Riis, C. (2016). An equilibrium characterization of an all-pay auction with certain and uncertain prizes. *Games and Economic Behavior, 100*(6), 431-436
http://dx.doi.org/10.1016/j.geb.2016.09.007

An equilibrium characterization of an all-pay auction with certain and uncertain prizes

Christian Riis
Norwegian Business School*

October 3, 2016

Abstract

In the important contribution "All pay auctions with certain and uncertain prizes" published in Games and Economic Behavior May 2014, the authors analyze an all pay auction with multiple prizes. The specific feature of the model is that all valuations are common except for the valuation of one of the prizes, for which contestants have private valuations. However, the equilibrium characterization derived in the paper is incorrect. This paper provides the correct equilibrium characterization of the model.

Key words: All pay auctions, uncertain prizes
JEL codes: D 44, D82, J31, J41

Minchuk and Sela (2014) (hereafter MS) consider an all pay auction with multiple prizes. The specific feature of their model is that all valuations are common except for the valuation of one of the prizes. For this particular prize contestants have private valuations, independently drawn from a common distribution.

The authors claim that the equilibrium bid function is symmetric and monotone in the valuation of the uncertain prize. However this is only the case if the uncertain prize has the highest or lowest value. It is not if the uncertain prize has an intermediate value, which is MS' main case.

*Department of Economics, Norwegian Business School, Oslo, Norway; Email: christian.riis@bi.no. I am grateful to Zongwei Lu and Erlend S. Riis for outstanding research assistance.

Let us first provide intuition for why MS' result fails. We follow MS' notation. There are n contestants competing for m different prizes. The highest bidder wins the most valuable prize, with a common value of v_n. The second highest bidder wins the second most valuable prize v_{n-1} and so on. The uncertain prize is indexed $n-j+1$, with private value, denoted by a, drawn independently from the interval $[v_{n-j}, v_{n-j+2}]$ according to a distribution function $F(\cdot)$. We refer to $f(\cdot)$ as the density of $F(\cdot)$.

Suppose that the equilibrium bid function $\beta(a)$ is strictly increasing. The probability that a player who bids according to valuation s will win the uncertain prize is then

$$\frac{(n-1)!}{(j-1)!\,(n-j)!} F(s)^{n-j}\,(1-F(s))^{j-1}\,.$$

Obviously this probability is a non-monotonic function of s, strictly increasing in s for low bid levels, and strictly decreasing for high bid levels. The probability of winning the uncertain prize reaches a maximum at a valuation $\hat{a}$ implicitly determined by

$$F(\hat{a}) = \frac{n-j}{n-1}$$

It follows from standard single crossing conditions[1] that a strictly increasing bid function on a bid segment is part of a separating equilibrium only if the win probability is increasing in valuation – as the player with a high valuation has a stronger incentive to bid aggressively than a bidder with a lower valuation. In other words, a strictly increasing bid function is incompatible with optimal bidding behavior if the probability of winning the uncertain prize declines with the bid level – in which case a bidder with a particularly high valuation for the uncertain prize will lower her bid.

Thus, a separating equilibrium cannot be monotone in the valuation of the uncertain prize in our setting. We will now by construction derive the separating equilibrium.

Suppose the number of players strictly exceeds the number of prizes, making a generalization of the result straightforward. A standard characteristic of a separating equilibrium in all pay auctions with ex ante identical contestants is the following:

[1] For at detailed exposition see Athey (2001)

- The player with the lowest possible valuation for the uncertain prize obtains zero payoff.

An implication of this is that the bid function must have upper support equal to v_n, the value of the highest prize. Otherwise the player with the lowest possible valuation obtains a strictly positive rent by jumping to the upper support. As the lower support must be zero, a separating equilibrium will be a mapping from valuation $a \epsilon [v_{n-j}, v_{n-j+2}]$ to bids β on $[0, v_n]$.[2]

Observe that the equilibrium can be anchored in the following observation: For high bids, the probability of winning the uncertain prize declines as the bid level increases. Therefore, single crossing indicates that the bid function must have a declining segment at high valuations. For low valuations the equilibrium bid function must be increasing. A conjecture would be that the equilibrium bid function has the shape illustrated in figure 1, consisting of two bid segments: a declining segment, $\beta_H(a)$, for those bid levels at which the win probability (for the uncertain prize) declines with the bid; and an increasing segment $\beta_L(a)$ for bid levels at which the win probability increases. A contestant with valuation a then randomizes between the two bid levels, and chooses $\beta_L(a)$ with probability $q(a)$. With one exception: the contestant with the highest possible valuation for the uncertain prize, $a = v_{n-j+2}$, bids a certain amount, corresponding to the bid level that maximizes the probability of winning the uncertain prize. Note that the player with the lowest possible valuation for the uncertain prize, $a = v_{n-j}$, randomizes between bidding zero and v_n.

[2]It is also standard that the bid distribution must be atomless.

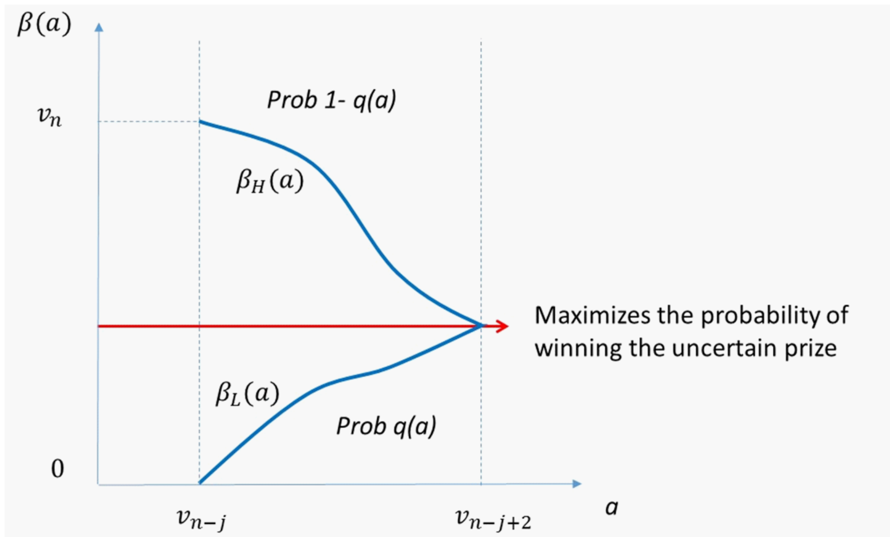

Figure 1

Figure 1: The equilibrium bid function

We will now show that the separating equilibrium indeed satisfies this pattern. Denote by $\tilde{F}_k^i(a)$ the equilibrium probability that a player with valuation a wins the prize indexed i, given that she bids according to bid segment $\beta_k(a)$, $k = L, H$. The probability that this player will win the uncertain prize, indexed $n-j+1$, is then

$$\tilde{F}_L^{n-j+1}(a) = \tag{1}$$

$$\frac{(n-1)!}{(i-1)!\,(n-i)!}\left(\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{n-j}\left(1-\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{j-1}$$

if she bids $\beta_L(a)^3$, and

$$\tilde{F}_H^{n-j+1}(a) = \tag{2}$$

$$\frac{(n-1)!}{(j-1)!\,(n-j)!}\left(1-\int_{v_{n-j}}^{a}(1-q(z))\,f(z)dz\right)^{n-j}\left(\int_{v_{n-j}}^{a}(1-q(z))\,f(z)dz\right)^{j-1}$$

if she bids $\beta_H(a)$.

Let us first characterize the equilibrium bid functions following MS' procedure, which is also the standard procedure.

A player with valuation a behaves as a player with valuation s in order to maximize $(k = L, H)$

$$\max_s \sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \tilde{F}_k^i(s)v_i + \tilde{F}_k^{n-j+1}(s)a - \beta_k(s)$$

The necessary conditions yields the following pair of differential equations

$$\beta_k'(s) = \sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \frac{d\tilde{F}_k^i(s)}{ds}v_i + \frac{d\tilde{F}_k^{n-j+1}(s)}{ds}a$$

The first order condition evaluated at $s = a$ yields the candidate bid functions[4]

$$\begin{aligned}
\beta_k(a) &= \beta_k(v_{n-j}) + \int_{v_{n-j}}^{a}\left[\sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \frac{d\tilde{F}_k^i(x)}{dx}v_i + \frac{d\tilde{F}_k^{n-j+1}(x)}{dx}x\right]dx \\
&= \beta_k(v_{n-j}) + \sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \tilde{F}_k^i(a)v_i - \sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \tilde{F}_k^i(v_{n-j})v_i + \int_{v_{n-j}}^{a}\left[\frac{d\tilde{F}_k^{n-j+1}(x)}{dx}\right]xdx
\end{aligned}$$

[3] To see this, note that a bid s wins the prize $n-j+1$ if an exact number $j-1$ of the bidder's contestants outbid her (note that there is a probability $1-\int_{v_{n-j}}^{s} q(z)f(z)dz$ that a single contestant will choose a bid above s), and the remaining $n-j$ contestants will bid below s.

[4] Note that $\tilde{F}^i(a) = \tilde{F}^i(v_{n-j}) + \int_{v_{n-j}}^{a} \frac{d\tilde{F}_k^i(x)}{dx}dx$.

If $k = L$ we have $\beta_L(v_{n-j}) = 0$ and thus $\tilde{F}^i_L(v_{n-j}) = 0$, as the player loses with certainty. If $k = H$ we have $\beta_L(v_{n-j}) = v_1$ and thus $\tilde{F}^n_L(v_{n-j}) = 1$, as the player in this case is certain to win the most valuable prize. In both cases, the candidate bid function is (where the second equality follows from integration by parts)

$$\begin{aligned}\beta_k(a) &= \sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \tilde{F}^i_k(a)v_i + \int_{v_{n-j}}^{a} \left[\frac{d\tilde{F}^{n-j+1}_k(x)}{dx}\right] xdx \\ &= \sum_{\substack{i=n-m+1\\ i\neq n-j+1}} \tilde{F}^i_k(a)v_i + \tilde{F}^{n-j+1}_k(a)a - \int_{v_{n-j}}^{a} \left[\tilde{F}^{n-j+1}_k(x)\right] dx\end{aligned}$$

This yields associated utilities

$$U_k(a) = \int_{v_{n-j}}^{a} \tilde{F}^{n-j+1}_k(x)dx$$

Thus the bidder's rent is associated with a valuation of the uncertain prize above the lower support v_{n-j}, exactly as described by MS.

It remains to determine $q(a)$. The equilibrium probability function, $q(\cdot)$, makes each contestant indifferent between choosing $\beta_L(a)$ and $\beta_H(a)$. This means that we have to find a function $q(\cdot)$ such that $U_L(a) = U_H(a)$ for every type a, thus

$$\int_{v_{n-j}}^{a} \tilde{F}^{n-j+1}_L(x)dx = \int_{v_{n-j}}^{a} \tilde{F}^{n-j+1}_H(x)dx \tag{3}$$

must always hold. Obviously this is equivalent to the following condition: for any $a\epsilon[v_{n-j}, v_{n-j+2}]$ we have

$$\tilde{F}^{n-j+1}_L(a) = \tilde{F}^{n-j+1}_H(a) \tag{4}$$

Hence, in a separating equilibrium, for each type a, the probability of winning the uncertain prize is independent of the choice between $\beta_L(a)$ and $\beta_H(a)$. Accordingly, the net cost of submitting a high bid, $\beta_H(a) - \beta_L(a)$, cancels out with the net gain from the higher probability of winning one of the certain and more valuable prizes. Note that $\beta_L(v_{n-j+2}) = \beta_H(v_{n-j+2})$ since $\tilde{F}^i_L(v_{n-j+2}) = \tilde{F}^i_H(v_{n-j+2})$ for all $i = n-m+1, .., n$[5], which confirms that the two bid segments meet at $a = v_{n-j+2}$.

[5] This can easily be checked by inserting v_{n-j+2} in the integral limits in (1) and (2), and generalizing to any i.

Substituting from (1) and (2) condition (4) can be written

$$\left(\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{n-j}\left(1-\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{j-1} \tag{5}$$
$$= \left(1-\int_{v_{n-j}}^{a}(1-q(z))\,f(z)dz\right)^{n-j}\left(\int_{v_{n-j}}^{a}(1-q(z))\,f(z)dz\right)^{j-1}$$

which must hold for all $a\epsilon[v_{n-j}, v_{n-j+2}]$. $q(\cdot)$ is the solution to the functional equation given by (5). We will now prove existence and uniqueness of $q(\cdot)$.

We prove this for the most general case where we assume no properties of f beyond being a probability density function. For notational simplicity and without loss of generality we set $[v_{n-j}, v_{n-j+2}] = [0,1]$.

Consider equation (6).

$$\tilde{\lambda}^{n-j}\left(1-\tilde{\lambda}\right)^{j-1} = \left(\tilde{\lambda}+1-F\right)^{n-j}\left(1-\left(\tilde{\lambda}+1-F\right)\right)^{j-1} \tag{6}$$

Note that replacing $\tilde{\lambda}$ with $\int_{v_{n-j}}^{a} q(z)f(z)dz$ yields equation (5).

Lemma 1 *There is a unique function $\tilde{\lambda}:[0,1]\to[0,1]$ that satisfies equation (6) for all $F\in[0,1]$ and that has the following properties:*

1. *$\tilde{\lambda}$ is strictly increasing;*
2. *$\tilde{\lambda}(F)\le F$ for all $F\in[0,1]$;*
3. *$\tilde{\lambda}$ is Lipschitz continuous with Lipschitz constant* 1.

Furthermore, this function is almost everywhere differentiable with $\tilde{\lambda}'(F)\in[0,1]$.

See appendix for proofs.

Theorem 1 *The function $q:[0,1]\to[0,1]$ defined by $q(a):=\tilde{\lambda}'(F(a))$ satisfies $\lambda(a)=\int_0^a f(z)q(z)dz$ for all a. Furthermore, this solution is unique for (almost all) $a\in[0,1]$ where $f(a)\neq 0$.*

One case is particularly simple. If the uncertain prize is the median prize, $j = (n+1)/2$, the equilibrium q is constant, $q(a) = 1/2$ for all a. To see this, insert $q(a) = 1/2$ in (5) and solve the integrals, which yields

$$\left(\frac{1}{2}F(a)\right)^{(n-1)/2}\left(1-\frac{1}{2}F(a)\right)^{(n-1)/2} = \left(1-\frac{1}{2}F(a)\right)^{(n-1)/2}\left(\frac{1}{2}F(a)\right)^{(n-1)/2}$$

Note that the equilibrium is fully revealing, despite the fact that players randomize bid levels, as each pair of possible bids is unique for the player's valuation.

1 Appendix

Proof Lemma 1. We will prove this in several steps.

Step 1: We first reformulate the problem, so that finding $\tilde{\lambda}(F)$ for a $F \in [0,1]$ becomes equivalent to finding a point of intersection between two graphs. Consider the function $G(x) := x^{n-j}(1-x)^{j-1}$ and the translation $H_F(x) := G(x+1-F)$. Note that $G(\tilde{\lambda})$ is the LHS of (6) and $H_F(\tilde{\lambda})$ is the RHS. Thus, the curve of the RHS, H_F, is just the curve of the LHS, G, shifted to the left by $1-F$. So for a given $F \in [0,1]$ we are looking for a $\tilde{\lambda}(F) \in [0,F]$ such that $G(\tilde{\lambda}(F)) = H_F(\tilde{\lambda}(F))$.

Step 2: We will now show that for each $F \in [0,1)$, there exists a unique point $\tilde{\lambda}(F) \in [0,F]$ such that $G(\tilde{\lambda}(F)) = H_F(\tilde{\lambda}(F))$. To do so, we shall show that the graphs of G and H_F have simple bell-shapes for which a unique intersection point is easy to derive. For the graph of G, note that a simple analysis tells us that:

1. G has a unique peak at $\dfrac{n-j}{n-1} =: \lambda_G$;
2. On $[0,\lambda_G]$ G is strictly increasing from 0 to $G(\lambda_G)$;
3. On $[\lambda_G,1]$ G is strictly decreasing from $G(\lambda_G)$ to 0.

Write $\lambda_{H_F} := \lambda_G - 1 + F$. Then, equivalently, H_F is strictly increasing on $[-1+F, \lambda_{H_F}]$ and strictly decreasing on $[\lambda_{H_F}, F]$.

Existence: Trivially $\tilde{\lambda}(0) = 0$. When $F \in (0,1)$, we have that $H_F(0) > 0 = G(0)$ and $H_F(F) = 0 < G(F)$. Thus, by the intermediate value property, there is an intersection point $\tilde{\lambda}(F) \in (0,F)$.

Uniqueness: We claim that for all F, the intersection point $\tilde{\lambda}(F)$ lies between the peaks λ_{H_F} and λ_G, and that there can be no other intersection points outside of the interval $(\lambda_{H_F}, \lambda_G)$. Once this is established, it is trivial to show that the intersection point must be unique.

- Suppose $\tilde{\lambda} \in [0, \lambda_{H_F}]$. We have $H_F(\tilde{\lambda}) = G(\tilde{\lambda} + 1 - F)$, and since G is strictly increasing on the interval $[0, \lambda_G] \supseteq [\tilde{\lambda}, \tilde{\lambda} + 1 - F]$, we have $G(\tilde{\lambda} + 1 - F) > G(\tilde{\lambda})$. Thus $H_F(\tilde{\lambda}) > G(\tilde{\lambda})$ and there are no points of intersection on this interval.

- Suppose $\tilde{\lambda} \in [\lambda_G, 1]$. By a similar analysis as for the above case, $H_F(\tilde{\lambda}) = G(\tilde{\lambda} + 1 - F) < G(\tilde{\lambda})$ as G is decreasing on the interval $[\lambda_G, 1]$. We again conclude that there are no points of intersection.

- Suppose $\tilde{\lambda} \in (\lambda_{H_F}, \lambda_G)$. We have $H_F(\lambda_{H_F}) > G(\lambda_{H_F})$ and $H_F(\lambda_G) < G(\lambda_G)$, so by continuity, there must be a point of intersection. Furthermore, as H_F is strictly decreasing and G is strictly increasing on this interval, the crossing point must be unique.

So for each $F \in [0, 1)$, we have a mapping $\tilde{\lambda} : F \mapsto \tilde{\lambda}(F) \in [0, F]$. For $\tilde{\lambda}$ at $F = 1$, we note that as $\tilde{\lambda}(F) \in (\lambda_{H_F}, \lambda_G)$ for $F \in (0, 1)$ and $\lim_{F \to 1} \lambda_{H_F} = \lambda_G$, we set $\tilde{\lambda}(1) = \lim_{F \to 1} \tilde{\lambda}(F) = \lambda_G$ for continuity.

Step 3: We will now show that $\tilde{\lambda}$ is strictly increasing. To do so, we will show that for all F and $\varepsilon > 0$, we have $\tilde{\lambda}(F) \leq \tilde{\lambda}(F + \varepsilon)$. We know that $\tilde{\lambda}(F + \varepsilon) \in (\lambda_{H_{F+\varepsilon}}, \lambda_G)$, so if $\lambda_{H_{F+\varepsilon}} \geq \tilde{\lambda}(F)$, we are done. If $\lambda_{H_{F+\varepsilon}} < \tilde{\lambda}(F)$ then $H_{F+\varepsilon}$ is decreasing on $[\tilde{\lambda}(F), \tilde{\lambda}(F) + \varepsilon]$, so

$$H_{F+\varepsilon}(\tilde{\lambda}(F)) > H_{F+\varepsilon}(\tilde{\lambda}(F) + \varepsilon) = H_F(\tilde{\lambda}(F)) = G(\tilde{\lambda}(F)).$$

Thus, $H_{F+\varepsilon}$ has yet to intersect with G at $\tilde{\lambda}(F)$ and so $\tilde{\lambda}(F + \varepsilon) > \tilde{\lambda}(F)$ and $\tilde{\lambda}$ is strictly increasing.

Step 4: Then we will show that $\tilde{\lambda}$ is Lipschitz continuous with Lipschitz constant 1, which is equivalent to showing that $\tilde{\lambda}(F + \varepsilon) \leq \tilde{\lambda}(F) + \varepsilon$. This follows from a similar analysis to that of Step 3. It will be enough to show that $H_{F+\varepsilon}(\tilde{\lambda}(F) + \varepsilon) < G(\tilde{\lambda}(F) + \varepsilon)$. We have

$$H_{F+\varepsilon}(\tilde{\lambda}(F) + \varepsilon) = H_F(\tilde{\lambda}(F)) = G(\tilde{\lambda}(F)) < G(\tilde{\lambda}(F) + \varepsilon)$$

since G is increasing on $[0, \lambda_G]$. Thus $\tilde{\lambda}(F) + \varepsilon$ is past the intersection point $\tilde{\lambda}(F + \varepsilon)$ of $H_{F+\varepsilon}$ and G, which is what we wanted to show.

Step 5: We conclude the proof by deducing that $\tilde{\lambda}$ is almost everywhere differentiable and that $\tilde{\lambda}'(F) \in [0,1]$. This follows immediately from the fact that Lipschitz continuous functions are almost everywhere differentiable, that $\tilde{\lambda}$ is increasing and that $\tilde{\lambda}(F+\varepsilon) \leq \tilde{\lambda}(F) + \varepsilon$. ■

Proof Theorem 1. We will prove this in three steps.

Step 1: First we show that λ is almost everywhere differentiable, and that $\lambda'(a) = f(a)\,\tilde{\lambda}'(F(a))$. λ is the composition $\lambda(a) := \tilde{\lambda} \circ F(a)$. By the chain rule, if $\tilde{\lambda}$ and F are differentiable at a, then λ is differentiable at a, and $\lambda'(a) = \tilde{\lambda}'(F(a))\,F'(a)$. We established in Lemma 1 that $\tilde{\lambda}$ is differentiable almost everywhere. Furthermore, since f is a probability density function, it is absolutely integrable, so by the Lebesgue differentiation theorem, $F(a) = \int_0^a f(z)dz$ is continuous, almost everywhere differentiable and $F'(x) = f(x)$. This concludes Step 1.

Step 2: Secondly we show that $\lambda(a) = \lambda(b) + \int_b^a \lambda'(z)dz$. By the second fundamental theorem for absolutely continuous functions, this is equivalent to showing that λ is absolutely continuous. We have by that if F and $\tilde{\lambda}$ are absolutely continuous and F is monotone, then the composition $\tilde{\lambda} \circ F$ is also absolutely continuous. Absolute continuity of $\tilde{\lambda}$ follows directly from Lipschitz continuity. F is clearly monotone, and absolute continuity of F follows from absolute integrability of f . Therefore, λ is absolutely continuous.

Step 1 and 2 together show that $\lambda(a) = \int_0^a f(z)\,\tilde{\lambda}'(F(z))\,dz$, so we can set $q(a) := \tilde{\lambda}'(F(a))$. We have from Lemma 1 that $\tilde{\lambda}'(a) \in [0,1]$, which implies that q is a function from $[0,1]$ to $[0,1]$. In other words, we have the existence of a solution $q : [0,1] \to [0,1]$.

Step 3: Now it remains to show uniqueness of the solution q on the subset of $[0,1]$ where $f(a) \neq 0$. We suppose that $\tilde{q} : [0,1] \to [0,1]$ also solves $\lambda(a) = \int_0^a f(z)\tilde{q}(z)dz$ and show that $\tilde{q}(a) = q(a)$ for almost all a for which $f(a) \neq 0$. The integrands $f(z)q(z)$ and $f(z)\tilde{q}(z)$ are absolutely integrable, so by the Lebesgue differentiation theorem (alternative formulation),

$$f(a)q(a) = \lim_{h \to 0^+} \frac{1}{h} \int_{[a,a+h]} f(z)q(z)dz, \qquad f(a)\tilde{q}(a) = \lim_{h \to 0^+} \frac{1}{h} \int_{[a,a+h]} f(z)\tilde{q}(z)dz,$$

for almost all $a \in [0,1]$. However, by our assumption,

$$\int_{[a,a+h]} f(z)q(z)dz = \int_{[a,a+h]} f(z)\tilde{q}(z)dz$$

so $f(a)q(a) = f(a)\tilde{q}(a)$ almost everywhere, and we are done. ■

2 References

Athey, Susan. "Single crossing properties and the existence of pure strategy equilibria in games of incomplete information." Econometrica 69.4 (2001): 861-889.

Minchuk, Yizhaq, and Aner Sela. "All-pay auctions with certain and uncertain prizes." Games and Economic Behavior 88 (2014): 130-134.