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All pay auctions with certain and uncertain prizes – a comment

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Abstract

In the important contribution "All pay auctions with certain and uncertain prizes" published in Games and Economic Behavior May 2014, Minchuk and and Sela analyze an all pay auction with multiple prizes. The specific feature of the model is that all valuations are common except for the valuation of one of the prizes, for which contestants have private valuations. However, the equilibrium characterization derived in the paper is incorrect. This note derives the correct equilibrium of the model.

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JEL codes: D 44, D82, J31, J41

Minchuk and Sela (2014) (hereafter MS) consider an all pay auction with multiple prizes. The specific feature of their model is that all valuations are common except for the valuation of one of the prizes. For this particular prize contestants have private valuations, independently drawn from a common distribution.

The authors claim that the equilibrium bid function is symmetric and monotone in the valuation of the uncertain prize. However this is only the case if the uncertain prize has the highest or lowest value. It is not if the uncertain prize has an intermediate value, which is MS' main case.

Let me first provide intuition for why MS' result fails. I follow MS' notation. There are n contestants competing for m different prizes. The highest bidder wins the most valuable prize, with a common value of v_n.

The second highest bidder wins the second most valuable prize v_{n-1} and so on. The uncertain prize is indexed $n-j+1$, with private value, denoted by a, drawn independently from a distribution $F(\cdot)$ with support on $[v_{n-j}, v_{n-j+2}]$. Suppose now that the equilibrium bid function $\beta(a)$ is strictly increasing. The probability that a player who bids according to valuation s will win the uncertain prize is then

$$\frac{(n-1)!}{(j-1)!\,(n-j)!} F(s)^{n-j} \left(1 - F(s)\right)^{j-1}.$$

Obviously this probability is a non-monotonic function of s, strictly increasing in s for low bid levels, and strictly decreasing for high bid levels. The probability of winning the uncertain prize reaches a maximum at a valuation $\hat{a}$ implicitly determined by

$$F(\hat{a}) = \frac{n-j}{n-1}$$

It follows from standard single crossing conditions[1] that a strictly increasing bid function on a bid segment is part of a separating equilibrium only if the win probability is increasing in valuation – as the player with a high valuation has a stronger incentive to bid aggressively than a bidder with a lower valuation. In other words, a strictly increasing bid function is incompatible with optimal bidding behavior if the probability of winning the uncertain prize declines with the bid level – in which case a bidder with a particularly high valuation for the uncertain prize will lower her bid.

Thus, a separating equilibrium cannot be monotone in the valuation of the uncertain prize in our setting. I will now by construction derive the separating equilibrium.

Suppose the number of players strictly exceeds the number of prizes, making a generalization of the result straightforward. A standard characteristic of a separating equilibrium in all pay auctions with ex ante identical contestants is the following:

- The player with the lowest possible valuation for the uncertain prize obtains zero payoff.

An implication of this is that the bid function must have upper support equal to v_n, the value of the highest prize. Otherwise the player with the

[1] For at detailed exposition see Athey (2001)

lowest possible valuation obtains a strictly positive rent by jumping to the upper support. As the lower support must be zero, a separating equilibrium will be a mapping from valuation $a \epsilon [v_{n-j}, v_{n-j+2}]$ to bids β on $[0, v_n]$.[2]

Observe that the equilibrium can be anchored in the following observation: For high bids, the probability of winning the uncertain prize declines as the bid level increases. Therefore, single crossing indicates that the bid function must have a declining segment at high valuations. For low valuations the equilibrium bid function must be increasing. A conjecture would be that the equilibrium bid function has the shape illustrated in figure 1, consisting of two bid segments: a declining segment, $\beta_H(a)$, for those bid levels at which the win probability (for the uncertain prize) declines with the bid; and an increasing segment $\beta_L(a)$ for bid levels at which the win probability increases. A contestant with valuation a then randomizes between the two bid levels, and chooses $\beta_L(a)$ with probability $q(a)$. With one exception: the contestant with the highest possible valuation for the uncertain prize, $a = v_{n-j+2}$, bids a certain amount, corresponding to the bid level that maximizes the probability of winning the uncertain prize. Note that the player with the lowest possible valuation for the uncertain prize, $a = v_{n-j}$, randomizes between bidding zero and v_n.

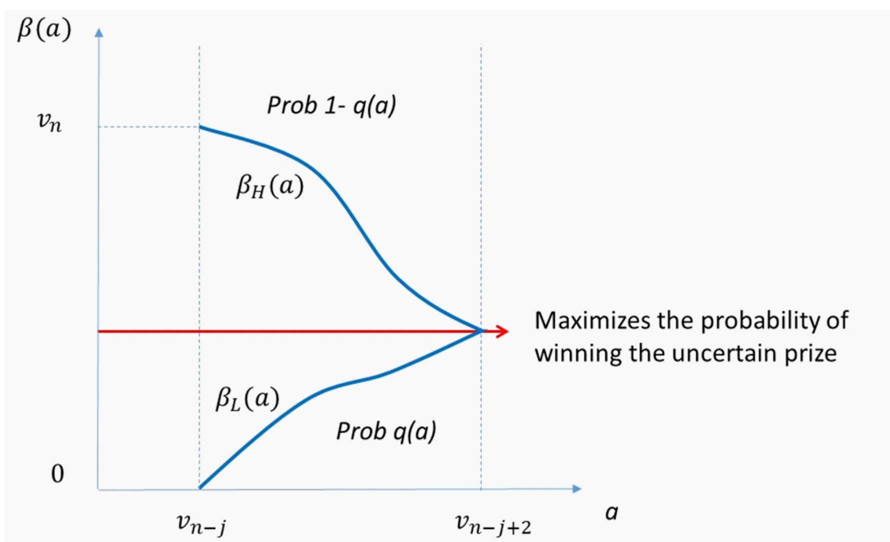

Figure 1

I will now show that the separating equilibrium indeed satisfies this pat-

[2]It is also standard that the bid distribution must be atomless.

tern. Denote by $\tilde{F}_k^i(a)$ the equilibrium probability that a player with valuation a wins the prize indexed i, given that she bids according to bid segment $\beta_k(a)$, $k = L, H$. The probability that this player will win the uncertain prize, indexed $n-j+1$, is then

$$\tilde{F}_L^{n-j+1}(a) = \tag{1}$$

$$\frac{(n-1)!}{(i-1)!\,(n-i)!}\left(\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{n-j}\left(1-\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{j-1}$$

if she bids $\beta_L(a)$[3], and

$$\tilde{F}_H^{n-j+1}(a) = \tag{2}$$

$$\frac{(n-1)!}{(j-1)!\,(n-j)!}\left(1-\int_{v_{n-j}}^{a}(1-q(z))\,f(z)dz\right)^{n-j}\left(\int_{v_{n-j}}^{a}(1-q(z))\,f(z)dz\right)^{j-1}$$

if she bids $\beta_H(a)$.

Let us characterize the equilibrium bid functions following MS' procedure, which is also the standard procedure. A player with valuation a behaves as a player with valuation s in order to maximize $(k = L, H)$

$$\max_s \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}} \tilde{F}_k^i(s)v_i + \tilde{F}_k^{n-j+1}(s)a - \beta_k(s)$$

The necessary conditions yields the following pair of differential equations

$$\beta_k'(s) = \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}} \frac{d\tilde{F}_k^i(s)}{ds}v_i + \frac{d\tilde{F}_k^{n-j+1}(s)}{ds}a$$

[3] To see this, note that a bid s wins the prize $n-j+1$ if an exact number $j-1$ of the bidder's contestants outbid her (note that there is a probability $1-\int_{v_{n-j}}^{s} q(z)f(z)dz$ that a single contestant will choose a bid above s), and the remaining $n-j$ contestants will bid below s.

The first order condition evaluated at $s = a$ yields the candidate bid functions[4]

$$\begin{aligned}
\beta_k(a) &= \beta_k(v_{n-j}) + \int_{v_{n-j}}^{a} \left[\sum_{\substack{i=n-m+1 \\ i \neq n-j+1}} \frac{d\tilde{F}_k^i(x)}{dx} v_i + \frac{d\tilde{F}_k^{n-j+1}(x)}{dx} x \right] dx \\
&= \beta_k(v_{n-j}) + \sum_{\substack{i=n-m+1 \\ i \neq n-j+1}} \tilde{F}_k^i(a) v_i - \sum_{\substack{i=n-m+1 \\ i \neq n-j+1}} \tilde{F}_k^i(v_{n-j}) v_i + \int_{v_{n-j}}^{a} \left[\frac{d\tilde{F}_k^{n-j+1}(x)}{dx} \right] x dx
\end{aligned}$$

If $k = L$ we have $\beta_L(v_{n-j}) = 0$ and thus $\tilde{F}_L^i(v_{n-j}) = 0$, as the player loses with certainty. If $k = H$ we have $\beta_L(v_{n-j}) = v_1$ and thus $\tilde{F}_L^n(v_{n-j}) = 1$, as the player in this case is certain to win the most valuable prize. In both cases, the candidate bid function is (where the second equality follows from integration by parts)

$$\begin{aligned}
\beta_k(a) &= \sum_{\substack{i=n-m+1 \\ i \neq n-j+1}} \tilde{F}_k^i(a) v_i + \int_{v_{n-j}}^{a} \left[\frac{d\tilde{F}_k^{n-j+1}(x)}{dx} \right] x dx \\
&= \sum_{\substack{i=n-m+1 \\ i \neq n-j+1}} \tilde{F}_k^i(a) v_i + \tilde{F}_k^{n-j+1}(a) a - \int_{v_{n-j}}^{a} \left[\tilde{F}_k^{n-j+1}(x) \right] dx
\end{aligned}$$

This yields associated utilities

$$U_k(a) = \int_{v_{n-j}}^{a} \tilde{F}_k^{n-j+1}(x) dx$$

Thus the bidder's rent is associated with a valuation of the uncertain prize above the lower support v_{n-j}, exactly as described by MS. Note that $\beta_L(v_{n-j+2}) = \beta_H(v_{n-j+2})$ since $\tilde{F}_L^i(v_{n-j+2}) = \tilde{F}_H^i(v_{n-j+2})$ for all $i = n - m + 1, .., n$[5], which confirms that the two bid segments meet at $a = v_{n-j+2}$.

It remains to determine $q(a)$. The equilibrium probability function, $q(\cdot)$, makes each contestant indifferent between choosing $\beta_L(a)$ and $\beta_H(a)$. This

[4] Note that $\tilde{F}^i(a) = \tilde{F}^i(v_{n-j}) + \int_{v_{n-j}}^{a} \frac{d\tilde{F}_k^i(x)}{dx} dx$.

[5] This can easily be checked by inserting v_{n-j+2} in the integral limits in (1) and (2), and generalizing to any i.

means that we have to find a function $q(\cdot)$ such that $U_L(a) = U_H(a)$ for every type a, thus

$$\int_{v_{n-j}}^{a} \tilde{F}_L^{n-j+1}(x)dx = \int_{v_{n-j}}^{a} \tilde{F}_H^{n-j+1}(x)dx \tag{3}$$

must always hold. Obviously this is equivalent to the following condition: for any $a\epsilon[v_{n-j}, v_{n-j+2}]$ we have

$$\tilde{F}_L^{n-j+1}(a) = \tilde{F}_H^{n-j+1}(a) \tag{4}$$

Hence, in a separating equilibrium, for each type a, the probability of winning the uncertain price is independent of the choice between $\beta_L(a)$ and $\beta_H(a)$. Accordingly, the net cost of submitting a high bid, $\beta_H(a) - \beta_L(a)$, cancels out with the net gain from the higher probability of winning one of the certain and more valuable prizes.

Substituting from (1) and (2) condition (4) can be written

$$\begin{aligned} &\left(\int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{j-1}\left(1 - \int_{v_{n-j}}^{a} q(z)f(z)dz\right)^{n-j} && (5)\\ = &\left(1 - \int_{v_{n-j}}^{a} (1-q(z))\, f(z)dz\right)^{j-1}\left(\int_{v_{n-j}}^{a} (1-q(z))\, f(z)dz\right)^{n-j} \end{aligned}$$

which must hold for all $a\epsilon[v_{n-j}, v_{n-j+2}]$. $q(\cdot)$ is the solution to the functional equation given by (5). One case is particularly simple. If the uncertain prize is the median prize, $j = (n+1)/2$, the equilibrium q is constant, $q(a) = 1/2$ for all a. To see this, insert $q(a) = 1/2$ in (5) and solve the integrals, which yields[6]

$$\left(\frac{1}{2}F(a)\right)^{(n-1)/2}\left(1-\frac{1}{2}F(a)\right)^{(n-1)/2} = \left(1-\frac{1}{2}F(a)\right)^{(n-1)/2}\left(\frac{1}{2}F(a)\right)^{(n-1)/2}$$

[6] An intuitive argument for existence more generally goes as follows: note that the equilibrium condition has a recursive structure in that (5) determines $q(\cdot)$ up to any arbitrary a', independent of the shape of $q(\cdot)$ above a'. Furthermore since $\tilde{F}_L^{n-j+1}(a) = \tilde{F}_H^{n-j+1}(a)$ is an identity, their derivatives are equal, $d\tilde{F}_L^{n-j+1}(a)/da = d\tilde{F}_H^{n-j+1}(a)/da$, and since $d\tilde{F}_L^{n-j+1}(a)/da$ is proportional to $q(a)$ and $d\tilde{F}_H^{n-j+1}(a)/da$ is proportional to $1 - q(a)$, and given the equilibrium function $q(\cdot)$ up to a, $q(a)$ is uniquely determined by the condition $d\tilde{F}_L^{n-j+1}(a)/da = d\tilde{F}_H^{n-j+1}(a)/da$.

Note that the equilibrium is fully revealing, despite the fact that players randomize bid levels, as each pair of possible bids is unique for the player's valuation.

1 References

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