* | 30-60 | 60> | *Total average* | *<30* | 30-60 | 60> | *Total average* |
| *OTM* | $ -0,81 | $ 2,05 | $ -1,19 | $ 0,11 | $ -1,02 | $ 2,26 | $ -1,22 | $ -0,10 |
| *ATM* | $ -0,74 | $ 1,12 | $ -3,96 | $ -0,98 | $ -1,05 | $ 2,50 | $ -1,54 | $ -0,07 |
| *ITM* | $ -1,19 | $ 2,57 | $ -2,28 | $ -0,17 | $ -1,21 | $ 2,69 | $ -1,96 | $ 0,05 |
| *Total sample* | | | | $ -0,35 | | | | $ 0,04 |

***Table 7***

We begin by looking at the actual dollar value. We see that fB&S ranks first. The average error is quite small for fB&S, overpricing on average only 4cents. It looks promising for the fB&S, but we do see that the dollar errors range around an error from negative 1 to a positive 3 dollars. Both models show the same pattern to underprice options for maturities <30 and >60, and overprice options 30-60.

The same can be seen in the mean absolute percentage error where the fB&S is again ranked first with an error of 11,6% compared to 12%. For the results for different maturities we see that fB&S performs better than the B&S except for the category in the money (*ITM*). Still it's a promising result for the extended model as it reduces the errors.

## *5.3 Delta hedging with a single instrument*

The delta hedging is conducted in the same way as in chapter 3. The position is calculated using the parameters at time *t*, with the implied volatility from the call option value at *t-1*. The position is rebalanced at a daily frequency. This procedure is repeated for a total time period of 60 days, rebalancing for 41 trading days. The absolute average dollar hedging errors are estimated at each point of rebalancing. The results reported in table 8.

***Delta hedging results***

*The table shows the portfolio value of a delta hedging strategy with a single instrument, the spot price. Each model is run from 70 days to maturity and rebalanced every trading day to maturity. The results show absolute average hedging error both reported in dollar value for each maturity and for the whole period.*

| | B&S | | | | FB&S | | | |
|---|---|---|---|---|---|---|---|---|
| *Maturity* | *<30* | *30-60* | *>60* | *Total average* | *<30* | *30-60* | *>60* | *Total average* |
| *OTM* | 20,05 | 8,94 | NA | 12,83 | *20,05* | 8,96 | *NA* | 12,84 |
| ATM | NA | 8,74 | 2,79 | 7,25 | NA | 8,74 | 2,79 | 7,25 |
| ITM | 15,14 | 23,65 | 14,55 | 17,65 | 15,14 | 23,65 | 14,55 | 17,65 |
| *Total sample* | | | | 14,20 | | | | 14,20 |

***Table 8***

Based on the absolute average hedging errors we see that the models perform equally, with a negligible due to rounding. We see that on average both models price at the money options better and in the money options the worst. We see that across maturities the models also prices longer term options better than short term options. Thus we can't distinguish which model is the best for hedging options. The hedging results do kind of conflict with our out of sample results as the fB&S performed better, however the B&S did have closer implied volatility. This could be due to that the H parameter is so close to H=0,5 that the models perform similar.

# 6. Conclusion

We believe the fractional extension of the B&S has potential. For our empirical tests, the fB&S performed, if only slightly, better than the regular B&S in the out of sample, yielding the same result for the dynamic hedging. The empirical evidence gathered from the in-sample test show us that our model is still misspecified with regards to the volatility smile. Comparing the implied volatility for the fractional model and the regular B&S, it is not able to explain the moneyness effect on volatility any better.
From our out of sample test we saw the model was a minor improvement over the regular B&S, reducing the average error by 0,4%. The dynamic hedging results yielded no difference. The indication is unfortunately that the extension does not yield any extra benefit in accurately pricing the options, but only adds another uncertainty in the Hurst parameter. This is further reinforced by the fact that the Hurst value were close to 0,5. It is then a strong assumption to say that this can be interpreted as persistence in the S&P500.

We found the properties of fractional Brownian motion interesting, as we saw how the models implied volatility is heavily affected by the value of the Hurst and the difference in time to maturity. Hurst estimates have shown to tend to the real value of Hurst as the dataset increases (Rea 2009), so even if our data consist of the five years of previous stock prices, it may not be enough to capture the real (anti)persistence of each series. The Hurst may correctly identify if the stock or index is persistent or anti-persistent, but not the correct value of long-range dependence.

This is troubling, as adding the Hurst parameter seems to only add to the uncertainty of the implied volatility. Current statistical tools are struggling with estimating the Hurst parameter (Cajueiro and Tabak 2008). Writing this thesis we found a dozen different Hurst models, who converge to the correct values for a simulated series, but estimate quite different parameters when applied to market data. The choice of R/S was made because of its ability to find a Hurst with limited data, and is a common choice in financial research.

Our results are based on the framework created by Bakshi, Cao & Chen (1997). The limitations of our dataset was a drawback performing these empirical tests. The fact that S&P500 is an index with a Hurst parameter close to that of an independent series is a good test for the model if it could perform under any conditions. Further empirical tests on even longer time to maturity, different stocks and European and Asian indexes would be a welcome extension to this thesis as these markets have shown stronger persistence than the American market. It would also be interesting to see how the model's performance would compare with some even more extensive tests.

We believe that the fBm is warranted a place in finance and asset pricing in general, as there are financial assets that have shown to exhibit long-range dependence. Mixed models, where the underlying is driven by a combination of fBm and the gBm could be another place to test the fractional properties. One could also try to create models incorporating stochastic volatility and stochastic interest rates as it has been shown to better capture leptokurtic features.

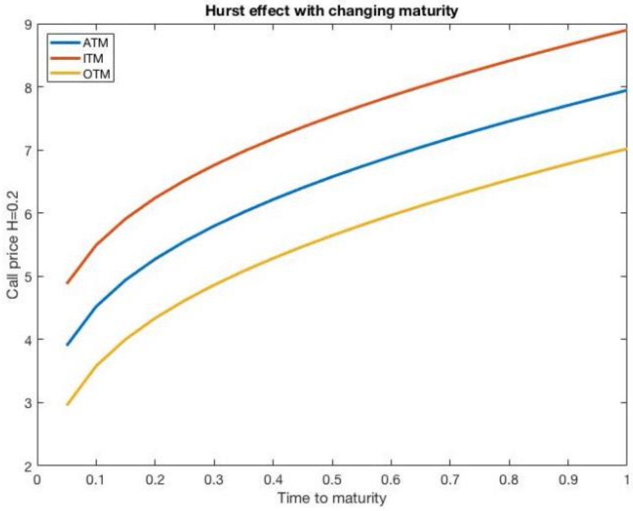
# 7. Appendix

## *Appendix 1.*

| Jarque-Bera test | |
|---|---|
| *The Jarque-Bera tests how well the data skewness and kurtosis matches that of an normal distribution.* | |
| Rejection rule: | |
| H0: log returns are normally distributed | |
| H1: log returns are non-normal | |
| | *S&P500* |
| *P-value* | *0,001* |

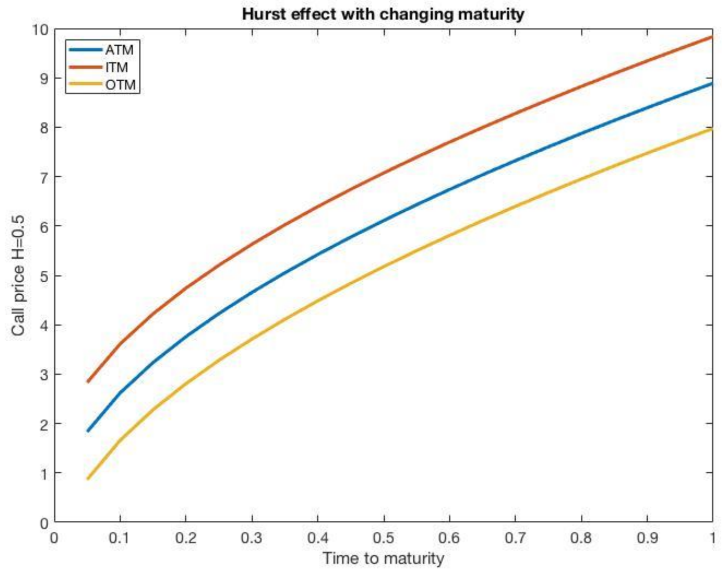
## *Appendix 2.*

*Call price for ATM, ITM and OTM call options with changing maturity for a Hurst=0,2*



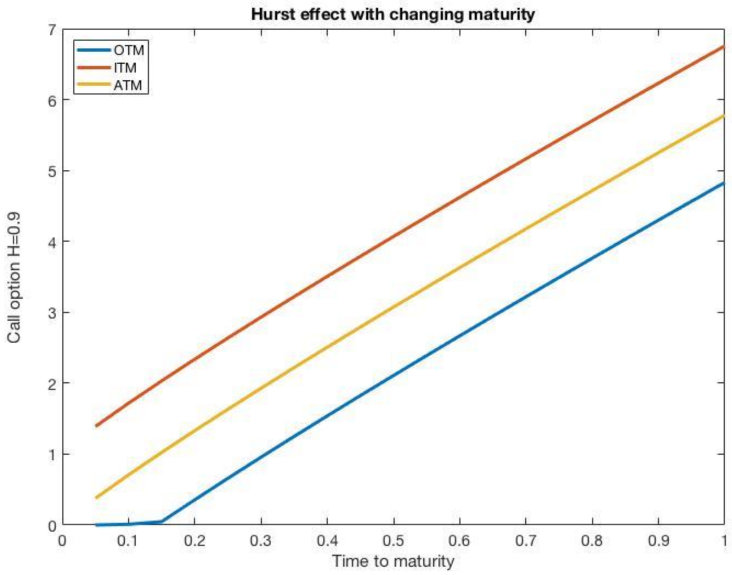
# *Appendix 3.*

*Call price for ATM, ITM and OTM call options with changing maturity for a Hurst=0,5*




# *Appendix 4.*

*Call price for ATM, ITM and OTM call options with changing maturity for a Hurst=0,9*



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