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Consumption with Liquidity Constraints: An Analytical Characterization*

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Abstract

How do liquidity constraints affect households? This is a well-researched subject with remarkably few theoretical results. This paper bridges this gap by providing analytical results on how a liquidity constraint affects optimal consumption. I first provide an implicit analytical solution to consumption with liquidity constraints. Using this expression, I prove that the consumption function is strictly concave in wealth if there exists a relevant liquidity constraint. Since uninsurable income risk also makes consumption concave in wealth, it is not possible to separate the effects of income risk from the effects of liquidity constraints. Next, I prove that for utility functions that satisfy decreasing absolute risk aversion, a tighter liquidity constraint induces households to reduce consumption, become more sensitive to wealth changes, have a "more" concave consumption function in wealth, and exhibit greater absolute risk aversion and absolute prudence. This result thus provide theoretical results for how credit crunches affect households.

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1 Introduction

The question of how liquidity constraints affect households attracts a lot of attention. Numerous important papers discuss how credit constraints affect consumption (e.g. Zeldes

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1989, Jappelli and Pagano 1994, and Ludvigson 1999), amplify business cycles (e.g. Kiyotaki and Moore 1997 and Iacoviello 2005), affect the cost of business cycles (e.g. Imrohoroğlu 1989), and prolong crises (e.g. Eggertsson and Krugman (2012) on deleveraging and Guerrieri and Lorenzoni (2017) on credit crunches). However, despite the widespread interest, there are remarkably few analytical results on liquidity constraints. This paper attempts to bridge this gap by providing analytical results on two important questions: How does the presence of a liquidity constraint affect household behavior? And how does changes in liquidity constraints affect household behavior?

To address these questions, I study a household problem where infinitely-lived, impatient, and risk-averse households save and borrow, have time-separable hyperbolic absolute risk aversion (HARA) utility, and face a liquidity constraint on wealth holdings. The key to my results relies on the observation that the household problem with a liquidity constraint is a solvable first-order differential equation in optimal consumption. I derive an implicit closed form solution to the consumption function where the liquidity constraint enters as a boundary constraint. Using this framework, I prove how the presence of and shifts in liquidity constraints affect consumption and risk preferences.

The first contribution of this paper is to show how liquidity constraints affect consumption. I compare the consumption function with and without a liquidity constraint, and show that households respond to the constraint in three ways. First, they reduce consumption. The liquidity constraint forces consumption to be equal to total income at the constraint. Since households are impatient, they reduce consumption at the constraint, but also for all other levels of wealth since they want to smooth consumption. Second, households become more sensitive to wealth changes. Households want to avoid the constraint, making each unit of wealth more valuable. They therefore increase the marginal propensity to consume (MPC) out of wealth. Third, the consumption function is strictly concave in wealth. The effects of the constraint are weaker the wealthier households are. The reduction in consumption is therefore greatest for households close to the constraint and less so for wealthy households. The result of this gradual decrease in influence of the constraint is a strictly concave consumption function in wealth.

Importantly, the results here rely on a continuous time version of the consumption-saving problem. There is a notable difference between the results in this paper and comparable results for discrete time versions of the same model (Carroll and Kimball 2001). In discrete time, the liquidity constraint introduces a kink point in wealth in the optimal consumption function. Below this threshold, households are constrained and consumption is equal to total income. Above this threshold, households are unconstrained. The continuous time version of the model introduces no such kink points. Indeed, as long as households have wealth above the constraint, they are never constrained. Moreover, the derivatives of the optimal consumption function are well-defined for all levels of wealth. As a result, I can prove how the liquidity constraint affects the MPC and the concavity of the consumption function for all households.

A liquidity constraint and uninsurable income risk have the same qualitative effects on consumption. Households reduce consumption, increases the MPC out of wealth, and has a concave optimal consumption function in the presence of either a liquidity constraint or uninsurable income risk (Kimball 1990a and Carroll and Kimball 1996) However, there is one important difference. The effects of a liquidity constraint are independent of the third derivative of the utility function (prudence). Even imprudent households have lower consumption, are more sensitive to wealth changes, and have a concave consumption function in wealth if a liquidity constraint exists. Indeed, since the liquidity constraint generates buffering behavior similar to that of an income risk model and this effect is independent of prudence, we can conclude that prudence is not necessary for precautionary saving even at the household level. This result is stronger than Huggett and Ospina (2001) who show that prudence is not necessary for aggregate precautionary saving.

The second question I ask is how does a tighter liquidity constraint affect consumption and risk preferences? Credit crunches and changing credit conditions are important components of the business cycle. Recently, Guerrieri and Lorenzoni (2017) derive important implications of credit crunches on interest rates and output. This paper complements this research by providing theoretical results on how a tighter liqudity constraint affects households. I show, in Proposition 2, that risk-averse, impatient, and infinitely-lived households with decreasing absolute risk aversion (DARA) utility respond to a tighter liquidity constraint by reducing consumption, becoming more sensitive to wealth changes, having a "more" concave consumption function in wealth, and increasing absolute risk aversions and absolute prudence.

In a sense, a tighter liquidity constraint is very similar to a demand shock. All else equal, households respond by consuming less. However, there are further implications on consumption dynamics and policy transmission. First, tighter liquidity constraints limit households' ability to borrow. An interest rate reduction that is supposed to induce households to consume more has less influence when borrowing is more restricted. Expansionary monetary policy is therefore less influential when liquidity constraints are tighter. Second, tighter liquidity constraints make households more sensitive to short-run wealth changes. Short-run changes in disposable income, such as tax rebates or transfer policies, influence households more if the marginal propensity to consume is higher. As a result, fiscal policy is more influential with a tighter liquidity constraint. Changes in credit conditions thus alter the optimal mix of monetary and fiscal policy.

This paper is not the first to analytically characterize how liquidity constraints affect consumption. Seater (1997) and Park (2006) are similar to my paper in the sense that they use a continuous time approach to solve the consumption-saving problem with liquidity constraints. Seater (1997) analyzes the case where time-discounting equals the interest rate and income fluctuates. Compared with the unconstrained case, he shows that consumption must be lower when the constraint binds and higher in other periods for the budget constraint to hold. Park (2006) solves the problem with CRRA utility and shows that the liquidity constraint affects optimal consumption even for households far away from the liquidity constraint. Carroll and Kimball (2001) analyzes a discrete time version of the model and find that for either quadratic, CRRA or CARA utility, the consumption function is concave in wealth in the neighborhood of the liquidity constraint. Moreover, Nishiyama and Kato (2012) show that the consumption function is concave in wealth with quadratic utility if a liquidity constraint exists. Since the quadratic utility has zero prudence, this implies that prudence is not necessary for the consumption function to be concave in wealth. In a related paper, Carroll (2004a) derives the conditions for a well-defined solution with a liquidity constraint, but limits the analysis to CRRA utility. The contributions of this paper are to characterize the global effects of liquidity constraints on consumption for a general class of utility functions and further to characterize how changes in liquidity constraints affect households. Moreover, I explicitly derive how liquidity constraints and changes in these affect risk preferences.

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 presents the results on how a liquidity constraint affect consumption and risk preferences. In Section 4, I present the result on how changes in the liquidity constraint affect consumption and risk preferences.

2 The Model

I solve the consumption-saving problem for an infinitely-lived impatient household with constant income and a fixed liquidity constraint. The household maximizes its lifetime utility flow from future consumption c_t discounted at rate $\rho \geq 0$

$$\int_t^{\infty} e^{-\rho s} u(c(w_s)) ds \tag{1}$$

where $u(c)$ is a utility function of the HARA class

$$u(c) = \begin{cases} \frac{1}{a-1}(ac+b)^{1-1/a} & a \neq 0, 1 \\ -be^{-c/b} & a = 0 \\ \log(c+b) & a = 1 \end{cases} \tag{2}$$

with $b > \max\{-ac, 0\}$. Note that $u'(c) > 0$ and $u''(c) < 0$ is always true for HARA utility. Household wealth takes the form of risk-free bonds and evolves according to

$$dw_t = (rw_t + y - c(w_t))dt \tag{3}$$

where r is the real interest rate. The household also faces an exogenous liquidity constraint on wealth holdings

$$w_t \geq \underline{w} \tag{4}$$

where $\underline{w}$ is a scalar satisfying $\underline{w} > -\frac{y}{r}$ (natural borrowing limit).

For the rest of this paper, I analyze the household problem in a stationary setting and present the equations without time subscripts. The following Hamilton-Jacobi-Bellman (HJB) equation fully describes the solution to the household problem[1]

$$\rho V(w) = \max_c u(c) + V'(w)(rw + y - c(w)) \tag{5}$$

where $V(w)$ is the value function of a household with assets w. Note that equation (5) can also represent a model with deterministic income growth, see Appendix B. The first order necessary condition is

$$u'(c) = V'(w) \tag{6}$$

To obtain a differential equation in consumption, I first use the envelope theorem on equation (5) to obtain

$$c(w) = rw + y - \frac{V'(w)}{V''(w)}(\rho - r)$$

and then use the first order condition (6) to replace all value function expressions with utility functions to obtain

$$c(w) = rw + y - \frac{u'(c(w))}{u''(c(w))c'(w)}(\rho - r)$$

Furthermore, the definition of HARA utility is that risk tolerance is linear in consumption,

[1]See e.g. Chang (2004) for how to derive this expression.

$-\frac{u'(c(w))}{u''(c(w))} = ac(w) + b$. We can then simplify further to get

$$c(w) = rw + y + \frac{ac(w) + b}{c'(w)}(\rho - r) \tag{7}$$

The optimal solution to the household problem is therefore a first-order non-linear differential equation in consumption.

3 Consumption with Liquidity Constraints

This section contains three main results. First, I show that the differential equation is solvable and we are able to obtain an implicit expression (8) for optimal consumption in Proposition 1. Using expression (8), it is straightforward to derive three results on the qualitative properties of consumption with liquidity constraints in Corollary 1. Third, I show how the liquidity constraint affects households' risk preferences in Corollary 2.

Proposition 1. *Suppose* $\underline{w} > -\frac{y}{r}$, $\rho > r > 0$, *then*

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho - r)}} \tag{8}$$

optimally solves (7) *for all HARA utility functions* (2).

Proof. See Appendix A.1. □

Proposition 1 provides an implicit expression for optimal consumption with a liquidity constraint. Note that (8) contains the special case with linear consumption when the liquidity constraint is the natural borrowing limit ($\underline{w} = -\frac{y}{r}$)

$$c_u(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} \tag{9}$$

By comparing consumption in the constrained and the unconstrained case, we can show what happens to optimal consumption when we add a liquidity constraint

Corollary 1. *For utility functions of the HARA class, if* $\underline{w} > -\frac{y}{r}$, $\rho > r > 0$, *and* $r + a(\rho - r) > 0$. *Then there exists an optimal consumption function for all* $w \in (\underline{w}, \infty)$ *where*

1. *Consumption is lower than in the unconstrained case:* $c_u(w) > c(w) > rw + y$.

2. *The MPC out of wealth is greater than in the unconstrained case:* $c'(w) > c'_u(w) > 0$.

3. *Consumption is strictly concave in wealth:* $c''(w) < 0$.

4. *If also* $a > 0$, *consumption is asymptotically linear:* $\lim_{w\to\infty} c(w) = c_u(w)$.

Proof. See Appendix A.2. □

Corollary 1 reveals four effects of introducing a liquidity constraint in the optimal consumption problem. First, the presence of a liquidity constraint reduces consumption for all levels of wealth. The liquidity constraint imposes a shadow cost on consumption because households know that the constraint binds at some point in the future. Due to the intertemporal substitution motive, households become more reluctant to consume and reduce their optimal consumption.

Second, the MPC out of wealth is always greater with a liquidity constraint than in the unconstrained case. Since households want to avoid the liquidity constraint, each unit of wealth is more valuable to households. They are therefore more sensitive to wealth changes, increasing their MPC out of wealth.

Third, the optimal consumption function is strictly concave in wealth in the presence of a liquidity constraint. The effects of the liquidity constraint are greatest at the constraint, but also affects all wealth levels through the intertemporal substitution motive. However, the effect is declining in the distance from the constraint. Poor households reduce consumption more than wealthy households. This asymmetric reduction in consumption results in an optimal consumption function that is strictly concave in wealth.

Fourth, if utility also satisfies DARA, consumption is asymptotically linear as wealth goes to infinity. However, note that the optimal consumption function is never linear for a finite value of wealth since $c_u(w) > c(w)$.

There are two notable differences between introducing liquidity constraint in continuous time compared to discrete time. In discrete time, there exists a threshold wealth level. Below this wealth level, households are constrained, while the consumption function is linear above this level (Carroll and Kimball 2001). The liquidity constraint therefore introduces a kink point in the optimal consumption function in discrete time. This does not happen in continuous time. Since $c(w) > rw + y$ when $w > \underline{w}$, the liquidity constraint never binds unless the household is exactly at the constraint. Moreover, since $c_u(w) > c(w)$ for all $w \in (\underline{w}, \infty)$, the optimal consumption function is never exactly equal to the linear unconstrained solution for a finite value of wealth. As a result, the liquidity constraint does not introduce a kink point and the optimal consumption is smooth and well-behaved.

This smoothness is a very important property and allows us to derive general results for the effects of the liquidity constraint. In particular, the first and second derivatives of the optimal consumption function exist for all wealth levels. In discrete time, on the other hand, the liquidity constraint only introduces concavity locally around the kink point (counterclockwise concavification, Carroll and Kimball 2001), but is linear otherwise. Moreover, the MPC is higher only below the kink point, and equal otherwise. The fact that liquidity constraints introduces this artificial non-smoothness in discrete time is an important reason why solving economic models in continuous time might be preferable.[2]

Remark 1. *The results in Corollary 1 only hold for $w \in (\underline{w}, \infty)$. For $w = \underline{w}$, $c(\underline{w}) = r\underline{w} + y$ while $c'(\underline{w})$ and $c''(\underline{w})$ are undefined.*

Remark 2. *The condition $\underline{w} > -\frac{y}{r}$ ensures that the liquidity constraint is tighter than the natural borrowing limit.*

Remark 3. *The results in Corollary 1 are independent of the third derivative of the utility function.*

Remark 4. *The condition $\rho > r > 0$ ensures that the liquidity constraint is relevant.*

Remark 5. *The condition $r + a(\rho - r) > 0$ ensures that $c'(w) > 0$ for all $w \in (\underline{w}, \infty)$.*

3.1 An Illustration of Corollary 1

In order to provide some intuition behind the results in Proposition 1, I illustrate how the liquidity constraint affects consumption for a power utility function.

Example 1. *Optimal Consumption with Power Utility.*
For households with power utility, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, the optimal consumption function is

$$c(w) = \frac{\gamma r - r + \rho}{\gamma r}(rw + y) - \left(\frac{\rho - r}{\gamma r}\right)(r\underline{w} + y)\left(\frac{r\underline{w} + y}{c(w)}\right)^{\frac{\gamma r}{\rho - r}}$$

Figure 1 illustrates how a liquidity constraint affects optimal consumption. The blue line shows consumption in the constrainted case and the red line shows consumption in the unconstrained case. The difference between the two lines illustrates how the liquidity constraint affects consumption. First, when households face a liquidity constraint, they cannot consume more than their total income at the constraint. Since consumption is always greater than total income when $\rho > r > 0$, households are forced to reduce consumption at the constraint. Since households smooth consumption through the intertemporal substitution motive, they also reduce consumption for wealth levels above the constraint. However, this effect is decreasing in the distance from the constraint.

[2]Other reasons are computational efficiency (Achdou et al. 2016) and the fact that one can more often derive closed form solutions to household problems in continuous time than in discrete time.

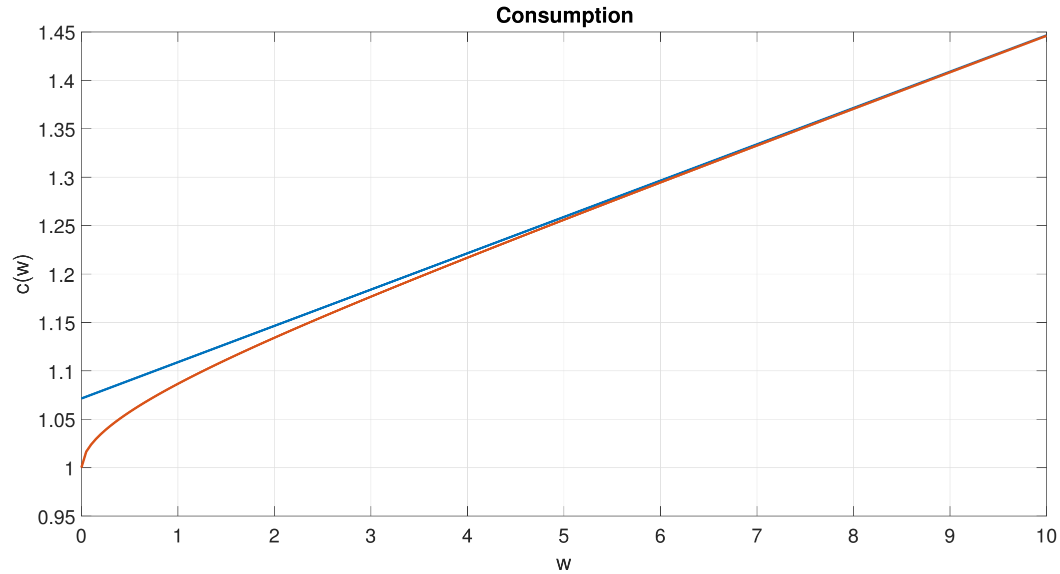

Figure 1: The effects of liquidity constraints for power utility.
Parameters: $\gamma = 2$; $y = 1$, $\rho = 0.04$, $r = 0.035$, and $\underline{w} = 0$.

Indeed, as wealth increases, the optimal constrained consumption function converges to the linear unconstrained consumption function. Figure 1 thus reveals all four effects of a liquidity constraint in Corollary 1: consumption is lower, the marginal propensity to consume out of wealth is higher, the optimal consumption function is concave in wealth, and the consumption function converges to the linear unconstrained consumption function as wealth goes to infinity.

3.2 Can we separate the effects of liquidity constraints from the effects of uninsurable income risk?

The results in this paper imply that the effects of liquidity constraints and uninsurable income risk are qualitatively similar. Corollary 1 shows that consumption is lower, the marginal propensity to consume out of wealth is higher, and the consumption function is concave in wealth if a liquidity constraint exists. Introducing uninsurable income risk has the same qualitative effects on optimal consumption (Kimball 1990a and Carroll and Kimball 1996). An important question is whether we can distinguish between the two channels. Indeed, many researchers have shown that the consumption function is concave in wealth, supporting that either liquidity constraints, income risk, or both are present.

However, there is a way to separate the two sources of consumption concavity. The effects of a liquidity constraint are independent of prudence while the effects of income risk crucially depend on the third derivative of the utility function. If one can find households that are imprudent in the data, but still have a concave consumption function, it suggests that a liquidity constraint is present and important. Moreover, a comparison of the shape of the consumption function between imprudent and prudent households can reveal the relative importance of the two channels. If the consumption functions are similar for the two groups, it suggests that liquidity constraints are the main culprit. While if the consumption functions are different between the two groups, it suggests that income risk is important for the shape of the consumption function.

3.3 Risk Preferences with Liquidity Constraints

Liquidity constraints also affect how households' behavior towards risk.

Corollary 2. *For utility function of the DARA class ($a > 0$), if $\underline{w} > -\frac{y}{r}$, and $\rho > r > 0$, the liquidity constraint has the following effects on risk preferences*

1. *Absolute risk aversion is higher than in the unconstrained case, $A(c(w)) \geq A(c_u(w))$.*

2. *Absolute prudence is higher than in the unconstrained case, $P(c(w)) \geq P(c_u(w))$.*

Proof. See Appendix A.3. □

Corollary 2 shows that if households have utility functions with decreasing absolute risk aversion (DARA), a liquidity constraint has two effects on risk preferences: households are more risk-averse in the sense that their absolute risk aversion is higher, and households are more prudent in the sense that their absolute prudence is higher.

Absolute risk aversion and absolute prudence are important sufficient statistics related to gambles and precautionary saving, respectively. Pratt (1964) shows that for small risks, the risk premium is increasing in absolute risk aversion. The introduction of a liquidity constraint therefore raises the risk premium of households. As a result, households reduce demand for risky assets and prices of risky assets should be lower when there exists a liquidity constraint than in the unconstrained case. Similarly, Kimball (1990b) shows that absolute prudence is a measure of the intensity of the precautionary saving motive. For small income risk, the precautionary saving premium is increasing in absolute prudence. One prediction is thus that constrained households are more sensitive to income risk and save more for precautionary reasons. Indeed, the liquidity constraint strengthens the precautionary saving motive, as emphasized by (Carroll and Kimball 2001).

4 Comparative Statics: Shifts in the Liquidity Constraint

In this section, I go beyond the effects of introducing a liquidity constraint to an unconstrained household and investigates the effects of changes in the liquidity constraint. The key observation is that the degree to which consumption is affected by a liquidity constraint is governed by a simple metric, the distance between consumption at the liquidity constraint in the unconstrained and the constrained cases. I define this metric as the cost of the liquidity constraint and characterize how changes in this cost affect consumption and risk preferences.

Definition 1. *The Cost of the Liquidity Constraint.*
The cost of the liquidity constraint is the distance between consumption at the liquidity constraint in the constrained and the unconstrained case

$$\kappa(\underline{w}) = c_u(\underline{w}) - c(\underline{w}) = \frac{\rho - r}{r}\Big(a(r\underline{w} + y) + b\Big) > 0$$

Note that the cost of the liquidity constraint is linear in the value of the constraint, $\underline{w}$. Thus one can equally well interpret the results in this section as the effects of shocks to the liquidity constraint. Indeed, this section provides theoretical results for a permanent credit crunch shock similar to that analyzed in e.g. Guerrieri and Lorenzoni (2017). In addition, it provides theoretical results for how other shocks affect consumption through changes in the liquidity constraint. For instance, if households face loan-to-value constraints on mortgage debt, a house price decline affects the tightness of the liquidity constrain in addition to many other channels. The results in this section thus characterize how households adjust due to the liquidity constraint channel in response to a house price decline. Indeed, the subsequent results indeed remain agnostic to the underlying cause of changes in the tightness of the constraint.

Proposition 2. *The Effects of Tighter Credit Conditions.*
For utility functions of the DARA class ($a > 0$), if $u' > 0$, $u'' < 0$, $a > 0$, $\underline{w} > -\frac{y}{r}$, $\rho > r > 0$, and $r + a(\rho - r) > 0$. Then for all $w \in (\underline{w}, \infty)$, the effects of an increase in the cost of the liquidity constraint $\hat{\kappa} > \kappa$ on consumption are

1. *Consumption decreases: $c(w; \hat{\kappa}) < c(w; \kappa)$.*

2. *The MPC out of wealth increases: $c'(w; \hat{\kappa}) > c'(w; \kappa)$.*

3. *The consumption function is "more" concave in wealth: $c''(w; \hat{\kappa}) < c''(w; \kappa)$.*

4. *The degree of absolute risk aversion increases: $A(c(w; \hat{\kappa})) > A(c(w; \kappa))$.*

5. *The degree of absolute prudence increases:* $P(c(w;\hat{\kappa})) > P(c(w;\kappa))$.

Proof. See Appendix A.4. □

Proposition 2 presents five qualitative effects of tighter credit conditions on household behavior. consumption decreases, the MPC out of wealth increases, consumption is "more" concave in wealth, households act more risk averse, and households act more prudent. The results in Proposition 2 thus imply that a tighter liquidity constraint always strengthen the effects of the liquidity constraint, inducing a greater deviation from the unconstrained (and linear) case.

In a sense, a tighter liquidity constraint is very similar to a negative demand shock. All else equal, households respond by consuming less. However, there are further important implications on consumption dynamics and policy transmission. First, a tighter liquidity constraint limits households' ability to borrow. Subsequently, interest rate cuts have smaller effects on consumption since borrowing is more restricted. Expansionary monetary policy is therefore less influential when liquidity constraints are tighter. Since tight credit conditions are common during recessions, this effect can help explain the observed procyclicality of the influence of monetary policy (see e.g. Tenreyro and Thwaites 2016). Second, a tighter liquidity constraint makes households more sensitive to short-run wealth changes. Since stimulus policies such as tax rebates influence consumption more if the MPC is higher, fiscal policy is therefore more influential during periods with tighter credit conditions. Since tighter credit conditions are common during recessions, this implies that fiscal policy is more influential during recessions. As a result, tighter credit conditions make monetary policy less influential and fiscal policy more influential and the optimal policy varies with credit conditions. This could have important implications for the coordination of optimal policy over the business cycle.

Proposition 2 also present two results on risk preferences. First, a tighter liquidity constraint makes households more risk-averse. Since more risk-averse households require higher risk premiums, a tighter liquidity constraint results in lower prices on risky assets. Subsequently, since credit conditions are tighter during recessions, we should observe that asset prices fall during recessions. Second, a tighter liquidity constraint also makes households more prudent and therefore more sensitive to uninsurable income risk. Households should therefore respond to tighter liquidity constraints by saving more, both because the liquidity constraint is tighter, but also because households are more prudent and therefore save more due to income risk.

5 Conclusion

In this paper, I derive an implicit analytical expression for optimal consumption for infinitely-lived, impatient, and risk-averse households with HARA utility. Using this expression, I show that the introduction of a liquidity constraint reduces consumption, increases the marginal propensity to consume out of wealth, and makes the consumption function concave in wealth. A liquidity constraint therefore has the same qualitative effects on consumption as prudence and uninsurable income risk - but the effect is independent of the sign of the third derivative of the utility function. Under the further assumption of DARA utility, I also show that the introduction of a liquidity constraint makes households more risk-averse and more prudent. In the second part of the paper, I show how tighter credit conditions affect optimal household behavior. For households with DARA utility, a tighter liquidity constraint induces them to reduce consumption, increase their marginal propensities to consume, make the consumption function ”more” concave in wealth, and households more risk-averse and prudent.

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A Appendix

A.1 Proof of Proposition 1

Proof. The first step is to solve for the closed form solution of the optimal value function. I show in Lemma 1 that equation (7) is solvable and the solution contains a free parameter which is pinned down by the liquidity constraint.

Lemma 1. *The analytical solution.*
Suppose that there exists a function $c(w)$ such that

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - K_0\left(ac(w) + b\right)^{-\frac{r}{a(\rho-r)}} \tag{10}$$

where K_0 is a constant. Then equation (7) holds for the HARA utility function

$$u(c) = \begin{cases} \frac{1}{a-1}\left(ac + b\right)^{1-1/a} & a \neq 0, 1 \\ -be^{-c/b} & a = 0 \\ \log(c + b) & a = 1 \end{cases}$$

The proof is as follows

Proof. Suppose that (10) holds. Differentiating both sides with respect to w, we get

$$c'(w) = r + a(\rho - r) + \frac{r}{a(\rho - r)}K_0\left(ac(w) + b\right)^{-\frac{r+a(\rho-r)}{a(\rho-r)}} ac'(w) \tag{11}$$

Solving (10) for the K_0-term

$$K_0\left(ac(w) + b\right)^{-\frac{r}{a(\rho-r)}} = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - c(w) \tag{12}$$

Substituting (12) into (11)

$$c'(w) = r + a(\rho - r) + \frac{r}{\rho - r}\left((rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - c(w)\right)\left(ac(w) + b\right)^{-1} c'(w) \tag{13}$$

Rearranging (13)

$$c(w) = \frac{r + a(\rho - r)}{r}\left(\frac{ac(w) + b}{c'(w)}(\rho - r) + rw + y\right) - (ac(w) + b)\frac{\rho - r}{r} + b\frac{\rho - r}{r}$$

$$c(w) = rw + y + \frac{ac(w) + b}{c'(w)}(\rho - r)$$

which is (7). □

The second step is to show that there exists a $c(w)$ that satisfies (10) together with the liquidity constraint $c(\underline{w}) \leq r\underline{w} + y$. Note that under the assumption of Proposition 1, $\rho > r > 0$ so $c(w) \geq rw + y$. The liquidity constraint therefore holds with equality, $c(\underline{w}) = r\underline{w} + y$. Solve (10) for K_0 using the liquidity constraint

$$K_0 = \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r + a(\rho - r)}{a(\rho - r)}} \tag{14}$$

Insert this K_0 into (10)

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho - r)}} \tag{15}$$

By the assumption in Proposition 1, $\rho > r > 0$ and $ac(w) + b > 0$, so (15) is continuous and exists for all $w \in [\underline{w}, \infty)$. Furthermore, it satisfies the liquidity constraint by construction.

The third step is to derive the solution for optimal consumption in the unconstrained case. This is very helpful because consumption in the constrained case is always bounded by the linear unconstrained solution and I will use this property to show that (15) is optimal. I define unconstrained consumption as the solution to (15) under the natural borrowing limit ($\underline{w} = -\frac{y}{r}$)

$$c_u(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} \tag{16}$$

The unconstrained solution is thus linear in total income. Moreover,

$$c_u(w) - c(w) = \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho - r)}} > 0$$

so constrained consumption is always bounded upward by the linear unconstrained consumption function.

The fourth step is to verify that the consumption rule (15) is indeed optimal. Since the first-order condition and the liquidity constraint hold by construction, it suffices to verify the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} V(w_t) = 0$$

First note that w is always bounded below by the liquidity constraint. We therefore need to show that w is bounded above. To show this note first that consumption is always greater or equal to total income

$$c(w) - (rw + y) = (a(rw + y) + b)\,\frac{\rho - r}{r} - \left(a(r\underline{w} + y) + b\right)\frac{\rho - r}{r}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} \geq 0$$

since $w \geq \underline{w}$ and $0 \leq \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}} \leq 1$ when $\rho > r > 0$. It follows that $w_{t+\Delta t} \leq w_t$ for all t such that the path of wealth is also bounded ($w_t \in [\underline{w}, w_0]$ for all t). Since all flow utility functions are continuous, the value function is also continuous and bounded such that the transversality condition is trivially satisfied.

□

A.2 Proof of Corollary 1

Proof. Corollary 1 follows by applying repeated total derivation on the solution for optimal consumption in the constrained problem and the unconstrained problem (16), and comparing the results. For convenience, I reiterate the optimal consumption functions in the constrained

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r}{a(\rho-r)}}$$

and the unconstrained cases

$$c_u(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r}$$

Part I: $c_u(w) > c(w) > rw + y$

Calculating the difference between consumption in the constrained and the unconstrained case, we get

$$c(w) - c_u(w) = -\frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r}{a(\rho-r)}} < 0$$

where the last inequality follows since $\rho > r > 0$ and $ac(w) + b > 0$ by assumption. To show the second part, calculate

$$c(w) - (rw + y) = (a(rw + y) + b)\,\frac{\rho - r}{r} - \left(a(r\underline{w} + y) + b\right)\frac{\rho - r}{r}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho - r)}} > 0$$

since $w \in (\underline{w}, \infty)$ and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}} < 1$ when $\rho > r > 0$.

Part II: $c'(w) > c_u'(w) > 0$
From (16), we know that

$$c_u'(w) = r + a(\rho - r) > 0$$

where the last inequality follows by the assumption in Corollary 1. We then need to find the marginal propensity to consume from the constrained solution. Apply total derivation on (8)

$$c'(w) = r + a(\rho - r) + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r + a(\rho - r)}{a(\rho - r)}} c'(w) \tag{17}$$

Note that since $r + a(\rho - r) > 0$ and $\rho > r > 0$ by assumption, we have

$$0 < \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r + a(\rho - r)}{a(\rho - r)}} < \left(\frac{a(r\underline{w} + y) + b}{a(r\underline{w} + y) + b}\right)^{\frac{r + a(\rho - r)}{a(\rho - r)}} = 1$$

It then follows that

$$c'(w) = \frac{r + a(\rho - r)}{1 - \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}} > r + a(\rho - r) = c_u'(w)$$

Part III: $c''(w) < c_u''(w) = 0$
Note first that since the unconstrained consumption function is linear, $c_u''(w) = 0$. We therefore only need to show that $c''(w) < 0$ in the constrained solution. Apply total derivation on (17)

$$c''(w) = \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c''(w) - \frac{r + a(\rho - r)}{\rho - r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r+2a(\rho-r)}{a(\rho-r)}} (c'(w))^2$$

$$c''(w) = -\frac{1}{1 - \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}}\,\frac{r + a(\rho - r)}{\rho - r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r+2a(\rho-r)}{a(\rho-r)}} (c'(w))^2 < 0$$

where the last inequality follows since $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$, $\rho > r > 0$, $r + a(\rho - r) > 0$, and $ac(w) + b > 0$ by the assumptions in Corollary 1.

Part IV: $\lim_{w\to\infty} c(w) = c_u(w)$

We know that $c'(w) > 0$ for all w so that $\lim_{w\to\infty} c(w) = \infty$. It thus follows that as long as $a > 0$

$$\lim_{w\to\infty} -\frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r}{a(\rho-r)}} = 0$$

such that

$$\lim_{w\to\infty} c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} = c_u(w)$$

□

A.3 Proof of Corollary 2

Proof. **Part I:** $A(c(w)) \geq A(c_u(w))$

Absolute risk aversion for HARA utility equals

$$A(c(w)) = -\frac{u''(c(w))}{u'(c(w))} = \frac{1}{ac(w) + b}$$

Under the DARA assumption, $\frac{A(c(w))}{dc(w)} < 0$, then since $c(w) \leq c_u(w)$, we get $A(c(w)) \geq A(c_u(w))$.

Part II: $P(c(w)) \geq P(c_u(w))$

Absolute prudence for HARA utility equals

$$P(c(w)) = -\frac{u'''(c(w))}{u''(c(w))} = \frac{1 + a}{ac(w) + b}$$

Under the DARA assumption, $\frac{P(c(w))}{dc(w)} < 0$, then since $c(w) \leq c_u(w)$, we get $P(c(w)) \geq P(c_u(w))$. □

A.4 Proof of Proposition 2

Proof. Since we know that $\kappa(\underline{w})$ is monotonically increasing and linear in $\underline{w}$, all results for $\underline{w}$ also hold for $\kappa\underline{w}$. I therefore show all results for $\underline{w}$

Before we start, let us reiterate the optimal consumption function

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r}{a(\rho-r)}}$$

Part I: $\frac{dc(w)}{d\underline{w}} < 0$
Apply total derivation of the consumption expression with respect to $\underline{w}$ gives

$$\frac{dc(w)}{d\underline{w}} = -(r + a(\rho - r))\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} + \frac{dc(w)}{d\underline{w}}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 0$$

where the last inequality follows since $r+a(\rho-r) > 0$, $\rho > r > 0$, and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$.

Part II: $\frac{dc'(w)}{d\underline{w}} > 0$
Apply total derivation on (8) with respect to w and obtain

$$c'(w) = r + a(\rho - r) + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c'(w) \tag{18}$$

Apply total derivation on (18) with respect to $\underline{w}$

$$\begin{aligned}\frac{dc'(w)}{d\underline{w}} = & \frac{r + a(\rho - r)}{\rho - r}\frac{r}{ac(w) + b}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} c'(w) \\ & - \frac{r + a(\rho - r)}{\rho - r}\frac{a(r\underline{w} + y) + b}{(ac(w) + b)^2}\frac{dc(w)}{d\underline{w}}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} c'(w) \\ & + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} \frac{dc'(w)}{d\underline{w}} > 0\end{aligned}$$

where the last inequality follows since $r + a(\rho - r) > 0$, $\rho > r > 0$, $ac(w) + b > 0$, $c'(w) > 0$, $\frac{dc(w)}{d\underline{w}} < 0$, and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$.

Part III: $\frac{dc''(w)}{d\underline{w}} < 0$
Apply total derivation on (18) with respect to w to obtain

$$c''(w) = -\frac{r + a(\rho - r)}{\rho - r}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} \frac{(c'(w))^2}{ac(w) + b} + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c''(w) \tag{19}$$

Apply total derivation on (19) with respect to $\underline{w}$

$$
\begin{aligned}
\frac{dc''(w)}{d\underline{w}} = &-\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}} \underbrace{\left(\frac{r}{ac(w)+b} - \frac{a(r\underline{w}+y)+b}{(ac(w)+b)^2}\frac{dc(w)}{d\underline{w}}\right)}_{>0} \frac{(c'(w))^2}{ac(w)+b} \\
&- \frac{r+a(\rho-r)}{\rho-r}\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} \underbrace{\left(\frac{2c'(w)\frac{dc'(w)}{d\underline{w}}}{ac(w)+b} - \frac{a(c'(w))^2}{(ac(w)+b)^2}\frac{dc(w)}{d\underline{w}}\right)}_{>0} \\
&+ \frac{r+a(\rho-r)}{a(\rho-r)}\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}} \underbrace{c''(w)}_{<0} \underbrace{\left(\frac{ar}{ac(w)+b} - \frac{a(r\underline{w}+y)+b}{(ac(w)+b)^2}a\frac{dc(w)}{d\underline{w}}\right)}_{>0} \\
&+ \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} \frac{dc''(w)}{d\underline{w}} < 0
\end{aligned}
$$

where the last inequality follows since $r + a(\rho - r) > 0$, $\rho > r > 0$, $ac(w) + b > 0$, $\frac{dc(w)}{d\underline{w}} < 0$, $a > 0$, and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$.

Part IV: $\frac{dA(c(w))}{d\kappa} > 0$
Absolute risk aversion for HARA utility equals

$$
A(c(w)) = -\frac{u''(c(w))}{u'(c(w))} = \frac{1}{ac(w)+b}
$$

Under the DARA assumption, $\frac{A(c(w))}{dc(w)} < 0$, then since $\frac{dc(w)}{d\kappa} < 0$ we get $\frac{dA(c(w))}{d\kappa} > 0$.

Part V: $\frac{dP(c(w))}{d\kappa} > 0$
Absolute prudence for HARA utility equals

$$
P(c(w)) = -\frac{u'''(c(w))}{u''(c(w))} = \frac{1+a}{ac(w)+b}
$$

Under the DARA assumption, $\frac{P(c(w))}{dc(w)} < 0$, then since $\frac{dc(w)}{d\kappa} < 0$ we get $\frac{dP(c(w))}{d\kappa} > 0$. □

B The Household Problem with Income Growth

In this appendix, I rewrite the household problem in Section 2 to include income growth. The difference between the two problems is that income now varies such that income is a state variable. However, by some simplifying assumptions similar to on the nature of the liquidity constraint and the income process, we can normalize wealth by income and keep the problem in its tractable form with one state variable.

The new budget constraint, income process, and liquidity constraint are

$$\frac{dw_t}{dt} = rw_t + y_t - c(w_t) \tag{20}$$

$$\frac{dy_t}{dt} = \mu y_t \tag{21}$$

$$w_t \geq \underline{w} y_t \tag{22}$$

where $\underline{w}$ is a scalar satisfying $\underline{w} > -\frac{1}{r}$ (natural borrowing limit).

Similarly to Carroll (2004b), I normalize the problem above by income. All normalized variables are denoted by a hat. The rewritten budget constraint and liquidity constraint are

$$\frac{d\hat{w}}{dt} = \frac{d}{dt}\frac{w_t}{y_t} = \frac{dw_t}{dt}\frac{1}{y_t} - \frac{dy_t}{dt}\frac{w_t}{y_t^2} = (r-\mu)\hat{w}_t + 1 - \hat{c}_t(\hat{w}) \tag{23}$$

$$\hat{w}_t \geq \underline{w} \tag{24}$$

Similar to the model in Section 2, the stationary solution is given by the HJB-equation where the utility function defined in terms of $\hat{c}$.

$$\rho V(\hat{w}) = \max_{\hat{c}} u(\hat{c}) + V'(\hat{w})\left((r-\mu)\hat{w} + 1 - \hat{c}(\hat{w})\right) \tag{25}$$

where $V(\hat{w})$ is the value function for a household with assets $\hat{w}$. The first order necessary condition is

$$u'(c) = V'(\hat{w}) \tag{26}$$

The first order differential equation is now equivalent to Equation (7) in Section 2

$$(u')^{-1}\left(V'(\hat{w})\right) = r\hat{w} + 1 - \mu\hat{w} - \frac{V'(\hat{w})}{V''(\hat{w})}(\rho + \mu - r)$$

or similarily, by defining $\hat{r} = r - \mu$, we get

$$(u')^{-1}\left(V'(\hat{w})\right) = \hat{r}\hat{w} + 1 - \frac{V'(\hat{w})}{V''(\hat{w})}(\rho - \hat{r}) \tag{27}$$

As a result, all the Propositions in the paper also holds in this growth model for the modified consumption function, $\hat{c}(\hat{w})$. Notably, since y is always positive, the results on positive consumption, higher marginal propensity to consume, and concavity also holds for the original consumption function, $c(w)$:

$$
\begin{aligned}
c &= \hat{c}(\hat{w})y > 0 \\
\frac{dc}{da} &= \frac{d}{da}\left(\hat{c}\left(\hat{w}\right)y\right) = \hat{c}'(\hat{w})\frac{1}{y}y = \hat{c}'(\hat{w}) > 0 \\
\frac{d^2c}{da^2} &= \frac{d}{da}\left(\hat{c}'\left(\hat{w}\right)\right) = \hat{c}''(\hat{w})\frac{1}{y} < 0
\end{aligned}
$$