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Complete and incomplete financial markets in multi-good economies

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Abstract

We investigate conditions for endogenous incompleteness and completeness in continuous-time financial markets driven by diffusion processes with multiple consumption goods and heterogeneous agents. We show that for a class of utility functions the financial market is endogenously incomplete. A sufficient condition for market completeness is that the dividend diffusion matrix in units of the numeraire good is invertible. Further, financial market completeness can depend on the choice of the numeraire good since changing the numeraire good implies a change of the risk-free asset and the asset structure.

Keywords: Multi-good economies; Financial market incompleteness; Financial market completeness

JEL Classification: G10; G11

1 Introduction

We consider a continuous-time Lucas tree economy driven by diffusion processes with multiple consumption goods and agents with heterogeneous preferences. We derive sufficient conditions for financial market incompleteness and completeness without having to calculate the equilibrium stock price diffusion matrix. Instead, the conditions rely solely on the utility function of the representative agent and an invertibility condition on dividends.

First, we define a class of utility functions for which the span of the risky assets is strictly smaller than the span of the dividends. Hence, the financial market is incomplete. This class of utility functions covers the preferences employed, among others, in the following papers: Cole and Obstfeld (1991), Zapatero (1995), Serrat (2001), Cass and Pavlova (2004), and Berrada et al. (2007). Within this class is, for instance, the widely used Cobb-Douglas utility function. When the representative agent has Cobb-Douglas utility, as in Cole and Obstfeld (1991), then the commodity price is proportional to the relative dividends and, consequently, dividends measured in units of the numeraire correlate perfectly with each other. Hence, in equilibrium stock prices are linearly dependent.[1]

Second, we define a class of utility functions for which the financial market is complete. Specifically, our completeness condition only requires verification of an invertibility condition on dividends in terms of the numeraire good for one realization of the state variables at the terminal time. In contrast, to verify market completeness without such a condition one has to calculate the equilibrium stock price diffusion matrix and to check if the matrix is invertible for every possible realization of the state variables at every point in time. Without an explicit closed-form solution for the equilibrium stock price diffusion matrix, this is a hopeless task.

In a Lucas tree economy the typical asset structure consists of claims on the Lucas trees

[1] Rosenberg and Ohlson (1976) study a related problem in multi-asset purely financial models with fixed asset supplies. Assuming that the aggregate investor has constant relative risk aversion utility and that asset prices follow a joint lognormal process implies that asset proportions are constant. In equilibrium, this can only be the case if risky asset returns are perfectly correlated.

and a locally risk free asset in units of the numeraire good. We show that financial market completeness depends on the choice of the numeraire good, since changing the numeraire good and keeping the number of available assets fixed implies that the original risk-free asset is non-tradable under the new numeraire good. Therefore, changing the numeraire good can move an economy from incomplete to complete and vice versa. Numeraire good irrelevance holds[2] when our sufficient condition for incompleteness is satisfied; thus, for this class of utility functions the market remains incomplete under any numeraire good.

Even if the financial market is endogenously incomplete, the numeraire good can be important. We show that the choice of numeraire good can determine whether trading in the available assets implements the Arrow-Debreu equilibrium. For example, for a certain choice of the numeraire good the equilibrium is of the peculiar type as in Cass and Pavlova (2004), even though agents do not have log-linear utility functions. For any other choice of the numeraire good, the endogenous incompleteness has real effects, as agents cannot implement the Arrow-Debreu equilibrium.

The departure point for our work is that in several multi-good models in the literature the span of the stocks drops relative to the span of the dividends. Specifically, Serrat (2001) solves a continuous-time international Lucas (1978) tree economy with multiple consumption goods and derives an explicit formula for stock price diffusion coefficients. Yet, it appears that even with an explicit formula for stock price diffusion coefficients it can be difficult to detect an inherently incomplete market. In the end, Kollmann (2006) shows that the economy studied by Serrat (2001) has incomplete financial markets. Importantly, Serrat (2001) claims that the presence of non-traded goods leads to portfolio home bias in stocks that are claims to traded goods and, hence, apparently proposes a solution to the portfolio home bias puzzle. Unfortunately, given that portfolios are indeterminate, we cannot learn anything about portfolio home bias from Serrat (2001). To our knowledge, each model where the span of the stocks drops relative to the span of the dividends is a multi-good variant of the

[2]We use the term *numeraire good irrelevance* to mean that the financial market does not switch from complete to incomplete when we change the numeraire good.

Lucas (1978) model. To us, it seems important to obtain imperfectly correlated stock market returns in frictionless multi-good economies, when the span of the dividends allows for completeness. In this regard and since the asset pricing literature with multiple consumption goods continues to grow,[3] our simple conditions for verification of financial market incompleteness and completeness can guide future modeling assumptions about utility functions in economies with multiple consumption goods and heterogeneous agents.

Our paper relates to Anderson and Raimondo (2008), who derive conditions for market completeness in a continuous time economy with a single consumption good. They prove that imposing certain smoothness conditions on the primitives of an economy in addition to invertibility of the payoff matrix imply invertibility of the stock return covariance matrix. Prior to Anderson and Raimondo (2008), every single model without a closed form solution for the stock return covariance matrix assumed in some form completeness (Duffie and Huang (1985); Duffie and Zame (1989)). More recently, Hugonnier et al. (2010), Riedel and Herzberg (2013), and Kramkov and Predoiu (2014) work out generalizations of the conditions for market completeness in a continuous time economy with a single consumption good. However, with multiple consumptions goods, these results do not rule out that generic incompleteness may arise.

Our paper also relates to Magill and Shafer (1990).[4] They show that in the real asset model of financial equilibrium theory the market is generically complete as long as the aggregate endowment satisfies a regularity condition, i.e., it spans all the uncertainty in the economy.[5] However, Cass and Pavlova (2004) show that the Lucas tree economy, although a special case of the real asset model, has some embedded structures that make it significantly different from the real asset model. Importantly, the generic existence result of Magill and

[3]Recent general Lucas (1978) type asset pricing models with multiple consumption goods include Ait-Sahalia et al. (2004), Yogo (2006), Piazzesi et al. (2007), and Lochstoer (2009) among others. International asset pricing models with multiple consumption goods that are Lucas (1978) trees include Lucas (1982), Cole and Obstfeld (1991), Zapatero (1995), Baxter et al. (1998), Serrat (2001), Kollmann (2006), Pavlova and Rigobon (2007, 2008, 2010), and Li and Muzere (2011) among many others.

[4]See Hart (1975) for an early contribution.

[5]See also Duffie and Shafer (1985) and Duffie and Shafer (1986).

Shafer (1990) does not apply, as one cannot perturb endowments independently of the cash flows of assets.

2 The economy

We consider a frictionless continuous time pure exchange economy over a finite time span $[0, T]$.[6] Uncertainty is represented by a filtered probability space, $\left(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t \geq 0}\right)$,[7] on which is defined a N-dimensional Brownian motion $Z = (Z_1, ..., Z_N)^\top$. In the following, all stochastic processes are assumed to be progressively measurable and all equalities are assumed to hold in the almost surely sense.

There are $N + 1$ securities, of which N are dividend paying stocks and one asset that is locally risk-free in units of the numeraire good. All dividend-paying stocks are in unit supply, while the risk-free asset is in zero net supply. There are N different consumption goods, where stock $i = 1, ..., N$ pays out dividends in consumption good i.[8] Dividends of stock i are paid at a rate $\delta_i(X(t))$, where δ_i denotes a nonnegative function and where $X(t)$ is a N-dimensional vector of state variables with dynamics

$$X(t) = X(0) + \int_0^t \mu_X(X(\tau), \tau) d\tau + \int_0^t \sigma_X(X(\tau), \tau) dZ(\tau). \tag{1}$$

Assumption 1. *The unique solution of Equation (1) takes values in $\mathcal{X} \subseteq \mathbb{R}^N$ and for all $(x, t) \in \mathbb{R}^N \times [0, T]$ the diffusion of the state variable process is invertible, i.e., $rank(\sigma_X(x, t)) = N$.*

Assumption 2. *The dividends, $\delta_i(x)$, are functions of class $\mathcal{C}^2$.*

Definition 1. *We define I to be the $N \times N$ identity matrix and for a vector y with $y \in \mathbb{R}^N$;*

[6] Our setup is close to the workhorse macro-finance model described in Pavlova and Rigobon (2013).

[7] The filtered probability space is defined over the finite horizon $[0, T]$, where Ω defines the state space, $\mathcal{F}$ denotes the σ-algebra, $\mathcal{P}$ is the probability measure, and the information structure or filtration $\mathcal{F}_{(.)}$ is generated by the Brownian motion of the state variable processes with $\mathcal{F} = \mathcal{F}_T$.

[8] It is easy to extended the economy to a setting where stocks pay out in more than one good.

and we define I_y to represent a $N \times N$-dimensional matrix with y_i as element (i,i) and zero elsewhere.

Given the process in Equation (1), an application of Ito's lemma to $\delta\left(X(t)\right) = (\delta_1(t), .., \delta_N(t))$ yields the diffusion of dividends: $\lambda(t) \equiv \lambda(X(t), t) = \frac{\partial \delta(X(t))}{\partial x} \sigma_X(X(t), t)$.[9]

Assumption 3. *The dividend diffusion matrix is invertible, i.e., $rank(\lambda(t)) = N$.*

Assumption 3 ensures that the market is potentially complete, i.e., that the dividends span all the uncertainty in the economy.

Let $P = \{p_1, p_2, ..., p_N\}$ denote the vector of the N commodity prices and let consumption good $l \in \{1, ..., N\}$ serve as the numeraire. Thus, the price of consumption good l is normalized to one, $p_l(t) = 1$, for all $t \in [0, T]$. Commodity prices are determined in equilibrium. The N-dimensional commodity price evolves according to[10]

$$P(t) = P(0) + \int_0^t I_P(\tau)\mu_P(\tau)d\tau + \int_0^t I_P(\tau)\sigma_P(\tau)dZ(\tau), \tag{2}$$

where μ_P and σ_P denote expected growth rates and diffusion coefficients in $\mathbb{R}^N$ and $\mathbb{R}^{N\times N}$, respectively.

Definition 2. *We define the N-dimensional dividend rate process, $\widetilde{\delta}(t)$, in units of the numeraire as*

$$\widetilde{\delta}(t) = I_P(t)\delta(t). \tag{3}$$

There are N stocks, each representing a claim to its respective dividend rate process. In a complete market equilibrium, the N-dimensional stock price processes, $S(t)$, are given by

$$S(t) + \int_0^t \widetilde{\delta}(\tau)d\tau = S(0) + \int_0^t I_S(\tau)\mu(\tau)d\tau + \int_0^t I_S(\tau)\sigma(\tau)dZ(\tau). \tag{4}$$

[9]We will occasionally drop the explicit reference to the state variable X and write $F(t)$ rather than $F(X(t), t)$.

[10]To simplify notation, we do not explicitly indicate the numeraire good, i.e., $P(t) = P^l(t)$ denotes the relative commodity prices when good l serves as numeraire. Yet, it is important to recognize that equilibrium quantities depend on the choice of numeraire good.

The diffusion term $\sigma(t)$ denotes a $N \times N$ matrix with the i'th row given by $\sigma_i(t)^\top$.[11] Both the N-dimensional drift rates and the $N \times N$-dimensional diffusion terms in Equation (4) represent endogenous quantities.

A locally risk-free asset in zero net supply pays out in the numeraire good; and, thus, it is only risk-free in the numeraire good. Its price process, $B(t)$, is

$$B(t) = B(0) + \int_0^t r(\tau)B(\tau)d\tau, \tag{5}$$

with $B(0) = 1$. The risk-free rate, r, is to be determined endogenously in equilibrium.

The economy is populated by $J \geq 1$ agents indexed by j. The utility function of agent j, U_j, is

$$U_j\left(C^j\right) = E\left[\int_0^T e^{-\rho\tau}u_j(C^j(\tau))d\tau\right], \tag{6}$$

where $\rho > 0$, u is a classical time-additive von Neumann-Morgenstern utility function, and $C^j = \left\{c_1^j, c_2^j, \ldots, c_N^j\right\}$ denotes the vector containing the N consumption goods.

Assumption 4. *The utility function $u_j : (0, \infty)^N \to \mathbb{R}$ is assumed to be increasing, strictly concave function of class $\mathcal{C}^3$, and to satisfy the multidimensional Inada conditions.*

Agent j maximizes $U_j\left(C^j\right)$ subject to the dynamic budget constraint

$$\begin{aligned} W^j(t) &= W^j(0) + \int_0^t W^j(\tau)r(\tau)d\tau - \int_0^t P(\tau)^\top C^j(\tau)d\tau \\ &\quad + \int_0^t \pi^j(\tau)^\top\left(\mu(\tau) - r(\tau)1_N\right)d\tau + \int_0^t \pi^j(\tau)^\top\sigma(\tau)dZ(\tau), \end{aligned} \tag{7}$$

where $\pi^j(t) = \left(\pi_1^j(t), \pi_2^j(t), ..., \pi_N^j(t)\right)$ is a vector process of amounts held in the stocks by agent j and $W^j(t)$ is the wealth of agent j in units of the numeraire good. Agents are endowed with initial shares $\eta^j = \left(\eta_1^j, \ldots, \eta_N^j\right)$ of each stock. Hence, $W^j(0) = \left(\eta^j\right)^\top S(0)$ and

[11] In Equation (4), $\mu(\tau)$ is the vector of instantaneous expected returns. To see this, note that the instantaneous return on stock i is $dR_i(t) = \frac{S_i(t)+\tilde{\delta}_i(t)dt}{S_i(t)} = \mu_i(t)dt + \sigma_i(t)^\top dZ(t)$, which is the differential form of Equation (4) divided by $S_i(t)$.

$\sum_{j=1}^{J} \eta_i^j = 1$ for $i = 1, \ldots, N$. We impose the conditions that for all $j = 1, .., J$ we have $\eta_i^j \geq 0$ for all $i = 1, ..., N$ and $\eta_i^j > 0$ for at least one $i = 1, ..., N$. These conditions imply $W^j(0) > 0$ and, therefore, if an agent never trades away from his initial portfolio, wealth remains positive for all times and states. This is similar to the condition in Equation (4) in Anderson and Raimondo (2008). However, in their setting, agents receive an endowment stream in addition to initial shares of equity; thus, they impose a less strict condition as agents may be endowed with short positions.

Definition 3. *Equilibrium is a collection of allocations* (C^j, π^j) *for* $j = 1, 2, ..., J$, *and a price system* (B, S, P) *or price coefficients* $(r, \mu, \sigma, \mu_P, \sigma_P)$, *such that* $\left(C^j, \pi^j, \pi_B^j\right)$ *denote optimal solutions to agent* j*'s optimization problem and good and financial markets clear*

$$\sum_j C^j(t) = \delta(t), \ \sum_j \pi^j(t) = S(t), \ \sum_j \pi_B^j(t) = 0,$$

for $t \in [0, T]$ *where* π_B^j *is the amount held in the bond market.*

To derive sufficient conditions for incompleteness and completeness, we start with a social planner's problem and then ask whether we can decentralize it.

Definition 4. *Define the following social planner's problem*

$$u(a, \delta) = \max_{\sum_j C^j = \delta} \sum_j a_j u_j(C^j), \tag{8}$$

where $a \in \varsigma$ *denotes the Pareto weights and* ς *is the unit simplex of* $\mathbb{R}^J$.

For a given set of Pareto weights, a, the social planner maximizes the weighted average of the individual agents utility functions state by state and time by time, subject to the feasibility constraint. It is well know that an allocation $(C^j)_{j=1}^J$ is Pareto efficient if and only if it solves the social planner's problem in Equation (8). By the second welfare theorem, there exist Arrow-Debreu prices that imply a competitive equilibrium. However, agents in our economy cannot directly trade Arrow-Debreu securities, but only trade the N stocks and the locally risk-free asset. Hence, one needs to show that agents can use the available securities to implement the allocations. A sufficient condition for implementability of Pareto efficient allocations is that the financial market is complete, which in our setting is equivalent to $\sigma(t)$ being invertible.

Next, we define an Arrow-Debreu equilibrium.

Definition 5. *An Arrow-Debreu equilibrium is defined as a state price density, ξ, commodity prices, P, and consumption allocations, $(C^j)_{j=1}^J$, such that C^j maximizes U_j given the static budget constraint $E\left[\int_0^T \xi(\tau)\left(P(\tau)^\top C^j(\tau) - (\eta^j)^\top \widetilde{\delta}(\tau)\right) d\tau\right] \leq 0$, and all markets clear.*

Relative to the social planner's problem where only the aggregate feasibility constraints are imposed, the Arrow-Debreu equilibrium also requires the allocations to be affordable for a given initial wealth allocation, i.e., we require that the individual budget constraints are satisfied. For a given set of exogenous Pareto weights, a, we can define the equilibrium as finding the initial wealth allocations such that the solution to the social planner's problem in Equation (8) also satisfies the individual agents' budget constraints. For such an initial wealth allocation, the solution to the social planner's problem corresponds to an Arrow-Debreu equilibrium. In Section 3, we derive sufficient conditions for incomplete markets. In this case, we start with the social planner problem in Equation (8), which is always well defined even if financial markets are incomplete, and show that for a given set of Pareto weights, a, the candidate stock prices do not complete the market. When discussing sufficient conditions for completeness in Section 4, we solve the competitive equilibrium by first solving the Arrow-Debreu equilibrium. Next, we show that the candidate stock price diffusion matrix is invertible, and hence agents can implement the Arrow-Debreu equilibrium by trading the N-stocks and the money market account.

Assumption 5.

$$\sum_{j=1}^{J} E\left[\int_0^T e^{-\rho\tau} \nabla u_j \left(\delta(\tau)/J\right)^\top \delta(\tau) d\tau\right] < \infty. \tag{9}$$

By imposing Assumption 5, Proposition 1 shows that for a given initial wealth allocation, there exists an Arrow-Debreu equilibrium.

Proposition 1. *There exists an Arrow-Debreu equilibrium in which the state price density, $\xi(t)$, is*

$$\frac{\xi(t)}{\xi(0)} \equiv \frac{\xi(a,t)}{\xi(a,0)} = e^{-\rho t} \frac{\frac{\partial u(a,\delta(t))}{\partial \delta_l}}{\frac{\partial u(a,\delta(0))}{\partial \delta_l}}. \tag{10}$$

Moreover, the N-dimensional equilibrium commodity price vector, $P(t)$, is

$$P(t) \equiv P(a,t) = \frac{\nabla u(a,\delta(t))}{\frac{\partial u(a,\delta(t))}{\partial \delta_l}}. \tag{11}$$

The utility weights, a, correspond to solutions to

$$E\left[\int_0^T \xi(a,\tau)\left(P(a,\tau)^\top C^j(a,\tau) - \left(\eta^j\right)^\top \widetilde{\delta}(a,\tau)\right) d\tau\right] = 0, \tag{12}$$

where the above is evaluated at the optimal solution for $j = 1, ..., J$.

In Proposition 1, the state price density, ξ, is proportional to the representative agent's marginal utility of the numeraire good.

Lemma 1. *The commodity price diffusion coefficients, $\sigma_P(t)$, are given by*

$$\sigma_P(t) = \varepsilon(t)\lambda(t), \tag{13}$$

where $\varepsilon(t)$ is a $N \times N$ matrix with element (i,k) given by

$$\varepsilon_{i,k}(t) = \delta_k(t) \frac{\partial \ln MRS_{i,l}(t)}{\partial \delta_k}, \tag{14}$$

where $MRS_{i,l}(t) = \frac{\frac{\partial u(a,\delta(t))}{\partial \delta_i}}{\frac{\partial u(a,\delta(t))}{\partial \delta_l}}$ stands for the marginal rate of substitution and $k = 1, ..., N$.

Proposition 2. *The diffusion coefficients of the dividend rate processes in units of the numeraire good,* $\sigma_{\tilde{\delta}}(t)$, *are*

$$\sigma_{\tilde{\delta}}(t) = (I + \varepsilon(t))\, \lambda(t). \tag{15}$$

From Proposition 2, we see that even though the dividend diffusion matrix, λ, is invertible, the diffusion of the dividend processes in units of the numeraire, $\sigma_{\tilde{\delta}}$, might be non-invertible due to the dynamics of relative prices in equilibrium.

Given an Arrow-Debreu equilibrium, the natural candidate for a stock price, $S_i(t)$, is the discounted future value of dividends

$$S_i(t) = E_t \left[\int_t^T \frac{\xi(\tau)}{\xi(0)} p_i(\tau) \delta_i\left(X(\tau)\right) d\tau \right], \tag{16}$$

for $i = 1, .., N$.

3 Incomplete markets

In this section, we study conditions for when the market is endogenously incomplete. Before we present the theory, it is useful to consider a simple economy with one agent and two goods.

Example 1. The representative agent has Cobb-Douglas utility over the two goods

$$u(C(t)) = c_1(t)^{\alpha_1} c_2(t)^{\alpha_2}, \tag{17}$$

with $0 < \alpha_1 + \alpha_2 < 1$. Let good one be the numeraire good. In equilibrium, the commodity price satisfies

$$P(t) = \frac{\delta_1(t)}{\beta \delta_2(t)}, \tag{18}$$

where $\beta = \frac{\alpha_1}{\alpha_2}$. The value in units of the numeraire of dividends at time t of the two goods are $\tilde{\delta}_1(t) = \delta_1(t)$ and $\tilde{\delta}_1(t) = P(t)\delta_2(t) = \frac{\delta_1(t)}{\beta}$, respectively. We see that the value of the second dividend is proportional to the first dividend for any time and state of the economy. Hence, the second stock is co-linear with the first stock and, therefore, financial markets are incomplete. △

The example shows that commodity prices can render a stock price to not depend at all on its own dividends. Instead, the stock price depends only on the numeraire good. Thus, the two stocks in the example correlate perfectly. To answer the question whether this example generalizes to a larger set of utility functions, we define a class of utility functions for which this will indeed be the case.

Definition 6. *A utility function $u : \mathbb{R}_+^N \to \mathbb{R}$ is in U^{IC}, where IC stands for incompleteness, if it has a representation*

$$u(c_1, ..., c_N) = \varphi\left(c_2 c_1^{\beta_2}, ..., c_N c_1^{\beta_N}\right), \tag{19}$$

where $\varphi : \mathbb{R}_+^{N-1} \to \mathbb{R}$ is such that $u(c_1, ..., c_N)$ satisfies Assumption 4.

For the two-good case, Definition 6 corresponds to a constraint on the elasticity of substitution. If the elasticity of substitution equals one, then the market is incomplete. This is the argument put forward in Proposition 2 in Berrada et al. (2007). Yet, moving from the $two-good$ case to the $N-good$ case implies that unit elasticity of substitution between any two goods is not necessary for the utility function to satisfy the Definition 6.

Therefore, how can we interpret Definition 6 more generally? It turns out that Definition 6 imposes a particular structure on the marginal utilities. For a utility function $u \in U^{IC}$, the directional derivative in the direction $(-c_1(t), \beta_2 c_2(t), \ldots, \beta_N c_N(t))$ is zero for all times and states. This feature of utility functions in U^{IC} has important implications for financial market completeness. To see this, consider the social planner's problem in Equation (8) and assume that the corresponding utility function, $u(a, \delta)$, satisfies the condition in Definition 6. Then, for any time and state, the directional derivative of the utility function in the direction $v(t) = (-\delta_1(t), \beta_2\delta_2(t), \ldots, \beta_N\delta_N(t))$ is $\frac{\partial u(a,\delta(t))}{\partial \delta}\frac{v(t)}{\|v(t)\|} = 0$. Using the expression for commodity prices in Proposition 1, this is equivalent to $\sum_{n=2}^{N} \beta_n p_n(t)\delta_n(t) = p_1(t)\delta_1(t)$, and, consequently, stock one can be perfectly replicated by a buy-and-hold portfolio consisting of $(\beta_2, \ldots, \beta_N)$ units of stock $2, \ldots, N$. Hence, the market must be incomplete. The next theorem summarizes this result.

Theorem 1. *If the utility function from the social planner's problem in Equation (8), u, is such that $u \in U^{IC}$, then $\sigma(t)$ is non-invertible and the financial market is incomplete.*

Theorem 1 is a statement about the utility function derived from the social planner's problem in Definition 4. Hence, to apply Theorem 1, one first needs to calculate the utility function from the social planer's problem, then verify if it is in U^{IC}. Yet, this is a simple task relative to calculating the endogenous stock price diffusion matrix and checking its invertibility for all possible states and times. As the next proposition illustrates, for certain utility functions for the individual agents one can directly infer that the utility function of the social planner's problem belongs to U^{IC} without explicitly solving the social planner's problem.

Proposition 3. *Let $u_j(C) = \varphi_j\left(c_2 c_1^{\beta_2}, ..., c_N c_1^{\beta_N}\right)$ for $j = 1, ..., J$, i.e., every agent's utility function is in U^{IC} and agents share the preference parameters $\beta = (\beta_2, ..., \beta_N)$, then the financial market is incomplete.*

The proposition states that financial markets are incomplete even if the functional form of the aggregator, φ_j, differ across all agents as long as preferences are in U^{IC} and agents share the preference parameters $\beta = (\beta_2, ..., \beta_N)$. However, this is only sufficient for the market to be incomplete. For example, log-linear preferences such as in Cass and Pavlova (2004), which are in U^{IC}, lead to incomplete markets even with heterogeneity in β across agents.

Since our condition for market incompleteness is only sufficient, preferences other than those that are in U^{IC} can lead to endogenously incomplete financial markets, as the example below shows.

Example 2. Let the utility function of the representative agent be $u(c_1, c_2) = log(c_1) + \frac{c_2^{1-\gamma}}{1-\gamma}$ for $\gamma > 1$. It is easy to verify that this utility function is not in U^{IC}. Further, assume that $\delta_i(x) = e^{X_i(t)}$ for $i = 1, 2$, with $dX_1(t) = \mu_{X_1} dt + \sigma_{X_1} dZ_1(t)$ and $dX_2(t) = \mu_{X_2} dt + \sigma_{X_2} dZ_2(t)$. If we use good 2 as numeraire good, then the dividends in units of the numeraire are $\widetilde{\delta}_1(t) = P(t)\delta_1(t) = \delta_2(t)^\gamma$ and $\widetilde{\delta}_2(t) = \delta_2(t)$, respectively. Hence, the dividend diffusion matrix in units of the numeraire, $\sigma_{\delta_{\tilde{\delta}}}$, is non-invertible. Since X_1 and X_2 are independent, it can be shown that the price of stock one and two only depend on Z_2. Therefore, the financial market is incomplete. $\triangle$

A natural question to ask is what distinguishes the above example from utility functions in U^{IC}. As we illustrate in Section 4, if we instead choose good one as numeraire in Example 2, then the market is in fact complete. Hence, financial market completeness can depend crucially on the definition of the available assets. For instance, in Example 2, the agents can trade claims to the two dividend streams in addition to an asset that is locally risk-free in units of the numeraire. If we instead use good one as a numeraire, and assume that agents trade the two stocks and an asset that is locally risk-free in units of good one, then the market is complete. The reason for why the financial market switches from incomplete to complete when we change the numeraire is that we also change the set of available assets.

To formalize these observations, consider that there is a change in the numeraire good and the number of available assets is fixed. To maintain the assumption that the locally risk-free asset pays out in the numeraire good, we change the asset structure to accommodate this requirement. This implies that under a new numeraire good the previous risk-free asset is not tradeable. Using these assumption, we now provide a definition for numeraire good irrelevance.

Definition 7. *A multi-good economy, with an asset structure as in Section 2, exhibits numeraire good irrelevance when an arbitrary choice of the numeraire good does not affect financial market incompleteness or completeness.*

Definition 7 can also be expressed in terms of the stock price diffusion matrix, σ. Specifically, a multi-good economy exhibits numeraire good irrelevance when an arbitrary choice of the numeraire good does not effect the rank of the stock price diffusion matrix σ. These definitions turn out to be equivalent in our setting. In Example 2, the stock price diffusion matrix, σ, has full rank when good one is used as numeraire, but the rank drops when good two is used as numeraire. If Defintion 7 holds, i.e., there is a locally risk-free asset in the numeraire good in addition to the N stocks, then a drop in the rank of the stock price diffusion matrix is equivalent to incomplete markets.

We now present a proposition that, unlike the utility function in Example 2, shows that for any utility function in U^{IC} the market is incomplete regardless of the choice of numeraire.

Proposition 4. *If $u \in U^{IC}$, then the financial market exhibits numeraire good irrelevance.*

It might be that, although financial markets are incomplete, the agents can still implement the efficient allocations. Indeed, Cass and Pavlova (2004) show that when agents have log-linear utility functions over the different consumption goods, then the market is incomplete. Yet, the agents can implement the Arrow-Debreu equilibrium. They label such an equilibrium as peculiar financial equilibrium. However, as the next example shows, endogenous incompleteness, in general, has real effects as the agents cannot implement the Arrow-Debreu equilibrium by trading in the available assets.[12]

Example 3. Let $J = N = 2$. The utility function of agent $j = 1, 2$ is

$$u_j(c_1^j, c_2^j) = \varphi_j \left(c_2^j \left(c_1^j \right)^\beta \right). \tag{20}$$

By Proposition 3, we know that the financial market is incomplete. Now we state, through a

[12]It can be shown that introducing N pure discount bonds with maturity T, where bond i pays out one unit of good i, always completes the financial market. While the availability of N pure discount bonds does not resolve the counter factual equilibrium property that stock market returns are perfectly correlated, it does allow for solving models in which the market is incomplete without the bond contracts. Hence, such a complete market equilibrium with N pure discount bonds differs, in general, from the incomplete market equilibrium.

proposition, the result that endogenous incompleteness of financial markets matter in general for more than just portfolio indeterminacy.

Proposition 5. *Let preferences be as in Equation (20) and assume that* $\delta_i(x) = e^{X_i(t)}$ *for* $i = 1, 2$*, with* $dX_1(t) = \mu_{X_1} dt + \sigma_{X_1}^\top dZ(t)$ *and* $dX_2(t) = \mu_{X_2} dt + \sigma_{X_2}^\top dZ(t)$*. Then, we have the following:*

- *Independently of whether good one or two is used as a numeraire good, the Arrow-Debreu equilibrium cannot be implemented by trading in the available assets, unless* φ_j *is such that the optimal consumption is* $c_i^j(t) = f\delta_i(t)$ *for all* t *for some* $f \in (0, 1)$*.*

- *If the consumption basket* $\delta_2(t)\delta_1(t)^\beta$ *is used as numeraire good, then the Arrow-Debreu equilibrium can be implemented by trading in the available assets.*

Thus, it is difficult to model an economy that yields a peculiar financial equilibrium. △

We close this section by applying Theorem 1 to the economy in Serrat (2001).

Example 4. Serrat (2001) studies portfolio policies in an economy with two countries and argues that the model can rationalize the portfolio home bias puzzle. Kollmann (2006), however, proves that the diffusion matrix in the Serrat (2001) economy is non-invertible and, therefore, portfolio policies are in fact indeterminate. Hence, the model of Serrat (2001) cannot explain the portfolio home bias puzzle. In this example, we apply Theorem 1 to show that the preferences in Serrat (2001) are in U^{IC} and, therefore, the market is incomplete. Specifically, the utility function of the representative agent in Serrat (2001) can be expressed, with a small simplification, as follows

$$u(c_1, c_2, c_3, c_4) = \frac{1}{q} \left(c_1^q + c_2^q\right) \left(a c_3^\alpha + b c_4^\beta\right). \tag{21}$$

We rewrite Equation (21) as

$$
\begin{aligned}
u(c_1, c_2, c_3, c_4) &= \frac{1}{q}\left(1+\left(\frac{c_2}{c_1}\right)^q\right)\left(a\left(c_3 c_1^{\frac{q}{\alpha}}\right)^\alpha + b\left(c_4 c_1^{\frac{q}{\beta}}\right)^\beta\right) \\
&= \varphi\left(c_2 c_1^{\beta_2}, c_3 c_1^{\beta_3}, c_4 c_1^{\beta_4}\right), \qquad (22)
\end{aligned}
$$

where $\varphi(v, w, z) = \frac{1}{q}(1+v^q)\left(aw^\alpha + bz^\beta\right)$, $\beta_2 = -1$, $\beta_3 = \frac{q}{\alpha}$, and $\beta_4 = \frac{q}{\beta}$. From Equation (22), we see that the utility function in Serrat (2001) implies incomplete financial markets since it satisfies the sufficient condition for incompleteness in Theorem 1. $\triangle$

Therefore, our theory should help avoiding such unfortunate modeling assumptions about utility functions in economies with multiple consumption goods.

4 Complete markets

To derive sufficient conditions for market completeness, we introduce additional assumptions on the primitives of the economy. The first assumption imposes conditions on the state vector X. It corresponds to assumptions $A(c)$ and $A(d)$ in Hugonnier et al. (2010).

Assumption 6. *The solution to Equation (1) admits a transition density $\phi(t, x, \tau, y)$ that is smooth for $t \neq \tau$. Moreover, there are locally bounded functions (K, L), a metric d that is locally equivalent to the Euclidean metric, and constants ϵ, ι, $\varrho > 0$ such that $\phi(t, x, \tau, y)$ is analytic with respect to $t \neq \tau$ in the set*

$$
\mathcal{P}^2_\epsilon \equiv \{(t, \tau) \in \mathbb{C}^2 : \Re(t) \geq 0, 0 \leq \Re(\tau) \leq T \text{ and } |\Im(\tau - t)| \leq \epsilon \Re(\tau - t)\}, \qquad (23)
$$

and satisfies

$$
|\phi(t, x, \tau, y)| \leq K(x)L(y)|\tau - t|^{-\iota} e^{\varrho|\tau - t| - d(x,y)^2/|\tau - t|} \equiv \bar{\phi}(t, x, \tau, y), \qquad (24)
$$

for all $(t, \tau, x, y) \in \mathcal{P}^2_\epsilon \times \mathcal{X}^2$.

Assumption 7. *The dividends, $\delta_i(x)$, are real analytic functions.*

Assumption 8. *The utility function of agent $j = 1, ..., J$, u_j, is analytic and there are constants $R \leq \rho$ and $\nu > 1$ such that*

$$\int_0^T \sum_{j=1}^J \left(\int_{\mathcal{X}} e^{-R\tau} \frac{\partial u_j\left(\delta(y)/J\right)}{\partial \delta}^\top \delta(y) \bar{\phi}(0, x, \nu\tau, y) dy \right) d\tau < \infty. \tag{25}$$

Assumptions 6, 7, and 8 are imposed to guarantee that the candidate prices in Equation (16) are jointly real analytic in $(t, x) \in (0, T) \times \mathcal{X}$. Specifically, Assumption 6 covers a wide range of processes typically used in the finance literature. For instance, arithmetic Brownian motions and vector autoregressive processes both satisfy Assumption 6.[13] The requirement that dividends are real analytic functions is satisfied by most Lucas tree economies in the literature, where it is typically assume that dividends are exponential functions of the state variables, i.e, $\delta_i(X) = e^{A_i + B_i^\top X}$ for $A_i \in \mathbb{R}$ and $B_i \in \mathbb{R}^N$. However, option like payoffs such as $(X_i - A)^+$ are in general not real analytic. In Assumption 8, we assume that every agent has real analytic utility functions. This is satisfied for all conventional utility functions.

As for the case of incomplete financial markets, we start out with a definition for when a utility function leads to complete financial markets. The following theorem presents the main result of this section.

Definition 8. *A utility function is in $U^{C,l}$, where C stands for completeness and l denotes the numeraire good, if*

$$I + \varepsilon(T, x), \tag{26}$$

is invertible for at least one $x \in \mathcal{X}$.

Theorem 2. *If the utility function of the representative agent, u, is such that $u \in U^{C,l}$, then $\sigma(t)$ is invertible and the financial market is complete when good l serves as the numeraire good.*

[13] Hugonnier et al. (2010) discuss several examples of processes that satisfy Assumption 6.

Our condition for completeness, unlike the invertibility condition on the dividend diffusion matrix in Anderson and Raimondo (2008), Hugonnier et al. (2010), Riedel and Herzberg (2013), and Kramkov and Predoiu (2014) involves marginal rates of substitution, ε, derived from endogenous commodity prices.[14] Nevertheless, $I + \varepsilon(T, x)$, is easy to compute. Since market completeness is guaranteed if $I + \varepsilon(T, x)$ is invertible at one point the verification of completeness is easy. It is easy even in a situation where the utility of the representative agent is not known in closed form. Using standard aggregation techniques, one can check if the resulting financial market is complete by numerically solving for the utility function of the representative agent and applying the theorem. In contrast, without the condition for market completeness in Theorem 2, one has to calculate the stock price diffusion coefficients, which are conditional expectations, for every possible realization $(t, x) \in [0, T] \times \mathcal{X}$. Clearly, this seems, in general, computationally infeasible.

The condition in Anderson and Raimondo (2008) represents, a long anticipated, missing building block in the theory of continuous time asset pricing with heterogeneous agents. Yet, at least from Duffie and Huang (1985), Duffie and Zame (1989), Huang (1987), and Karatzas et al. (1990), it is expected that such equilibrium exists. Further, although essentially all models before Anderson and Raimondo (2008) assume complete markets in one way or another, many papers in this literature, especially the applied ones, contain examples or numerical work that demonstrate that equilibrium holds at least for some parameter values.

In contrast, the literature on multi-goods contains a series of models that imply incomplete financial markets (e.g. Cole and Obstfeld (1991), Zapatero (1995), and Serrat (2001)). For instance, as shown by Cole and Obstfeld (1991), the popular Cobb-Douglas utility function implies that markets are incomplete. Hence, it appears that our condition for completeness might prove useful for future modeling of utility functions in economies with multiple consumption goods.

Further, the next proposition shows that one can extend the result in Theorem 2 to a

[14] Theorem 2 in Hugonnier et al. (2010) also depends on preferences as it involves a second order expansion of the stock return volatility.

condition on the utility functions of individual agents.

Proposition 6. *If $u_j \in U^{C,l}$ for some $j = 1, \ldots, J$, then the market is complete when good l serves as the numeraire good for almost every vector of Pareto weights $a \in \varsigma$.*

Proposition 6 shows that as long as one agent has a utility function in $U^{C,l}$, then the market is complete for all Pareto weights outside a set of measure zero. Since the dividend diffusion matrix in Equation (15) is jointly real analytic in (t, x, a) and, therefore, if it is invertible for one x and a, then it is always invertible. Hence, if one can show that the market is complete when the social planner puts all weight on one agent, then the market is complete for almost every other set of Pareto weights due to real analyticity.

Theorem 2 covers a larger set of economies than implied by Proposition 6, as it is not necessary that any of the individual agents utility functions are in $U^{C,l}$, but only that the representative agent's utility function is in $U^{C,l}$. The next Proposition presents such a case.

Proposition 7. *Let $J = N = 2$. The utility function of agent $j = 1, 2$ is*

$$u_j(c_1^j, c_2^j) = \varphi_j\left(c_1^j\left(c_2^j\right)^{\alpha_j}\right), \tag{27}$$

for some real analytic function φ_j. Assume that $\alpha_1 \neq \alpha_2$ and $\varphi_j(x) \neq A_j ln(x) + B_j$ for some constants A_j, B_j, then the market is complete for almost all $a \in \varsigma$.

The reason for why we cannot use Proposition 6 is that none of the agents have utility functions in $U^{C,l}$. In fact, both agents have utility functions in U^{IC} and, therefore, the market would be incomplete if the social planner puts all the weight on one of the two agents. However, incomplete financial markets only occur at the boundary points $a_j = 1$ for $j = 1, 2$. For any other set of Pareto weights the market is complete as the utility function of the representative agent satisfies the condition in Theorem 2.

Our condition for financial market completeness is only sufficient. The next example, which builds on Example 2, presents an economy where the utility function is not in $U^{C,l}$, but the financial market is complete when good l serves as the numeraire good.

Example 5. Let the utility function of the representative agent be

$$u(c_1, c_2) = log(c_1) + \frac{c_2^{1-\gamma}}{1-\gamma}, \tag{28}$$

for $\gamma > 1$. If good one is used as the numeraire, then the market is complete since $det(1 + \varepsilon(T, x)) = 1 - \gamma \neq 0$. Let $\delta_i(X(t)) = e^{X_i(t)}$ with

$$dX_1(t) = \alpha\left(\bar{X}_1 - X_1(t)\right)dt + \sigma_{X_1}dZ_1(t), \tag{29}$$

$$dX_2(t) = \alpha(X_1(t) - X_2(t))\,dt + \sigma_{X_2}dZ_2(t). \tag{30}$$

This specification of the dividend streams satisfies Assumption 3, i.e., the market is potentially complete. However, if good two is used as numeraire, then $det(1 + \varepsilon(T, x)) = 0$ and, hence, the utility function does not satisfy the sufficient condition for completeness as it is not in $U^{C,2}$. In this case, the span of the dividends in units of the numeraire drops relative to the span of the dividends. Still, in this example, the financial market is in fact complete, as the next proposition shows.

Proposition 8. *Let $N = 2$ and $J = 1$ with utility given as in Equation (28). Moreover, assume that $\delta_i(X(t)) = e^{X_i(t)}$ with the dynamics of $X = (X_1, X_2)$ given by Equation (29) and (30). If good two is used as numeraire, then $\sigma_{\tilde{\delta}}$ is not invertible, yet the market is complete.*

Therefore, it is not necessary that the dividend diffusion matrix in units of numeraire, $\sigma_{\tilde{\delta}}$, is invertible for the market to be complete. △

In Example 5, the first stock only depends on δ_2^γ, while the second stock depends on both the current value of $\delta_2(t)^\gamma$ and the conditional expectation of $\delta_2(s)^\gamma$ for $t < s$. The distribution of $\delta_2(s)$ given the information at time $t < s$ depends on both X_1 and X_2 and, therefore, the second stock price not only correlates with the Z_2 as the first stock, but also Z_1. Thus, the financial market is complete.

5 Conclusion

In this paper, we investigate the determinants of financial market incompleteness and completeness in a continuous time Lucas (1978) model with multiple consumption goods and agents with heterogeneous preferences. First, we show that for a class of utility functions, including the Cobb-Douglas utility function and the preferences in Serrat (2001), the financial market is endogenously incomplete. While it is possible that even when the financial market is endogenously incomplete agents can implement the Pareto optimal allocations, as for example in Cass and Pavlova (2004), in general endogenous incompleteness prevents the agents from implementing the Pareto optimal allocations. Consequently, endogenous financial market incompleteness usually has real effects.

Second, we derive a sufficient condition for market completeness that only depends on the properties of aggregate output and the utility function of the representative agent. The condition is easy to verify, even in cases in which the utility function of the representative agent is not known in closed form. The major advantage of our condition is that it only requires verification of an invertibility condition for one realization of the state variables at the terminal time. In contrast, to verify market completeness without such a condition requires the calculation of the stock price diffusion matrix and checking whether the matrix is invertible for every possible realization of the state variables at every point in time.

In an economy with multiple consumption goods, one has to take a stand on the numeraire good. In turn, the numeraire good determines how the risk free asset is defined. We illustrate that whether a market is complete or not, might depend on the specification of the risk-free asset, and, hence, the numeraire good. In addition, we illustrate that even in the case when the market is guaranteed to be endogenously incomplete, the choice of numeraire good might be crucial in terms of the real effects of incompleteness. In particular, depending on the choice of numeraire, the agents might or might not be able to implement the Pareto optimal allocations by trading in the available assets.

Our conditions for market incompleteness and completeness should be useful for applied work in asset pricing theory. A large body of the literature in asset pricing theory is based on the Lucas tree economy. Recently, more papers consider economies with multiple consumption goods, even in settings outside of international finance. Multiple consumption goods usually lead to complex models. Thus, it can be difficult to explore the properties of the stock price diffusion matrix. Simple conditions for verifying market incompleteness and completeness will be useful in such cases. Serrat (2001) is an example for which our condition for market incompleteness would have been useful.

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6 Proofs

Proof. Proposition 1: The result follows from standard aggregation in complete markets extended to a multiple good setting (see Huang (1987), Duffie and Zame (1989), Karatzas et al. (1990), Dana and Pontier (1992), and Hugonnier et al. (2010)). For completeness, we sketch parts of the proof. The details can be found in Anderson and Raimondo (2008), Huang (1987), Karatzas et al. (1990), Dana and Pontier (1992), and Hugonnier et al. (2010), with slight modification to accommodate for multiple consumption goods, and in Lakner (1989) for the case of multiple commodities. First, consider agent j's optimization problem in Equations (6) and (7) when prices are given and financial markets are complete. The dynamic optimization problem can be reduced to the static optimization problem (see Cox and Huang (1989), Karatzas et al. (1990)):

$$\max_{C^j} E\left[\int_0^T e^{-\rho s} u_j(C^j(\tau))d\tau\right] \quad \text{s.t.} \quad E\left[\int_0^T \xi(\tau)P(\tau)^\top C^j(\tau)d\tau\right] \leq W^j(0) = \left(\eta^j\right)^\top S(0). \tag{A.1}$$

The utility gradient, $\nabla u_j : (0,\infty)^N \to (0,\infty)^N$, has an inverse $D_j : (0,\infty)^N \to (0,\infty)^N$ that inherits the basic properties of ∇u_j. We have that

$$e^{-\rho t} u_j\left(D_j(e^{\rho t}\varpi)\right) - \varpi^\top D_j(e^{\rho t}\varpi) = max\left\{e^{-\rho t}u_j(C(t)) - \varpi^\top C(t)\right\}, \tag{A.2}$$

holds for any non-negative ϖ. Then, $C^j(t) = D_j\left(y_j e^{\rho t}\Psi(t)\right)$ solves the maximization problem in Equation (A.1), where $\Psi(t) = \xi(t)P(t)$ and y_j is the solution to

$$E\left[\int_0^T \xi(\tau)P(\tau)^\top D_j\left(y_j e^{\rho s}\Psi(\tau)\right)d\tau\right] = W^j(0). \tag{A.3}$$

Since equilibrium is Pareto optimal, we can consider the following social planner problem

$$u(\delta; a) = \max_{\sum_j C^j = \delta} \sum_j a_j u_j(C^j), \tag{A.4}$$

where $a \in \varsigma$ is the vector of utility weights. Next, it can be shown that the maximization in Equation (A.4) is achieved by

$$C^j(t) = D_j\left(\frac{e^{\rho t}\Psi(t)}{a_j}\right), \tag{A.5}$$

where $\Psi(t) = e^{-\rho t}\nabla u(\delta(t); a)$ and the weights, a, are solutions to

$$E\left[\int_0^T e^{-\rho s}\nabla u(\delta(\tau); a)^\top D_j\left(\frac{e^{\rho s}\nabla u(\delta(\tau); a)}{a_j}\right)d\tau\right] = \sum_{i=1}^N \eta_i^j E\left[\int_0^T e^{-\rho s}\frac{\partial u(\delta(\tau); a)}{\partial \delta_i}\delta_i(\tau)d\tau\right] \tag{A.6}$$

Comparing Equation (A.6) with Equation (A.3), we can identify $a_j = \frac{1}{y_j}$. Defining $e_j(a) = \frac{1}{a_j}E\left[\int_0^T e^{-\rho s}\left(\nabla u(\delta(\tau); a)^\top D_j\left(\frac{e^{\rho s}\nabla u(\delta(\tau);a)}{a_j}\right) - \sum_{i=1}^N \eta_i^j \frac{\partial u(\delta(\tau);a)}{\partial \delta_i}\delta_i(\tau)\right)d\tau\right]$ as the excess utility map, one can show that $e(a) = \left(e_1(a), \ldots, e_J(a)\right)$ has all the properties of a finite dimensional demand function (see Lemma C.1 in Hugonnier et al. (2010) with slight modification to accommodate for multiple consumption goods) and, consequently, there exists some strictly positive a^* such that $e(a^*) = 0$. $\square$

Proof. Lemma 1: In equilibrium, the commodity price vector, $P(t)$, is given by

$$P(t) = \frac{\nabla u(\delta(t))}{\frac{\partial u(\delta(t))}{\partial \delta_l(t)}} = \left[MRS_{1,l}(t), ..., MRS_{N,l}(t)\right]^\top . \quad (A.7)$$

Applying Ito's lemma to $P(t)$ yields the lemma. □

Proof. Proposition 2: The lemma follows directly from applying Ito's lemma to the consumption process in units of the numeraire. □

Proof. Theorem 1: Assume that there exists a solution to the following equation

$$\frac{\partial u\left(\delta(t)\right)}{\partial \delta_1}\delta_1(t) = \beta_2 \frac{\partial u\left(\delta(t)\right)}{\partial \delta_2}\delta_2(t) + ... + \beta_N \frac{\partial u\left(\delta(t)\right)}{\partial \delta_N}\delta_N(t). \quad (A.8)$$

Integrate from t to T to get

$$\int_t^T \frac{\partial u\left(\delta(\tau)\right)}{\partial \delta_1}\delta_1(\tau)d\tau = \int_t^T \left(\beta_2 \frac{\partial u\left(\delta(\tau)\right)}{\partial \delta_2}\delta_2(\tau) + ... + \beta_N \frac{\partial u\left(\delta(\tau)\right)}{\partial \delta_N}\delta_N(\tau)\right) d\tau. \quad (A.9)$$

Take conditional expectation on both sides

$$E_t \int_t^T \frac{\partial u\left(\delta(\tau)\right)}{\partial \delta_1}\delta_1(\tau)d\tau = E_t \int_t^T \left(\beta_2 \frac{\partial u\left(\delta(\tau)\right)}{\partial \delta_2}\delta_2(\tau) + ... + \beta_N \frac{\partial u\left(\delta(\tau)\right)}{\partial \delta_N}\delta_N(\tau)\right) d\tau. \quad (A.10)$$

From dividing the above equation by $\xi(t)$ and comparing it with the pricing formula in Equation (16), we infer that the following equation is satisfied

$$S_1(t) = \beta_2 S_2(t) + ... + \beta_N S_N(t), \quad (A.11)$$

for all t. Hence, the stock prices are linearly dependent and the financial market is incomplete. Finally, note that any utility function $u \in U^{IC}$ satisfies the partial differential in Equation (A.8). □

Proof. Proof of Proposition 3. Let φ_{ji} be the partial derivative of φ_j with respect to argument i. Consider the first order conditions (FOC) from the social planner's problem

$$a_j \sum_{i=1}^{N-1} \varphi_{ji} c_{i+1}^j \beta_{i+1} \left(c_1^j\right)^{\beta_{i+1}-1} = y_1, \quad (A.12)$$

$$a_j \varphi_{ji} \left(c_1^j\right)^{\beta_{i-1}} = y_i, \quad i = 2, .., N. \quad (A.13)$$

We rewrite the above as

$$c_1^j = \sum_{i=1}^{N-1} \frac{y_{i+1}}{y_1} \beta_{i+1} c_{i+1}^j. \tag{A.14}$$

Summing over Equation (A.14) for $j = 1, ..., J$ and applying market clearing yields

$$\delta_1 = \sum_{i=1}^{N-1} \frac{y_{i+1}}{y_1} \beta_{i+1} \delta_{i+1}. \tag{A.15}$$

Finally, noting that $\frac{y_{i+1}}{y_1} = \frac{p_{i+1}}{p_1}$, we have that the following equation must hold

$$\sum_{i=1}^{N-1} \beta_{i+1} p_{i+1} \delta_i = p_1 \delta_1, \tag{A.16}$$

which implies that the market is incomplete. □

Proof. Proposition 4: This follows directly from noting that the PDE in Equation (A.8) does not depend on the choice of numeraire. □

Proof. Proof of Proposition 5. First, we derive the optimal consumption allocations in the Arrow-Debreu equilibrium. Next, we calculate the dynamics of the stock prices and check whether the Arrow-Debreu consumption allocations can be implemented by trading in the available assets. The FOC of the central planner problem is

$$\begin{aligned} a_j \varphi_j' \left(c_2^j \left(c_1^j \right)^{\beta} \right) c_2 \beta \left(c_1^j \right)^{\beta - 1} &= y_1, && \text{(A.17)} \\ a_j \varphi_j' \left(c_2^j \left(c_1^j \right)^{\beta} \right) \left(c_1^j \right)^{\beta} &= y_2, && \text{(A.18)} \end{aligned}$$

for $j = 1, 2$. This implies that

$$\frac{c_1^1}{c_2^1} = \frac{c_1^2}{c_2^2} = \frac{\delta_1}{\delta_2}, \tag{A.19}$$

where the last equality follows from the market clearing. Furthermore, this means that optimal consumption allocations take the form

$$\begin{aligned} c_i^1 &= f \delta_i, && \text{(A.20)} \\ c_i^2 &= (1 - f) \delta_i. && \text{(A.21)} \end{aligned}$$

Define $e = \delta_2 \delta_1^{\beta}$. Then, we have by Ito's lemma

$$de(t) = e(t) \left(\mu_e dt + \sigma_e^{\top} dZ(t) \right), \tag{A.22}$$

where $\sigma_e = \sigma_{X_2} + \beta\sigma_{X_1}$. Inserting e into the first order conditions, we get

$$a_1\varphi_1'\left(f^{1+\beta}e\right)f^{\beta} = a_2\varphi_2'\left((1-f)^{1+\beta}e\right)(1-f)^{\beta}. \tag{A.23}$$

Thus, by the implicit function theorem, the consumption share f is a function of the consumption basket e. Next, consider the stock prices

$$\begin{aligned} \xi(t)S_1(t) &= E_t\left[\int_t^T \xi(u)\delta_1(u)du\right] \\ &= a_1E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}e(u)du\right], \end{aligned} \tag{A.24}$$

and $\xi(t)S_2(t) = \frac{\xi(t)}{\beta}S_1(t)$. Hence, the financial market is incomplete. Now, assume that the first good is the numeraire good.[15] Then, in equilibrium, the state price density, ξ, is proportional to $\varphi_j'\left(c_2^j\left(c_1^j\right)^{\beta}\right)c_2\beta\left(c_1^j\right)^{\beta-1}$. Applying Ito's lemma on the left hand side of Equation (A.24) and Clark-Ocone Theorem (see Ocone and Karatzas (1991)) of Malliavin Calculus (see Nualart (1995), Detemple and Zapatero (1991) and Detemple et al. (2003)) on the right hand side of Equation (A.24) and, then, equating the diffusion coefficient on both sides, we get

$$\sigma(t) = A_e(t)\sigma_e + (\beta-1)\sigma_{X_1} + \sigma_{X_2} + B_S(t)\sigma_e, \tag{A.25}$$

where $A_e(t) = \frac{\partial}{\partial e}log\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}\right)$ and $B_S(t) = \frac{E_t\left[\int_t^T \frac{\partial}{\partial e}\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}e(u)\right)du\right]}{E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}e(u)du\right]}$. Next, consider the wealth of agent 1

$$\begin{aligned} \xi(t)W_1(t) &= E_t\left[\int_t^T \xi(u)\left(c_1^1(u) + p(u)c_2^1(u)\right)du\right] \\ &= a_1(1+\beta)E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{1+\beta}e(u)du\right]. \end{aligned} \tag{A.26}$$

Applying Ito's lemma on the left hand side and Clark Ocone Theorem on the right hand side, we get

$$\sigma_{W_1}(t) = A_e(t)\sigma_e + (\beta-1)\sigma_{X_1} + \sigma_{X_2} + B_{W_1}(t)\sigma_e, \tag{A.27}$$

where $B_{W_1}(t) = \frac{E_t\left[\int_t^T \frac{\partial}{\partial e}\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{1+\beta}e(u)du\right)\right]}{E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{1+\beta}e(u)du\right]}$. As B_S and B_{W_1} are not identical, the optimal wealth will load onto a different linear combination of the two Brownian motions than the stock price, and, hence, it is not possible to implement the Arrow-Debreu equilibrium if good one is the numeraire good.[16] Next, consider the case when e is the numeraire good.

[15] The proof when the second good is the numeraire good follows similarly.

[16] If $f(e(t)) = f$ for all $e(t)$, then the Arrow-Debreu equilibrium is implementable. However, we rule out any utility function that leads to such a no-trade equilibrium.

Then, in equilibrium the state price density, ξ, is proportional to $\varphi_j'\left(c_2^j\left(c_1^j\right)^\beta\right)$. Defining $\hat{A}_e(t) = \frac{\partial}{\partial e} log\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)\right)$ and following a similar calculation, we have that the stock price diffusion coefficients are

$$\sigma(t) = \left(\hat{A}_e(t) + B_S(t)\right)\sigma_e, \tag{A.28}$$

and the diffusion of the wealth of the first agent is

$$\sigma_{W_1}(t) = \left(\hat{A}_e(t) + B_{W_1}(t)\right)\sigma_e, \tag{A.29}$$

and, thus, effectively the market is complete as the optimal wealth and the stock prices depend on the same linear combination of the two Brownian motions. □

Proof. Theorem 2: We establish that the consumption prices $\Psi(t) = \xi(t)P(t)$ are jointly real analytic in $(t, x) \in (0, T) \times \mathcal{X}$. This follows from Theorem 2.3.5 (analytic implicit function theorem) in Krantz and Parks (2002) (Theorem B.2 in Anderson and Raimondo (2008)). According to the analytic implicit function theorem, the utility function of the representative agent is real analytic in $(t, x) \in (0, T) \times \mathcal{X}$. The fact that the candidate price functions S are jointly real analytic in (t,x) follows from Proposition 2 in Hugonnier et al. (2010). Following Hugonnier et al. (2010), adapted to the multiple good setting, one can show that the diffusion matrix of the candidate stock prices is

$$\sigma(t, x) = (T - t)\sigma_{\widetilde{\delta}}(T, x) + o(T - t) = (T - t)\left(I + \varepsilon(T, x)\right)\lambda(T, x) + o(T - t). \tag{A.30}$$

As $S(t, x)$ is jointly real analytic, it follows that $\sigma(t, x)$ is jointly real analytic. Equation (A.30) shows that the stock price diffusion coefficients, $\sigma(t, x)$, are proportional to the diffusion coefficients of the dividends in unit of the numeraire in a neighborhood of the terminal time. Following Theorem 1 in Hugonnier et al. (2010), one can show that a sufficient condition for the market to be complete is that $det\left(\left(I + \varepsilon(T, x)\right)\lambda(T, x)\right) = det\left(I + \varepsilon(T, x)\right) det\left(\lambda(T, x)\right)$ is non-zero. As $det\left(\lambda(T, x)\right) \neq 0$ by Assumption 3, it follows that $det\left(I + \varepsilon(T, x)\right) \neq 0$ for at least one $x \in \mathcal{X}$ is sufficient for market completeness. □

Proof. Proposition 6: Let $u_j \in U^{C,l}$ for some j and choose $a_j = 1$. This is equivalent to studying the economy with only agent j, and, hence, the market is complete for this choice of a as $u_j \in U^{C,l}$. Next, note that $\Psi(t) = \xi(t)P(t)$ is jointly real analytic in $(t, x, a) \in (0, T) \times \mathcal{X} \times \varsigma$. Again, this follows from Theorem 2.3.5 (analytic implicit function theorem) in Krantz and Parks (2002) (see Theorem B.2 in Anderson and Raimondo (2008)). The fact that $S(t, x, a)$ is jointly real analytic follows from Proposition 2 in Hugonnier et al. (2010), and it then follows that $\sigma(t, x, a)$ is jointly real analytic. Using similar arguments as the proof of Theorem 2, it then follows that the market is complete for almost every $a \in \varsigma$. $\square$

Proof. Proposition 7: From the FOC, it follows that

$$p_2(t) = \alpha_1 \left(\frac{c_1^1}{c_2^1} \right) = \alpha_2 \left(\frac{c_1^2}{c_2^2} \right). \tag{A.31}$$

Using the equilibrium commodity price and the market clearing in the second good, we have

$$\begin{aligned} p_2(t) \left(c_2^1(t) + c_2^2(t) \right) &= p_2(t)\delta_2(t) = \widetilde{\delta}_2(t), \\ \alpha_1 c_1^1(t) + \alpha_2 c_1^2(t) &= \widetilde{\delta}_2(t). \end{aligned} \tag{A.32}$$

From the market clearing in the first good, we have

$$c_1^1(t) + c_1^2(t) = \delta_1(t) = \widetilde{\delta}_1(t). \tag{A.33}$$

Combining Equation (A.32) and (A.33), we have

$$\begin{bmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} c_1^1(t) \\ c_1^2(t) \end{bmatrix} = \begin{bmatrix} \widetilde{\delta}_1(t) \\ \widetilde{\delta}_2(t) \end{bmatrix}. \tag{A.34}$$

Let

$$dc_1^1(t) = \phi_1(t)dt + \Sigma_1(t)^T dZ(t), \tag{A.35}$$

$$dc_1^2(t) = \phi_2(t)dt + \Sigma_2(t)^T dZ(t), \tag{A.36}$$

and

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t)^T \\ \Sigma_2(t)^T \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix}. \tag{A.37}$$

Applying Ito's lemma to both sides of Equation (A.34) yields

$$\begin{bmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix} \Sigma(t) = I_{\widetilde{\delta}}(t)(I + \varepsilon(t))\lambda(t). \tag{A.38}$$

Hence, for the financial market to be complete it is sufficient to show that the left hand side of the above equation is invertible for at least one $x \in \mathcal{X}$ at time T. If $a_i > 0$ for $i = 1, 2$, then $c_j^i(t, x) > 0$ for all $(t, x) \in (0, T) \times \mathcal{X}$. As $\alpha_1 \neq \alpha_2$ by assumption, $\begin{bmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}$ is invertible.

Hence, we need to show that $\Sigma(T)$ is invertible. Calculating $\Sigma(t)$ by Ito's lemma, we have

$$\Sigma(t) = J(c_1(t)) I_{\delta(t)} \lambda(t), \tag{A.39}$$

where $J(c_1(t))$ denotes the Jacobian of $c_1(t) = (c_1^1(t), c_1^2(t))$ and is given by

$$J(c_1(t)) = \begin{bmatrix} \frac{\partial c_1^1(t)}{\partial \delta_1} & \frac{\partial c_1^1(t)}{\partial \delta_2} \\ \frac{\partial c_1^2(t)}{\partial \delta_1} & \frac{\partial c_1^2(t)}{\partial \delta_2} \end{bmatrix}. \tag{A.40}$$

Note that $\det(\Sigma(T)) = \det(J(c_1(T))) \det\left(I_{\delta(T)}\right) \det(\lambda)$. By definition, $\det\left(I_{\delta(T)}\right) \neq 0$ and $\det(\lambda(T)) \neq 0$, implying that $\det(\Sigma(T)) \neq 0$ if and only if $\det(J(c_1(T))) \neq 0$. From the clearing of the commodity market, we obtain

$$c_1^1(t) + c_1^2(t) = \delta_1(t). \tag{A.41}$$

Taking the derivative with respect to δ_2, we get

$$\frac{\partial c_1^1(t)}{\partial \delta_2} = -\frac{\partial c_1^2(t)}{\partial \delta_2}. \tag{A.42}$$

Next, taking the derivative with respect to δ_1, we get

$$\frac{\partial c_1^2(t)}{\partial \delta_1} = 1 - \frac{\partial c_1^1(t)}{\partial \delta_1}. \tag{A.43}$$

Inserting Equation (A.42) and (A.43) into $J(c_1(T))$, we have

$$\det(J(c_1(T))) = -\frac{\partial c_1^1(t)}{\partial \delta_1}\frac{\partial c_1^1(t)}{\partial \delta_2} - (1 - \frac{\partial c_1^1(t)}{\partial \delta_1})\frac{\partial c_1^1(t)}{\partial \delta_2} = -\frac{\partial c_1^1(t)}{\partial \delta_2}, \tag{A.44}$$

and, therefore, if we can show that $\frac{\partial c_1^1(x,T)}{\partial \delta_2} \neq 0$ for some $x \in \mathcal{X}$ then $J(c_1(T))$ is invertible and the market is complete. Solving Equation (A.34) for $c_1^1(t)$, we have

$$c_1^1(t) = \frac{1}{\alpha_1 - \alpha_2}\left(\alpha_2 \tilde{\delta}_1(t) - \tilde{\delta}_2(t)\right). \tag{A.45}$$

Using the expression for the commodity price in Equation (A.31), one can rewrite Equation (A.45) as

$$c_1^1(t) = \frac{A}{1 + A\left(\frac{\delta_2(t)}{c_2^1(t)}\right)} \delta_1(t), \tag{A.46}$$

where $A = \frac{\alpha_1}{\alpha_1 - \alpha_2}$. Next, differentiating Equation (A.46) with respect to δ_2 yields

$$\frac{\partial c_1^1(t)}{\partial \delta_2} = -\frac{1}{\delta_1(t)} \left(c_2^1(t) - \delta_2(t) \frac{\partial c_2^1(t)}{\partial \delta_2} \right). \tag{A.47}$$

Note that the partial derivative above is only zero if

$$c_2^1(t) - \delta_2(t) \frac{\partial c_2^1(t)}{\partial \delta_2} = 0. \tag{A.48}$$

Solving Equation (A.48), we get that

$$c_2^1(t) = f_2 \delta_2(t), \tag{A.49}$$

where f_2 is a constant (not depending on δ_1) due to the symmetry of the problem. Hence, we have that the optimal consumption of the first agent of the first good can be written as $c_1^1(t) = f_1 \delta_1(t)$. Using the first order conditions together with the expressions for $c_1^1(t)$ and $c_2^1(t)$ we have

$$a_1 \varphi_1^{'} \left(f_1 \delta_1(t) (f_2 \delta_2(t))^{\alpha_1} \right) \left(f_2 \delta_2(t) \right)^{\alpha_1} = a_2 \varphi_2^{'} \left((1 - f_1) \delta_1(t) ((1 - f_2) \delta_2(t))^{\alpha_2} \right) \left((1 - f_2) \delta_2(t) \right)^{\alpha_2}. \tag{A.50}$$

Since dividends are less than perfectly correlated, the above cannot hold unless $\phi_j(x) = A + log(x)$, i.e., log-linear preference. Hence, as long as agents do not have log-linear preferences, then we have $\frac{\partial c_1^1(t)}{\partial \delta_2} \neq 0$ and, therefore, $\Sigma(T)$ is invertible and the market is complete. □

Proof. Proposition 8:

First, we show that the dividend diffusion matrix in units of the numeraire good is non-invertible when good two is the numeraire. The commodity price of the first good is $P(t) = \frac{\delta_2(t)^\gamma}{\delta_1(t)}$. Therefore, in units of numeraire the dividends are $\tilde{\delta}_1(t) = P(t)\delta_1(t) = \delta_2(t)^\gamma$ and $\tilde{\delta_1}(t) = \delta_1(t)$. Applying Ito's lemma to dividends in units of numeraire, we have

$$\sigma_{\tilde{\delta}}(t) = \begin{bmatrix} 0 & \gamma \sigma_2 \\ 0 & \sigma_2 \end{bmatrix}, \tag{A.51}$$

and, therefore, $\sigma_{\tilde{\delta}}$ is non-invertible. The stock price is jointly real analytic in $(t, x) \in (0, T) \times \mathcal{X}$ (see proof of Theorem 2) and it is sufficient to find one realization $x \in \mathcal{X}$ for which the stock price diffusion matrix is invertible. It can be shown that the value of the first stock is

$$S_1(t) = \frac{\delta_2(t)^\gamma}{\rho} \left(1 - e^{-\rho(T-t)} \right). \tag{A.52}$$

Applying Ito's lemma to Equation (A.52), we have that the diffusion coefficients are

$$\begin{aligned}\sigma_{1,1} &= 0,\\ \sigma_{1,2} &= \gamma\sigma_{X_2},\end{aligned}$$

and, therefore, the first stock only loads onto the second Brownian motion. To calculate the value of the second stock, we need the joint distribution of the state variables, $X(t)$. Note that $X(t)$ is jointly normal and one can show that the conditional variance of $X(s)$ at time t for $t < s$ is only a function of $s - t$ and not the current value of the state variable $X(t)$. Denote the conditional covariance matrix $V(s-t)$. The conditional mean of $X(s)$ is

$$E_t(X_1(s)) = e^{-\alpha(s-t)}X_1(t) + \bar{X}_1\left(1 - e^{-\alpha(s-t)}\right), \tag{A.53}$$

$$E_t(X_2(s)) = e^{-\alpha(s-t)}X_2(t) + \bar{X}_1\left(1 - e^{-\alpha(s-t)}\right) + \alpha e^{-\alpha(s-t)}\left(X_1(t) - \bar{X}_1\right)(s-t) \tag{A.54}$$

The value of the second stock is

$$\begin{aligned}S_2(t) &= e^{\gamma X_2(t)} E_t\left[\int_t^T e^{-\rho s + (1-\gamma)X_2(s)} ds\right]\\ &= e^{\gamma X_2(t)} \int_t^T e^{-\rho s + (1-\gamma)E_t(X_2(s)) + \frac{1}{2}(1-\gamma)V_2(s-t)} ds.\end{aligned} \tag{A.55}$$

Next, applying Ito's lemma to Equation (A.55), we have that the diffusion coefficients are

$$\sigma_{2,1} = (1-\gamma)e^{\gamma X_2(t)} \int_t^T e^{-\rho s + (1-\gamma)E_t(X_2(s)) + \frac{1}{2}(1-\gamma)V_2(s-t)} \alpha e^{-\alpha(s-t)}(s-t)\, ds\, \sigma_{X_1}, \tag{A.56}$$

$$\sigma_{2,2} = \left(\gamma + (1-\gamma)e^{\gamma X_2(t)} \int_t^T e^{-\rho s + (1-\gamma)E_t(X_2(s)) + \frac{1}{2}(1-\gamma)V_2(s-t)} e^{-\alpha(s-t)} ds\right)\sigma_{X_2}. \tag{A.57}$$

As $\sigma_{2,1}(t)$ and $\sigma_{1,2}(t)$ are non-zero for some $x \in \mathcal{X}$, the market is complete. Note that one can also apply the second order approximation in Hugonnier et al. (2010) (see Theorem 2) adapted to multiple goods. □

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