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Abstract

China's growth is characterized by massive capital accumulation, made possible by high and increasing domestic savings. In this paper we develop a model with the aim of explaining why savings rates have been high and increasing, and we investigate the general equilibrium effects on capital accumulation and growth. We show that increased savings and capital accumulation stimulates further savings and capital accumulation, through an intergenerational distribution effect and an old-age requirement effect. We introduce what we term the savings multiplier, and we discuss why and how the one-child policy, and the dismantling of the cradle-to-grave social benefits provided through the state owned enterprises, have stimulated savings and capital accumulation.

Keywords: China, One-child policy, Overlapping generations, Growth, Savings.
JEL: O11, D91, E21

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1 Introduction

Since 1978 real per capita GDP in China has increased tenfold. The high and sustained growth is characterized by massive capital accumulation. So why China, and why now? Obvious answers relate to reforms that took place in the Chinese society over the same period. These are clearly of first order importance, but nevertheless do not fully answer why, compared to other reforming countries, capital accumulation in China has been so strong and so persistent. Thus, what we need to understand is what makes China and Chinese reforms so special. In this paper we argue that a main effect of the policy changes in China, in particular the introduction of the one-child policy and the dismantling of the cradle-to-grave social benefits, was to set the economy off on a path where savings and capital accumulation increased, in turn increasing savings and capital accumulation further. We argue that capital accumulation in China is fueled by what we term a savings multiplier.

Graph (a) in Figure 1 shows savings and investment as a share of GDP in China. In tandem with policy changes in the late 1970's, savings and investment as a share of GDP increased sharply. From then on, interestingly, savings and investment shares continued to grow. More than 40% of GDP has been invested over the last years. The high investments have been made possible by high and increasing savings, and in recent years more than 50% of GDP has been saved. Unlike in most other fast growing Asian economies, domestic savings have exceeded domestic investments.

The high savings rate is the sum of high corporate savings and high household savings. High corporate savings can be explained by capital market imperfections, where profitable firms have financed their investments by retained profits (Song, Storesletten and Zilibotti 2011). A number of papers, that we discuss below, have investigated the maybe most puzzling fact, namely that households have increased their savings rate, despite being quite poor, having fast income growth, and receiving low returns on their savings. At present, household savings is the single largest component of total savings, and according to Yang (2012), the increase in the rate of household savings from 2000 to 2008 is also the most important contribution to the overall increase in the Chinese savings rate in the same period.

Our model, and its mechanisms, is motivated by two major policy reforms key to China's transformation; the one-child policy and the dismantling of state owned enterprises. The process of dismantling state owned enterprises has implied massive layoffs, where for instance each year between 1996 and 2001, 5 to 6 million employees were laid off nationwide (Xu 2011). The share of workers in these firms was halved from 1995 to 2005, and as a result the enterprise based cradle-to-grave social safety net shrank rapidly (Ma and Yi, 2010). According to Meng (2012), the state/collective share in industrial output value fell from 90 percent in 1990 to 30 percent in 2008. A particular implication of this is that for the majority of workers, state owned enterprises can no longer be relied on to provide old-age support.[1] As pointed out by Oksanen (2010), less

[1] The pre reform system where state owned enterprises provided old-age support is discussed by e.g. James

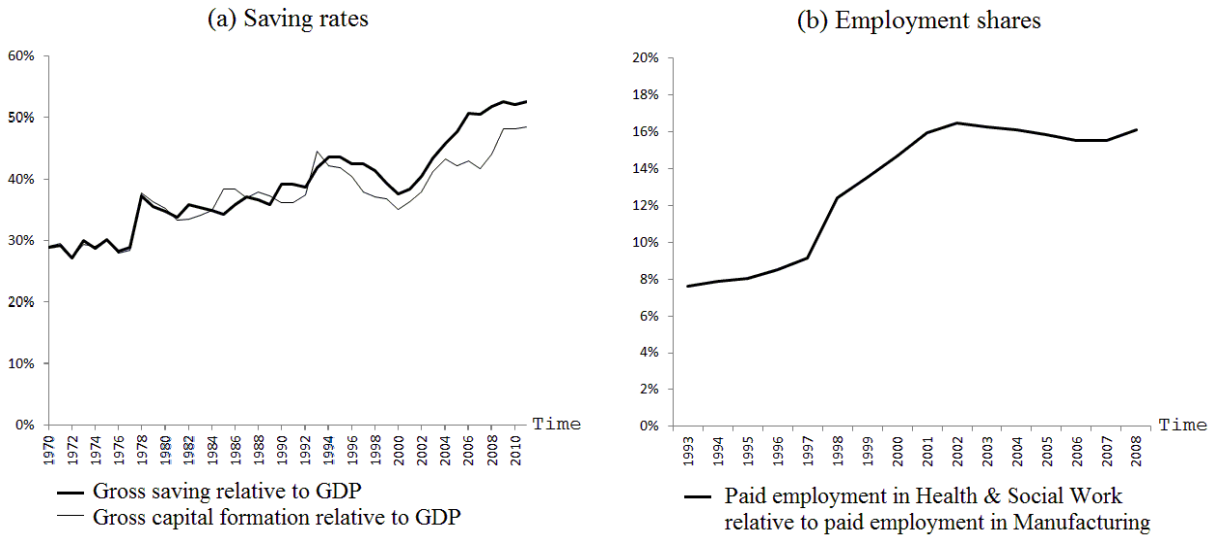

Figure 1: Graph (a): saving and investment shares of GDP in China 1970-2010 (source: World Bank). Graph (b): paid employment in Health and Social Work relative to paid employment in Manufacturing in China 1993-2008 (source: authors calculations on LABORSTA Table 2E, International Labor Organization).

than 30% of all employees are covered by pension schemes. Even for those covered, however, there are systemic deficiencies (p. 3) "due to problems in implementation (including fraud) the accounts are virtually empty as contributions were used to pay the pensions of current pensioners (or maybe even other expenditures)". Chinese workers need to find alternative ways to provide for their old-age care. As in most developing countries, the natural alternative was for such care to be provided within the family.

However, while the need for family based old-age care in China increased, the decision in 1978 to implement the one-child policy meant that the scope for such care dwindled. The one-child policy has had the obvious implication that there are fewer children to provide old-age care. China has entered the second generation of parents affected by this policy, with more and more families consisting of four grandparents, two parents and one child. According to Litao and Sixin (2009), the fertility rate decreased from 4.9 in 1975 to 1.7 in 2007, while life expectancy increased from 63 years to 73 years in the same period. Oksanen (2010, p. 4) finds that "Population aging seems to be the fastest in the world: the ratio of 65 year old to those aged 15-64 years is currently 11% and will increase to 38% by 2050".

As observed by Li et al. (2012), the number of people in the labor force may have peaked already in 2011, and since 1998 wage growth has exceeded GDP growth. This implies a shift in the income distribution towards workers (and as emphasized by Song and Yang (2010), towards

(2002), who note that (p. 56) "During the cultural revolution, the provision of old-age security (and other forms of social service) became a responsibility of each state enterprise, financed out of current revenues. Workers in the formal sector stayed at the same enterprise throughout their working lives. The enterprise provided housing, medical care and old-age security to its workers. The same services where provided to its pensioners".

the young). Zhong (2011) argues that a main reason for increased income inequality in recent years is the higher income differences between those working and those retired, and finds that (p. 103) "While the contributions of "ratio of household members in working age" to income inequality are relatively small in 1997 and 2000, it has increased dramatically in the first half of this decade". He argues that this is due to the one-child policy and population aging, which has induced labor shortages.

Despite massive investments in the manufacturing sector, the share of manufacturing employment out of total paid employment is decreasing, while that of service sectors is increasing. Graph (b) in Figure 1, based on numbers in the ILO database, which contains data from 1993 onwards, shows the labor share in health and social work relative to the labor share in manufacturing. Over 15 years it has doubled. From 1993 to 2008 the share of workers in manufacturing decreased from 37% to 29%, whereas the employment share of health and social work increased from 2.8% to 4.7%.

The Chinese reforms also has main implications for household expenditures. The share of health spending that households pay themselves increased from 16% in 1980 to 61% in 2001 (Blanchard and Giavazzi, 2006). Chamon and Prasad (2010) find that, among the households in their sample, expenditures on health and education grew from 2% of consumption expenditures in 1995 to 14% in 2005. Chou and Wang (2009, p. 137) conclude that the main challenges to China's health system are "the heavy reliance on private financing, dramatic drop in health insurance coverage, and rising health care costs". And according to Eggleston (2012, p. 4) "the growth in China's health spending has been one of the most rapid in world history".

In this paper, we develop an OLG-model, extended to take into account that agents need to purchase old-age care. We show how our framework produces a savings multiplier via two channels. The first is what we term the intergenerational distribution effect. Higher savings implies a higher capital stock, and increased potential for producing manufacturing goods. When substitution between manufacturing goods and old-age care is limited, a higher capital stock means more labor in the old-age care sector, and less labor in the manufacturing sector. Unlike in one-good OLG models, the income distribution then shifts in favor of the wage earners. Since the income distribution shifts away from the old and towards the young, savings increase, increasing the capital stock further. Thus savings and capital accumulation stimulate further savings and capital accumulation. The second reason for a savings multiplier is what we term the old-age requirement effect. Increased savings and capital accumulation pushes the future wage up, making old-age care more expensive. To compensate for the increased future costs of old-age care, agents increase their savings, contributing to further capital accumulation. Thus, this gives an additional channel whereby savings and capital accumulation stimulates further savings and capital accumulation.

In the case where substitution between old-age care and consumption of manufacturing goods is limited, the transitional dynamics in our model implies a growth process with increased savings and investment rates, wage growth exceeding GDP growth, a smaller fraction of the labor force

kids pool	1 girl	1 boy	2 girl	1 boy 1 girl	2 boys
rely on own savings	40.2	36.5	37.4	32.2	31.3
rely on children	44.9	50.0	54.7	61.8	63.7
"Yes I worry"	40.1	31.3	43.6	30.8	32.9

Table 1: As elderly, what will you rely on? Do you worry ?

in the manufacturing sector, income distribution shifting in favor of the young and in disfavor of the old, and an increasing share of private expenditures allocated to the purchase of old-age care services. Although highly simplified, we would argue that the mechanisms in the model, and their relation to the one-child policy and the dismantling of the cradle-to-grave social benefits provided by state owned enterprises, embraces important characteristics of savings and capital accumulation in China after 1978.

Our paper is related to, and motivated by, a large number of empirical papers that discuss the high and increasing household savings in China. Kraay (2000, p. 546) points out that households "once covered by generous cradle-to-grave benefits through employment in state enterprises, are finding their futures increasingly uncertain", and most studies see the lack of a public welfare system as key to explain household savings patterns. As argued by Modigliani and Cao (2004), a main effect of the one-child policy decided in 1978 was to strengthen the needs to save for retirement. Blanchard and Giavazzi (2006, p. 7) similarly argue that "The high savings rate reflects a high level of individual risk, related to health costs, retirement and the financing of education". Chamon and Prasad (2010) find that the increased savings rates are (p. 93) "best explained by the rising private burden of expenditures on housing, education and health care". Barnett and Brooks (2010) conclude that government expenditures on health has a strong impact on urban household savings, where (p. 8) "1 yuan of government health spending results in a 2 yuan decrease in saving". Song and Yang (2010) argue that a main reason for the increasing household savings rate is a change in the composition of income, where the income profile has flattened, so that the young workers earn a higher fraction of income than before, and, since the young have a high propensity to save, this increases aggregate savings. Wei and Zhang (2011) point to the rising number of boys relative to girls born (due to selective abortion), and find that parents of a boy save in order to increase the attractiveness of their son in the marriage market so as to increase the probability he finds a wife. This savings motive, in turn, spills over to other households, increasing savings further. Although these papers put different weight on different mechanisms, there seems to be some consensus in the literature that the high and increasing household savings results from the dismantling of state owned enterprises, the missing welfare system, the one-child policy, the aging of the population, and the increased need to provide for own retirement and old-age care.

Similar views can also be confirmed by household surveys. We have utilized the household survey that among others Wei and Zhang (2011) used. The data covers more that 9000 house-

age	≤ 44	45-54	≥ 55	all
children related	89.5	77.8	55.9	78.2
build house	21.6	18.4	11.2	18.2
retire	33.8	50.0	68.8	47.0
medical	11.4	18.4	35.0	18.9

Table 2: Self reported reason for savings

holds in 122 rural counties for the year 2002.[2] Table 1 presents the results when parents with one or two children are asked what funding they will rely on as old. They are also asked whether they worry about becoming old. We see that parents with one boy worries less than parents with one girl, probably due to the tradition of girls becoming part of the family of the husband. For those having more than one child, those with two girls worry the most. We also see that irrespective of the number of children, those with more boys rely more on their children and less on own savings. Table 2 shows frequencies when parents are asked to mention the **two** most important reasons for savings. The noteworthy pattern is that children related savings declines with age of head of household, while medical and retirement reasons increases with age of household head.

In addition to relating to the literature on savings and growth in China, our paper also relates to the debate on "communist capital accumulation". According to Acemoglu and Robinson (2012), growth in China has important similarities with growth in the former Soviet Union, based on high savings and massive capital accumulation, but being unsustainable if institutions are not reformed to be more inclusive. In fact, investments rates in China and the former Soviet Union are at similar levels, both exceeding 40% of GDP. In the Soviet Union the suppression and collectivization of agriculture was important in mobilizing the high required savings. In China, on the other hand, agriculture has been decollectivized after 1978. Our paper points out how the one-child policy, and the dismantling of state enterprises without replacing them with a welfare system, may be an alternative way to mobilize the savings required to fuel "communist capital accumulation".

The rest of the paper is organized as follows. In Section 2 we set up the model. We show the static equilibrium of the model in Section 3. In Section 4 we study transitional capital accumulation and growth, and introduce what we term the savings multiplier. We discuss how and why the steady-state capital stock in our model differs from standard OLG-models, and show how, again compared to standard OLG-models, the effects of savings and capital accumulation are magnified. We then discuss why the one-child policy has had such a massive impact on savings and capital accumulation. Section 5 presents extensions of the model. Subsection 5.1 investigates the introduction of a welfare state. A welfare state removes some of the incentives that has produced the high and increasing savings rates, and thus reduces capital accumulation and growth. Subsection 5.2 then discusses dynamic inefficiency, and Subsection 5.3 studies

[2]We are very grateful to Wei and Zhang for making their Stata do-files available and to ICPSR for giving access to data on http://www.icpsr.umich.edu/icpsrweb/ICPSR/studies/21741.

endogenous growth. Section 6 concludes. In the Appendix for online publication we provide the details of derivations that are left out from the main text.

2 The Model

In this section, we develop our model of savings and growth based on an overlapping-generations (OLG) structure that takes into account that when old, agents are in different needs from when young. In particular, due to the policy reforms discussed above, Chinese parents can rely less on their children to provide old-age care and less on state firms to act as a substitute for a welfare state. Thus, differently from the standard OLG framework pioneered by Diamond (1965) – henceforth termed the *canonical one-good model* – we separate between the production of goods and the production of care. One set of firms produces a *generic good*, used for investment and consumption of both young and old agents. The second set of firms provides *old-age care*.

2.1 Households

We consider an overlapping-generations environment where each agent lives two periods $(t, t+1)$. Total population, denoted N_t, consists of N_t^y young and N_t^o old agents, and grows at the exogenous net rate $n > -1$;

$$N_t = N_t^y + N_t^o, \quad N_t^y = N_t^o (1+n), \quad N_{t+1} = N_t (1+n). \tag{1}$$

Households purchase two types of goods over their life-cycle: a generic consumption good and old-age care services. The generic good is consumed in both periods of life. Old-age care services, instead, are exclusively purchased by old agents. The utility of an agent born at the beginning of period t takes the additive form

$$U_t \equiv u(c_t) + \beta v\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{2}$$

where c_t and d_{t+1} represent consumption levels of the generic good in the first and second period of life, respectively, h_{t+1} is the amount of old-age care consumed when old, $\bar{h} \geqslant 0$ is the *minimum requirement* – i.e., the minimum amount of old-age care required by old agents – and $\beta \in (0,1)$ is the private discount factor between young and old age. A constraint of the consumer problem is that the minimum requirement is at least weakly satisfied,

$$h_{t+1} - \bar{h} \geqslant 0. \tag{3}$$

As is standard, we first study existence and uniqueness of interior equilibria where old-age care obeys (3). We then verify ex-post the conditions under which $h_{t+1} > \bar{h}$ holds.[3] The case where

[3] In fact, in our main model which is the neoclassical case with constant returns to scale in generic-good production, there always exists a stable long-run equilibrium in which the allocation of labor between generic-good and health-care production exhibits stable shares consistent with the interior solution $h_{t+1} > \bar{h}$. We discuss cases where this may not be the case in Section 5.3, where we extend the model to allow for linear returns to capital at the aggregate level. Then, under certain conditions, the accumulation process may drive the economy towards long-run equilibria where labor is pushed away from the health-care sector so that the constraint $h_{t+1} - \bar{h} \geqslant 0$ becomes binding.

$\bar{h} = 0$, so that there is no minimum old-age care requirement, is of special interest. As we will see, this case transparently isolates what we term the intergenerational distribution effect in our model. For this reason, when we study the dynamics of the model in Section 4, we first put emphasis on this case, before we turn to the more general case of $\bar{h} \geqslant 0$, in which what we term the old-age requirement effect is also present.

We assume that only young agents work, supplying inelastically one unit of homogeneous labor. The only source of income in the second period of life is interest on previous savings. Personal lifetime income is entirely consumed at the end of the second period. Taking the consumption good as the numeraire in each period, the budget constraints read

$$c_t = w_t - s_t, \tag{4}$$

$$s_t R_{t+1} = d_{t+1} + p_{t+1} h_{t+1}, \tag{5}$$

where w_t is the wage rate, s_t is savings, R_{t+1} is the (gross) rate of return to saving, and p_{t+1} is the price of old-age care. Savings consist of physical capital, which as in the one-good OLG model is homogeneous with the generic consumption good. Assuming full depreciation within one period, market clearing requires that aggregate capital at the beginning of period $t+1$ equals aggregate savings of the young agents in the previous period, $K_{t+1} = N_t^y s_t$.

In order to make our new mechanisms as transparent as possible, we consider a specific, yet flexible form of preferences:

$$u(c_t) \equiv \log c_t, \tag{6}$$

$$v\left(d_{t+1}, h_{t+1} - \bar{h}\right) \equiv \log\left[\gamma\left(d_{t+1}\right)^{\frac{\sigma-1}{\sigma}} + (1-\gamma)\left(h_{t+1} - \bar{h}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{7}$$

where $\gamma \in [0,1]$ is a weighting parameter and $\sigma > 0$ is the elasticity of substitution between consumption goods and care services in the second period of life: d_{t+1} and h_{t+1} are strict complements if $\sigma < 1$, strict substitutes if $\sigma > 1$. In the limiting case $\sigma \to 1$, the term in square brackets reduces to the Cobb-Douglas form $(d_{t+1})^{\gamma}(h_{t+1})^{1-\gamma}$. When $\bar{h} > 0$, the income elasticity of old-age care falls short of unity, resembling the case in Acemoglu, Finkelstein and Notowidigdo (2013), who estimate the income elasticity of health spending to 0.7.

Assumptions (6)-(7) imply two fundamental properties. First, we can treat the canonical one-good model as a special case: letting $\gamma = 1$ (and $\bar{h} = 0$), old-age care services disappear from private utility and, hence, are not produced in equilibrium. Second, the utility functions (6)-(7) exhibit a unit elasticity of intertemporal substitution. This property allows us to describe the effects of old-age care on saving rates in the clearest way. Setting $\gamma = 1$, we obtain the logarithmic version of the canonical model, in which the saving rate is constant over time because consumption propensities are independent of the interest rate.[4] Hence, in the general case $0 < \gamma < 1$, any departure from constant saving rates in the model is exclusively due to the inclusion of old-age care services.

[4] More precisely, the savings rate of the young is constant with logarithmic preferences. When production is Cobb-Douglas, the income share of the young is constant, and thus also the aggregate savings rate is constant.

2.2 Production Sectors

Old-age care is labor intensive. In our framework this implies that the factor price of interest to old agents is not only the interest rate, but also the wage rate. This contrasts with standard one-good OLG models. There, old agents are on the supply side of the capital market, and the only relevant factor price when old is the (real) interest rate. In the present model, old agents are still on the supply side of the capital market, but since when old they need care, they are in addition on the demand side of the labor market. This implies that the wage rate is also important for old agents. To clarify this, and to capture in a simple way that care is more labor intensive than the production of the generic consumption and investment good, we assume that care services are produced with labor as the only factor of production.[5]

We denote by ℓ_t the fraction of workers employed in the generic sector, and by $1-\ell_t$ the fraction employed in the care sector. Perfect labor mobility and perfectly competitive conditions in the labor market ensure wage equalization in equilibrium. In the old-age care sector, there is a simple constant returns to scale production technology:

$$H_t \equiv \eta\,(1-\ell_t)\,N_t^y, \tag{8}$$

where H_t is the aggregate output of care services, and $\eta > 0$ is a constant labor productivity parameter.

In the generic good sector, we consider a specification displaying constant returns to scale at the firm level. A continuum of firms, indexed by $j \in [0, J]$, exploits the same Cobb-Douglas technology

$$X_t^j \equiv (k_t^j)^{\alpha}(a_t \ell_t^j N_t^y)^{1-\alpha} \text{ for each } j \in [0, J]\,, \tag{9}$$

where X_t^j is the output of the generic good produced by the j-th firm, k_t^j and $\ell_t^j N_t^y$ are the amounts of physical capital and labor employed at the firm level, $\alpha \in (0,1)$ is an elasticity parameter, and a_t is labor productivity in the generic-good sector.

2.3 Labor Productivity

Specification (9) assumes that the generic good technology displays constant returns to scale *at the firm level*, so that income shares are determined according to standard zero-profit conditions. In the main model of our paper we also make the standard neoclassical assumption of constant returns to scale at the *aggregate level*. We then impose that a_t equals an exogenous constant $B^{\frac{1}{1-\alpha}}$ in each period: the generic production sector exhibits strictly diminishing marginal returns to capital also at the aggregate level, and aggregate sectoral output $X_t \equiv JX_t^j$ is given by

$$X_t = B\,(K_t)^{\alpha}\,(\ell_t N_t^y)^{1-\alpha} \tag{10}$$

where $K_t \equiv Jk_t^j$ is aggregate capital and $\ell_t \equiv J\ell_t^j$ is aggregate labor employed in the generic sector. This is the setup of the canonical model since the seminal work of Diamond (1965).

[5] For a two-sector OLG model with capital in both sectors, as well as the existence and stability properties of such models, see Galor (1992).

In Subsection 5.3 we extend the model to allow for endogenous growth in its simplest fashion. Following Romer (1989), we include learning-by-doing whereby the productivity of workers employed in the generic sector increases with the amount of capital that each of these workers uses. In this case the labor productivity is governed by the spillover function $a_t = A^{\frac{1}{1-\alpha}} K_t / (\ell_t N_t^y)$, where A is an exogenous constant. Since a_t is taken as given at the firm level, income shares are still determined by the usual zero-profit conditions, but aggregate sectoral output is proportional to aggregate capital:

$$X_t = AK_t. \tag{11}$$

We next describe the equilibrium conditions that hold independently of the assumed technology for the generic good, then put our main emphasis on the neoclassical case, before returning to the extension of the model to endogenous growth in Subsection 5.3.

3 Static Equilibrium

This section discusses the static equilibrium conditions holding in each period for a given stock of capital per worker. We first study the profit-maximizing conditions for firms, the utility-maximizing conditions for households, the labor market equilibrium, and the goods market equilibrium. We then study the joint (static) equilibrium of all the markets, the implications for the aggregate savings rate, and finally the implied mapping to capital accumulation.[6]

3.1 Firms

In the service sector for old-age care, the technology (8) implies a wage that equals the market price of services times the labor productivity,

$$w_t = p_t \eta. \tag{12}$$

Market clearing requires that total output of old-age care services matches aggregate demand by old agents, $H_t = N_t^o h_t$. The existence of a minimum requirement, $h_t \geqslant \bar{h}$, requires that total production H_t exceeds $N_t^o \bar{h}$, which implies a constraint on sectoral employment shares: using the production function (8), we obtain

$$\ell_t \leqslant \frac{\eta (1+n) - \bar{h}}{\eta (1+n)} \equiv \ell^{\max}, \tag{13}$$

where $\ell^{\max}$ is the maximum level of employment in the generic sector that is compatible with a level of old-age care output equal to the minimum requirement.[7] In the remainder of the analysis, we will work under the parameter restriction

$$\bar{h} \leqslant \eta (1+n), \tag{14}$$

[6] Unless otherwise specified, all equations in this section are valid in the neoclassical case as well as in the AK-case. Thus, to avoid repetitions when we extend the model to endogenous growth in Subsection 5.3, in the present section we continue to use a_t for the labor productivity, without specifying if growth is neoclassical or endogenous (when not necessary).

[7] Formally, the level of health-care output equal to the minimum requirement is $H_t^{\min} \equiv \eta (1 - \ell^{\max}) N_t^y = N_t^o \bar{h}$.

which implies $\ell^{\max} \geqslant 0$. By construction, when the minimum requirement is $\bar{h} = 0$, we have $\ell^{\max} = 1$.

In the generic good sector each firm maximizes own profits $X_t^j - R_t k_t^j - w_t \ell_t^j N_t^y$ subject to technology (9). Denoting capital *per young* agent by $\kappa_t \equiv K_t/N_t^y$ and , respectively, the zero-profit conditions in the sector can be aggregated across firms and written as

$$w_t = a_t^{1-\alpha}(1-\alpha)(\kappa_t/\ell_t)^{\alpha} = (1-\alpha)(x_t/\ell_t), \tag{15}$$

$$R_t = a_t^{1-\alpha}\alpha(\ell_t/\kappa_t)^{1-\alpha} = \alpha(x_t/\kappa_t), \tag{16}$$

where $x_t \equiv X_t/N_t^y$ is sectoral output *per young* agent. Aggregating the incomes of both sectors, we thus have

$$\frac{Y_t}{N_t^y} = w_t + R_t\kappa_t = x_t\left(\frac{1-\alpha}{\ell_t} + \alpha\right), \tag{17}$$

where Y_t is aggregate income, which coincides with the total value of goods and services produced in the economy.[8]

3.2 Consumers

Each agent maximizes (2) subject to the budget constraints (4)-(5). Denoting the derivative of the u-function with respect to c_t by u_{c_t}, and so on, the solution to this problem yields two familiar first order conditions; the Keynes-Ramsey rule, $u_{c_t} = \beta R_{t+1} v_{d_{t+1}}$, and an efficiency condition establishing the equality between the price of care services and the marginal rate of substitution with second-period generic goods consumption, $v_{h_{t+1}}/v_{d_{t+1}} = p_{t+1}$. Under preferences (6)-(7), we show in the Appendix that these conditions result in the following relationships.

Consumption and savings of young agents are given by

$$c_t = \frac{1}{1+\beta}\left(w_t - \frac{p_{t+1}}{R_{t+1}}\bar{h}\right) \text{ and } s_t = \frac{1}{1+\beta}\left(\beta w_t + \frac{p_{t+1}}{R_{t+1}}\bar{h}\right). \tag{18}$$

Note that when $\bar{h} = 0$, these expressions are equivalent to those in the simplest version of the canonical OLG model, where young agents save a constant fraction of their wage income, which is then used to provide old age consumption.[9] When $\bar{h} > 0$, individual decisions on c_t and s_t are no longer fixed proportions of young age income. Young age consumption is lower, and savings higher, the larger is $\bar{h}$. More interesting, the strength of the effect is related to the *future* relative factor price, since $p_{t+1}/R_{t+1} = \eta w_{t+1}/R_{t+1}$. A high future wage w_{t+1}, and low returns on savings R_{t+1}, imply that much must be saved today in order to purchase the minimum amount of care tomorrow. We term this the *old-age requirement effect*. The old-age

[8] Defining the value of total output as $Y_t \equiv X_t + p_t H_t$, zero profits in both sectors implies $Y_t = w_t N_t^y + R_t K_t$ and therefore expression (17).

[9] As we will return to, however, this does not imply that the dynamics are equivalent to the canonical OLG model. As we will see, these are quite different also in the case where $\bar{h} = 0$, because in our model the *aggregate* savings rate is not constant due to our intergenerational income distribution effect.

requirement effect implies that future relative factor prices affect present savings.[10]

Turning next to generic consumption in the second period of life, each old agent purchases

$$d_t = (1+n)\left[\ell_t - (1-\alpha)\right] a_t^{1-\alpha} \left(\kappa_t/\ell_t\right)^{\alpha}, \tag{19}$$

which is the residual (per-old) output of the generic sector after consumption and savings of young agents have been subtracted. Result (19) implies that second-period consumption is positive only if $\ell_t > 1-\alpha$, which, as we will see, always turns out to be the case in equilibrium.

Finally, the relative demand for old-age care links the old agents' expenditure shares over the two goods to their relative price:

$$\frac{p_t \cdot \left(h_t - \bar{h}\right)}{d_t} = \left(\frac{1-\gamma}{\gamma}\right)^{\sigma} p_t^{1-\sigma}. \tag{20}$$

Expression (20) shows that the expenditure share of old agents on net health care, $h_t - \bar{h}$, increases (decreases) with the price when the two goods are complements (substitutes). The reason is that a ceteris paribus increase in p_t always reduces the 'physical consumption ratio' between net care and generic consumption, $\left(h_t - \bar{h}\right)/d_t$, but in the usual fashion the final effect on the 'expenditure ratio' $p_t\left(h_t - \bar{h}\right)/d_t$ depends on the elasticity of the relative demand for net care. Under complementarity, the demand is relatively rigid: if p_t increases, the price effect dominates the quantity effect and the expenditure share of net care increases. Under substitutability, instead, net old-age care demand is relatively elastic and the quantity effect dominates: an increase in p_t decreases the expenditure share of care. These substitution effects will imply that variations in the price of care have an impact on the labor allocation between the two production sectors.[11]

3.3 Labor Market

The labor demand schedules of both production sectors determine a unique equilibrium in the labor market. Combining (12) with (15), we obtain

$$p_t = (1/\eta)(1-\alpha) a_t^{1-\alpha} \left(\kappa_t/\ell_t\right)^{\alpha} \equiv \Phi\left(\ell_t, \kappa_t; a_t\right). \tag{21}$$

Condition (21) establishes that, in equilibrium, the wage rate must be equalized between the two production sectors. In particular, (21) defines p_t as the level of the price of care ensuring

[10] In particular, the feature that the future wage is relevant for individual consumption and savings decisions is in contrast to one-good versions of the OLG model, where the only future factor price relevant is the return to savings. Moreover, note that in general this feature is the result of old-age care in the model, and does not require $\bar{h} > 0$. For instance, with an intertemporal elasticity of substitution that falls short of one, a higher future wage would imply higher young age savings also in the case where $\bar{h} = 0$.

Also, to preview some intuition, note that since the future wage affects young age savings, it is already clear at this stage that the general equilibrium dynamics will be quite different from one-good OLG models. For instance, higher future wages implies higher savings and thus higher future capital stock, in turn increasing future wages even more.

[11] As usual substitution effects only disappear with Cobb-Douglas preferences: when $\sigma = 1$, the expenditure shares of generic goods and old-age care are independent of the relative price, and are exclusively determined by the relevant preference parameter γ.

equal wages between the two sectors for given levels of sectoral employment, capital per worker, and productivity.

The labor market equilibrium differs between the neoclassical case in our main model, and the extension to the AK case in Subsection 5.3. By substituting for the relevant value of labor productivity $a_t = B^{\frac{1}{1-\alpha}}$ the expression for the labor market equilibrium in the neoclassical case is given by

$$\Phi(\ell_t, \kappa_t) = (B/\eta)(1-\alpha)(\kappa_t/\ell_t)^{\alpha}, \tag{22}$$

while in the AK case this expression has to be replaced by

$$\Phi(\ell_t, \kappa_t) = (A/\eta)(1-\alpha)(\kappa_t/\ell_t). \tag{23}$$

In both cases the function $p_t = \Phi(\ell_t, \kappa_t)$ is strictly decreasing in ℓ_t; for a given capital per young κ_t, higher employment in the generic sector decreases the marginal productivity of labor, implying a lower wage, and thus a lower price of care.

3.4 Goods Markets

In the Appendix we show that solving the demand relationship (20) for the price of care, and substituting $p_t h_t/d_t$ with the market-clearing and zero-profit conditions holding for the producing firms, we obtain

$$p_t = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma}{\sigma-1}} \left[\frac{(1-\alpha)(\ell^{\max} - \ell_t)}{\ell_t - (1-\alpha)}\right]^{\frac{1}{1-\sigma}} \equiv \Psi(\ell_t). \tag{24}$$

This expression defines p_t as the price of care that ensures equilibrium in the goods market.[12] The most important insight of (24) is that the function $p_t = \Psi(\ell_t)$ is strictly decreasing when $\sigma < 1$, and strictly increasing when $\sigma > 1$. When $\sigma < 1$ the price of care is positively related to the employment share in the care sector $1 - \ell_t$. The reason is that a ceteris paribus increase in p_t increases the expenditure share old consumers devote to care services relative to generic consumption and, consequently, attracts labor in the care sector. When $\sigma > 1$, in contrast, a higher price of care means a lower expenditure share of care, and thus less labor in the care sector and more labor in the generic sector.[13]

3.5 Employment and Capital Co-Movements

Consider now the joint equilibrium of the markets for labor and for goods. The two relevant conditions, (22) and (24) in the neoclassical case, imply that the price of health care and the employment shares of the two sectors in each period t depend on the level of capital per worker

[12] Note that the term in square brackets only contains ℓ_t because, with Cobb-Douglas technologies, the sector allocation of labor alone determines the output ratio $X_t/p_t H_t$. If we deviate from Cobb-Douglas technologies, the term in square brackets would also contain capital employed in generic production: see the derivation of (24) in the Appendix.

[13] It should be noted that, in the special case of unit elasticity of substitution, $\sigma = 1$, expression (24) does not hold because price and quantity effects on the demand side balance each other. As a result, the equilibrium between demand and supply in the goods market is characterized by constant employment shares, with $\ell_t = \frac{(1-\alpha)(\gamma\ell^{\max}+1-\gamma)}{\gamma(1-\alpha)+1-\gamma}$ at each t.

κ_t. Formally, the employment share of the generic sector for a given level of κ_t, denoted by $\ell_t = \ell(\kappa_t)$, is the fixed point

$$\ell(\kappa_t) \equiv \arg\text{solve}_{\{\ell_t \in (1-\alpha, \ell^{\max})\}} \left[\Phi(\ell_t, \kappa_t) = \Psi(\ell_t)\right]. \tag{25}$$

Our assumptions guarantee the existence and uniqueness of this fixed point – a result that is shown in the Appendix and that can be verified in graphical terms in Figure 2. On the one hand, the function $\Phi(\ell_t, \kappa_t)$ is strictly decreasing in ℓ_t and exhibits positive vertical intercepts at the boundaries of the relevant interval $\ell_t \in (1-\alpha, \ell^{\max})$. On the other hand, the function $\Psi(\ell_t)$ is decreasing (increasing) under complementarity (substitutability) with limits

$$\lim_{\ell_t \to 1-\alpha} \Psi(\ell_t) = \quad \left\{ \ \infty \quad \text{if } \sigma < 1; \quad 0 \quad \text{if } \sigma > 1 \ \right\},$$

$$\lim_{\ell_t \to \ell^{\max}} \Psi(\ell_t) = \quad \left\{ \ 0 \quad \text{if } \sigma < 1; \quad \infty \quad \text{if } \sigma > 1 \ \right\},$$

These properties[14] ensure the existence and uniqueness of the fixed point $\Psi(\ell_t) = \Phi(\ell_t, \kappa_t)$, and that it is contained in the relevant interval $\ell \in (1-\alpha, \ell^{\max})$. The fixed point (25) simultaneously determines employment shares and the price of care, which is measured along the vertical axis of Figure 2. Substituting $\ell(\kappa_t)$ in $\Psi(\ell_t)$ or in $\Phi(\ell_t, \kappa_t)$ we obtain the equilibrium price of care for given capital per worker,

$$p(\kappa_t) \equiv \Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t). \tag{26}$$

Even though we have not yet specified whether and how capital grows, result (26) clarifies how capital accumulation affects the price of care and employment shares:

Proposition 1 *An equilibrium trajectory with positive accumulation implies a rising price of care. Under complementarity the employment share in the generic sector is decreasing. Under substitutability the employment share in the generic sector is increasing;*

$$\kappa_{t+1} > \kappa_t \qquad \Longleftrightarrow \qquad p_{t+1} > p_t$$

and

$$\kappa_{t+1} > \kappa_t \Rightarrow \left\{ \begin{array}{l} \ell_{t+1} < \ell_t \ \textit{if } \sigma < 1 \\ \ell_{t+1} > \ell_t \ \textit{if } \sigma > 1 \end{array} \right\}$$

Proof. The proposition is proved in graphical terms by means of a comparative-statics exercise.[15] Because $\Phi(\ell, \kappa)$ is positively related to κ, a higher stock of capital per young implies an up-rightward shift in the $\Phi(\ell, \kappa)$ curves in Figure 2. The new equilibrium price $p(\kappa)$ is higher in all cases, but sectoral employment shares react differently depending on the value of σ. The employment share of the generic sector $\ell(\kappa)$ increases under complementarity, decreases under substitutability:

$$\ell'_\kappa \equiv \frac{\mathrm{d}\ell(\kappa_t)}{\mathrm{d}\kappa_t} \ \begin{array}{l} < 0 \quad \text{if } \sigma < 1; \\ > 0 \quad \text{if } \sigma > 1 \ . \end{array}$$

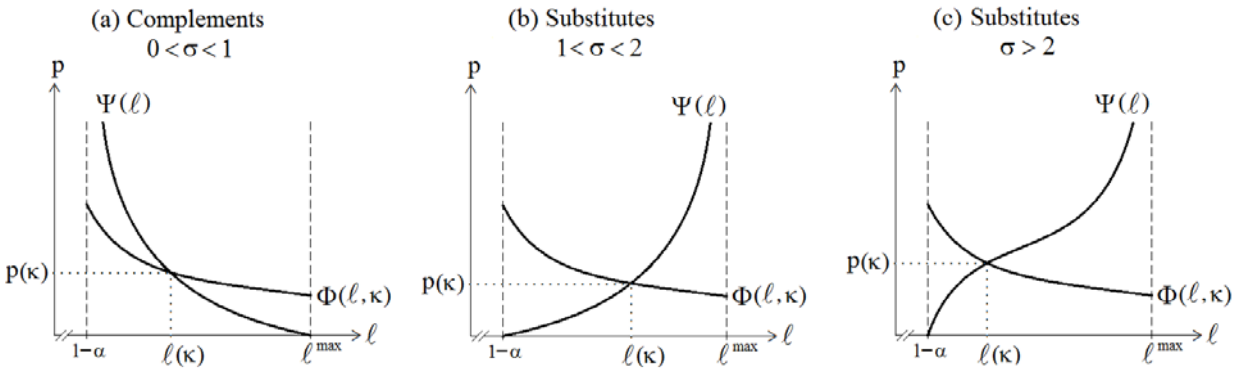

Figure 2: Static equilibrium: determination of ℓ_t and p_t for given κ_t. Qualitatively, the graphs do not change between the neoclassical and the AK model. The case of strong substitution ($\sigma > 2$) implies local concavity of $\Psi(\ell)$ for low ℓ without altering existence, uniqueness, and comparative-statics properties.

■

The intuition is that an increase in capital per young expands the production frontier of the generic good, and thereby increases the price of care. Under complementarity, old agents react to the price increase by raising the share of expenditure on net old-age care, which decreases the employment share in the generic sector $\ell(\kappa)$. Under substitutability, instead, old agents reduce the expenditure share on net care, and employment in the generic sector therefore grows. It is easily verified that the direction of these capital and employment co-movements is fully reversed when we consider an equilibrium trajectory with decumulation of capital per young – that is, when $\kappa_t < \kappa_{t-1}$.[16]

3.6 Static Equilibrium Comparative Statics

For a given capital stock, the static equilibrium labor allocation depends on the parameters in the model. In particular, for later use we investigate how it depends on productivity B, on population growth n, and on the level of the minimum requirement $\bar{h}$. The properties of $\ell(\kappa_t) = \ell\left(\kappa_t; B, n, \bar{h}\right)$ are summarized in the following Proposition:[17]

Proposition 2 *In the static equilibrium with given κ_t,*

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{dB} \equiv \ell'_B \ \begin{cases} < 0 & \text{if } \sigma < 1; \\ > 0 & \text{if } \sigma > 1 \end{cases}, \tag{27}$$

[14] Along with the further concavity properties of both curves described in the Appendix.

[15] Proposition 1 is equivalently proved by differentiating the equilibrium condition $\Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t)$. The exact relationship between κ and ℓ is reported in expression (36) below, and indeed implies that $\ell'_\kappa \equiv \mathrm{d}\ell(\kappa_t)/\mathrm{d}\kappa_t$ is strictly negative (positive) under complementarity (substitutability).

[16] Note that all the properties in this subsection, and therefore the identical results established in Proposition 1 as well as the proof, also hold in the AK case in our model: the co-movements of employment shares, price of health care and capital per worker are the same in both variants of the model.

[17] Again, this proposition is also valid if the productivity term B from the neoclassical version of the model is replaced by the productivity term A in the AK version of the model.

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{d\bar{h}} \equiv \ell'_{\bar{h}} < 0 \tag{28}$$

and

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{dn} \equiv \ell'_n > 0 \text{ if } \bar{h} > 0 \quad (= 0 \text{ if } \bar{h} = 0). \tag{29}$$

Proof. Also this proposition can be proved in graphical terms. An increase in B implies an upward shift in $\Phi(\ell, \kappa)$ in Figure 2. The employment share, ℓ, increases when $\sigma < 1$ while it decreases when $\sigma > 1$. Changes in population growth, n, and minimum care requirement, $\bar{h}$, operate through $\ell^{\max}$ that appears in the expression for $\Psi(\ell)$ in equation (24). An increase in $\ell^{\max}$ shifts $\Psi(\ell)$ to the right, increasing ℓ. As $\ell^{\max} \equiv 1 - \frac{\bar{h}}{\eta(1+n)}$, $\ell^{\max}$ increases with a lower $\bar{h}$ or with a higher n (provided that $\bar{h} > 0$). ■

A higher productivity B expands production possibilities of generic goods. When $\sigma < 1$, labor is pushed out of the generic sector, as consumers want to utilize the increased production possibilities to consume more services from the care sector. When $\sigma > 1$ in contrast, labor is drawn into the generic sector, since in this case old agents prefer less care but more generic goods.

The intuition for the effects working via $\ell^{\max}$ are intuitive. When a larger fraction of workers are needed in order to satisfy the minimum service requirement, the care sector will employ more workers.

3.7 Saving Rates and Accumulation

Before studying in detail the dynamics, it is instructive to describe the general relationships between saving rates, capital accumulation and sectoral employment shares. Considering the economy's aggregate income (17) and the wage rate (15), the total labor share accruing to young agents is given by

$$\frac{w_t N_t^y}{Y_t} = \frac{(1-\alpha)\frac{x_t}{\ell_t}}{x_t\left(\frac{1-\alpha}{\ell_t} + \alpha\right)} = \frac{1-\alpha}{1-\alpha(1-\ell_t)}, \tag{30}$$

Equation (30) shows that, in static equilibrium, an increase in the generic sector employment share ℓ_t reduces the total income share of young agents. The intuition is that if labor moves from the care sector to generic production, the return to capital increases relative to the wage rate. There is, thus, a shift in the income distribution away from the young towards the old. We term this effect the *intergenerational distribution effect*.

Since it is the young who save, the intergenerational distribution effect directly influences the economy's saving rate (and will, as we shall see, have important implications for capital accumulation). The savings rate, termed θ_t and defined as aggregate savings relative to the total value of production, is found by using the saving function in (18) and expression (30), and then inserting for $\ell^{\max}$ from (13):

$$\theta_t \equiv \frac{N_t^y s_t}{Y_t} = \underbrace{\frac{\beta(1-\alpha)}{1+\beta}}_{\text{Canonical model}} \cdot \underbrace{\frac{1}{1-\alpha\cdot(1-\ell_t)}}_{\text{Intergenerational Distribution}} \cdot \underbrace{\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right)}_{\text{Old-age Requirement}}, \tag{31}$$

where

$$\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right) \equiv \left[1 - \frac{(1-\alpha)}{\alpha\,(1+\beta)\,\eta(1+n)}\frac{\bar{h}}{\ell_{t+1}}\right]^{-1}, \quad \Gamma' > 0, \quad \Gamma\,(0) = 1. \tag{32}$$

Expression (31) is a semi-reduced form showing that the savings rate is negatively related to both ℓ_t and ℓ_{t+1}. The function Γ captures the savings induced if there is a the minimum health requirement. When $\bar{h} = 0$, Γ reduces to unity. The derivative is positive, and thus when $\bar{h} > 0$, then $\Gamma > 1$.

To explain the intuition is it instructive to compare the result in (31) to the savings rate in the canonical OLG model with logarithmic preferences and Cobb-Douglas technology. There, the young save a fraction $\beta/(1+\beta)$ of their income, and the income share of the young is $1-\alpha$. The savings rate is therefore, in this case, given by the first of the three terms on the right hand side of (31), and it is time independent.

The present model implies that the savings rate is, in general, not constant over time. Moreover, it is always higher than in the canonical model for two reasons; the intergenerational distribution effect and the old-age requirement effect. First, as seen by the second term on the right hand side of (31), the presence of employment in the care sector implies higher labor demand, shifting the income distribution in favor of the young, and thus increasing savings. Second, as seen by the third term on the right hand side of (31), with $\bar{h} > 0$, as we have seen from (18), the young have an additional savings motive in that they need some minimum amount of old-age care, increasing the savings rate further.[18] The old-age requirement effect on savings is stronger the lower is ℓ_{t+1}, because lower future employment in the generic sector implies higher future wages, increasing the cost of purchasing the minimum requirement of care. The expected increase in the cost of health care in period $t+1$ prompts young agents to save more in period t and, therefore, to accumulate more capital.

The natural question concerns the general-equilibrium impact of both these mechanisms on economic growth. In this respect, the market-clearing condition equating investment to savings implies that capital per worker obeys the dynamic law

$$\kappa_{t+1} = \frac{\theta_t Y_t}{1+n}, \tag{33}$$

Next periods capital per young is determined by this periods savings adjusted for population growth.

The next section discusses capital accumulation in the neoclassical variant of the model, while Subsection 5.3 extends the dynamics to the AK case.

4 Neoclassical Growth

In the neoclassical case labor productivity in the generic sector equals $a_t = B^{\frac{1}{1-\alpha}}$ in each period. In this framework, when the economy reaches a long-run equilibrium where capital per worker

[18] In the Appendix we show that restriction (14) and $\ell_{t+1} > 1-\alpha$ implies that $(1-\alpha)\,\bar{h} < \alpha\,(1+\beta)\,\eta(1+n)\ell_{t+1}$.

is constant, generic production grows at the exogenous rate of population growth. Subsections 4.1-4.3 derive the stability properties of the long-run steady state and show that, given an initial stock below the steady-state level, capital per worker grows monotonously. We also show that under complementarity, these transitional dynamics are characterized by increasing savings rates. Under substitutability, on the other hand, savings rates decrease during the transition to steady-state. The intergenerational distribution effect and the old-age requirement effect both contribute to these results.

Compared to the canonical OLG model the dynamics are more involved: since increased capital increases savings rates and thereby capital further, this opens for the possibility of (local) instability and multiple steady states. We show, however, that a departure from uniqueness and stability of the steady state can only occur under unreasonable high values of the elasticity of capital in generic production α.[19]

The case of complementarity is of particular interest when discussing growth in China, as it is already clear that capital accumulation in such a case involves increasing savings rates and increasing (share of) employment in the care sector. Subsection 4.4 clarifies further how, in this case, the intergenerational distribution effect and the old-age requirement effect give rise to a *savings multiplier*, where savings and capital accumulation stimulates further savings and capital accumulation.[20] Subsection 4.5 performs comparative-statics exercises suggesting that one-child policies may boost capital accumulation via two channels – the negative impact on population growth and the increased need to purchase care services in the market rather than relying on own children to provide them. The final positive effects on long-run capital per worker are magnified by the savings multiplier.

4.1 Accumulation Law

The equilibrium path of capital is determined by the saving decisions of young agents. Inserting from (31) and (17) in (33), we obtain a semi-reduced form of the accumulation law of capital per worker, which links κ_{t+1} to the previous stock κ_t and to the sectoral employment levels in the two periods:

$$\kappa_{t+1} = \underbrace{\frac{B\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}\kappa_t^{\alpha}}_{\text{Canonical model}} \cdot \underbrace{\ell_t^{-\alpha}}_{\text{Intergen. Distr.}} \cdot \underbrace{\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right)}_{\text{Old-age requirement effect}}. \tag{34}$$

This expression decomposes the accumulation law of capital in three parts. The first term on the right hand side of (34) is the dynamic law in the canonical one-good model: if we eliminate the care sector by setting $\ell_t = 1$ and $\bar{h} = 0$, capital per worker evolves according to this stable monotonic relationship, and the saving-output ratio is constant by virtue of constant income share of the young and logarithmic intertemporal preferences.

[19] Nevertheless, for completeness we also solve the dynamics for this case in the Appendix.

[20] Naturally, the convergence of this multiplier process is guaranteed exactly when the steady state is unique and stable.

The second and third terms on the right hand side of (34) again directly follow from the intergenerational distribution effect and the old-age requirement effect. An increase in ℓ_t reduces κ_{t+1} because a lower current wage reduces young agents' income, and thereby, current savings. An increase in ℓ_{t+1} reduces κ_{t+1} because a lower future wage reduces the expected future cost of health care, and thereby, current savings.

Recalling result (25), equilibrium employment shares are a function of the capital stock per worker in each period. Substituting $\ell_t = \ell(\kappa_t)$ and $\ell_{t+1} = \ell(\kappa_{t+1})$ into (34), we obtain the accumulation law

$$\kappa_{t+1} = \frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^{\alpha}\left[\ell(\kappa_t)\right]^{-\alpha}\Gamma\left(\frac{\bar{h}}{\ell(\kappa_{t+1})}\right). \tag{35}$$

Expression (37) implies that capital dynamics crucially depend on how sectoral employment shares react to variations in capital per worker. In this respect, the relevant elasticity is[21]

$$\frac{\ell'_{\kappa}(\kappa_t)\,\kappa_t}{\ell(\kappa_t)} = \frac{1}{1-\frac{1}{1-\sigma}\frac{1}{\alpha}Q_1}\left\{\begin{array}{l} <0 \text{ if } \sigma<1 \\ >0 \text{ if } \sigma>1 \end{array}\right\}, \tag{36}$$

where $Q_1 \equiv \frac{\ell_t}{\ell_t-(1-\alpha)}\cdot\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell_t} > 1$. The slope of the accumulation law can be found by taking the elasticity of (35) with respect to κ_t and κ_{t+1}, which yields[22]

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t}\frac{\kappa_t}{\kappa_{t+1}} = \frac{\alpha-\alpha\frac{\ell'_{\kappa}(\kappa_t)\kappa_t}{\ell(\kappa_t)}}{1+\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa_{t+1})}\frac{\ell'_{\kappa}(\kappa_{t+1})\kappa_{t+1}}{\ell(\kappa_{t+1})}}. \tag{37}$$

Starting with the numerator, we see that the direct effect on κ_{t+1} of an increase in κ_t is larger under complementarity, i.e. when $\ell'_{\kappa}(\kappa_t) < 0$. When $\bar{h} > 0$, there is also an indirect effect via the increase in $\ell(\kappa_{t+1})$, captured in the denominator.

To present the intuition in the most transparent way we first, in the next subsection, investigate the special case where $\bar{h} = 0$, and thus $\Gamma = 1$. This isolates the intergenerational distribution effect, and shows how this increases the steady state capital stock. In Subsection 4.3 we then expand the model to the case where $\bar{h} > 0$. This shows how the old-age requirement effect further increases the steady state capital stock.

4.2 Dynamics without Minimum Requirement

When there is no minimum health-care requirement for old agents, capital accumulation obeys a fairly simple dynamic law. In the main text, we assume that the elasticity of capital in generic production is not too high, that is:[23]

Assumption 1: $\alpha < \frac{3}{4}$.

This assumption is sufficient (but not necessary) for the steady state to be unique.[24] The next

[21] Expression (36) is obtained by differentiating the equilibrium condition $\Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t)$ and is fully derived in the Appendix. The fact that $Q_1 > 1$ directly follows from the requirement $1-\alpha < \ell_t < \ell^{\max}$ and it implies the signs reported in (36). Note that (36) yields an alternative proof of Proposition 1.

[22] Totally differentiating (35) yields $\frac{\mathrm{d}\kappa_{t+1}}{\kappa_{t+1}} = \alpha\frac{\mathrm{d}\kappa_t}{\kappa_t} - \alpha\frac{\partial\ell(\kappa_t)}{\partial\kappa_t}\frac{1}{\ell(\kappa_t)}\mathrm{d}\kappa_t - \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell_{t+1}}\frac{\partial\ell(\kappa_{t+1})}{\partial\kappa_{t+1}}\frac{1}{\ell(\kappa_{t+1})}\mathrm{d}\kappa_{t+1}$, which can be rearranged to obtain (37).

[23] In the Appendix part B, we solve the general model for the case in which Assumption 1 is not satisfied. For more on stability properties in OLG models with one capital stock, see e.g. Galor and Ryder (1989).

[24] Under substitutability the steady state is always unique and stable.

Proposition then establishes that the steady state is globally stable: under both complementarity and substitutability, the economy converges towards a long-run equilibrium in which capital per worker, the price of health care and employment shares are constant.

Proposition 3 *In the neoclassical case with $\bar{h} = 0$, capital per worker obeys*

$$\kappa_{t+1} = \frac{\beta\eta}{(1+n)(1+\beta)} p(\kappa_t), \tag{38}$$

where $p(\kappa_t)$ is the price of health care determined by (26). Under Assumption 1 the steady state $\kappa_{ss} = \frac{\beta\eta}{(1+n)(1+\beta)} p(\kappa_{ss})$ is unique and globally stable, implying

$$\lim_{t\to\infty} \kappa_t = \kappa_{ss}, \quad \lim_{t\to\infty} \ell_t = \ell(\kappa_{ss}), \quad \lim_{t\to\infty} p_t = p(\kappa_{ss}).$$

During the transition, given a positive initial stock $\kappa_0 < \kappa_{ss}$, both capital per worker and the price of health care increase, whereas employment in the generic sector declines (increases) and the saving rate increases (declines) under complementarity (substitutability):

$$\kappa_{t+1} > \kappa_t, \quad p_{t+1} > p_t, \qquad \left\{ \begin{array}{l} \ell_{t+1} < \ell_t \text{ and } \theta_{t+1} > \theta_t \text{ if } \sigma < 1 \\ \ell_{t+1} > \ell_t \text{ and } \theta_{t+1} < \theta_t \text{ if } \sigma > 1 \end{array} \right\}. \tag{39}$$

Proof. Expression (38) follows from setting $\bar{h} = 0$ in (35) and substituting (22) and (26). Result (39) follows from Proposition 1 combined with (31) that shows that, with $\bar{h} = 0$, θ_t is decreasing in ℓ_t. For κ_{ss} to be stable and unique, the elasticity (37) evaluated in κ_{ss} must be less than unity. Inserting $\kappa_t = \kappa_{t+1} = \kappa_{ss}$ and $\Gamma = 1$ and $\Gamma' = 0$ in (37), the elasticity reduces to

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \alpha - \alpha \frac{\ell'_\kappa(\kappa_{ss})\,\kappa_{ss}}{\ell(\kappa_{ss})},$$

where the right hand side is less than unity if and only if

$$m_1(\kappa_{ss}) \equiv -\frac{\ell'_\kappa(\kappa_{ss})\,\kappa_{ss}}{\ell(\kappa_{ss})} \frac{\alpha}{1-\alpha} < 1. \tag{40}$$

In the Appendix we show that Assumption 1 is a sufficient condition for (40) to be satisfied. In the Appendix we also prove existence. ■

Proposition 3 suggests three remarks. First, the dynamic law for capital (38) shows that, when there is no minimum requirement, investment per-young is proportional to the price of care. This is because savings only depend on current wages (w_t is proportional to p_t in each period). Second, given that capital per worker grows monotonically, both the wage and the price of care increase over time. Employment shares, however, move in opposite directions depending on the value of σ, which determines whether old agents increase or decrease their expenditure share on old-age care in response to increasing prices. The third remark is that, under complementarity, the savings rate θ_t increases during the transition because rising care prices attract labor in the care sector and the income share of young agents then grows – i.e., the intergenerational distribution effect.

The steady-state implications of the intergenerational distribution effect is immediate by comparing the steady-state level of the capital stock, κ_{ss}, with that in the canonical version of the model, which we term $\kappa_{ss}^{\text{canonical}}$. Starting from (34), and imposing $\bar{h}=0$ and $\kappa_{t+1}=\kappa_t=\kappa_{ss}$, we obtain

$$\kappa_{ss}=\frac{1}{\ell\left(\kappa_{ss}\right)^{\frac{\alpha}{1-\alpha}}}\left[\frac{B\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}\right]^{\frac{1}{1-\alpha}}=\frac{1}{\ell\left(\kappa_{ss}\right)^{\frac{\alpha}{1-\alpha}}}\kappa_{ss}^{\text{canonical}}, \tag{41}$$

where the steady-state level of capital per worker in the canonical model (which is obtained by setting $\ell_t=1$ in each period) is

$$\kappa_{ss}^{\text{canonical}}=\left[\frac{B\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}\right]^{\frac{1}{1-\alpha}}. \tag{42}$$

It immediately follows that $\kappa_{ss}>\kappa_{ss}^{\text{canonical}}$ always holds (as $\ell\left(\kappa_{ss}\right)<1$), i.e., capital per worker in our model is higher than in the canonical model independently of whether generic goods and care are complements or substitutes. The need for care increases the demand for labor, pushing income distribution in favor of the young, and therefore increases savings. (The size of the gap between κ_{ss} and $\kappa_{ss}^{\text{canonical}}$ depends, obviously, on the elasticity of substitution as well as the other parameters of the model through the term $\ell\left(\kappa_{ss}\right)$, which we return to below).

4.3 Dynamics with Minimum Care Requirement

When the minimum old-age care requirement is strictly positive, $\bar{h}>0$, the accumulation law (34) includes the dependency of current savings on future employment shares, i.e. the old-age requirement effect. This dynamic law determines the steady state(s) of the system and the associated stability properties. Under substitutability there is always a unique steady state. Under complementarity, i.e. $\sigma<1$, we again in the main text assume that the elasticity of capital in generic production is not too high, now that is:[25]

Assumption 2: $\alpha<\frac{1-\alpha}{1-\sigma}$.

This assumption is sufficient (but not necessary) for the steady state to be unique. We then have:

Proposition 4 *Under Assumption 2 equation (35) exhibits a unique steady state $\bar{\kappa}_{ss}$ that is globally stable. The transitional dynamics of $p\left(\kappa_t\right)$ and $\ell\left(\kappa_t\right)$ comply with Proposition 1.*

Proof. For $\bar{\kappa}_{ss}$ to be stable and unique, the elasticity (37) evaluated in $\bar{\kappa}_{ss}$ must be less than unity. Inserting $\kappa_t=\kappa_{t+1}=\bar{\kappa}_{ss}$ in (37), the elasticity reduces to

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t}=\frac{\alpha-\alpha\frac{\ell_\kappa'(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}}{1+\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\ell_\kappa'(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}},$$

where the right hand side is less than unity if and only if

$$m_1\left(\bar{\kappa}_{ss}\right)+m_2\left(\bar{\kappa}_{ss}\right)<1, \tag{43}$$

[25] Note that even in the limiting case where $\sigma\to 0$ this assumption is satisfied with the empirically plausible restriction $\alpha<\frac{1}{2}$. In the Appendix part B, we solve the model for the case in which Assumption 2 is not satisfied.

with

$$m_2\left(\bar{\kappa}_{ss}\right) \equiv -\frac{\ell'_\kappa\left(\bar{\kappa}_{ss}\right)\bar{\kappa}_{ss}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{1}{1-\alpha}\begin{cases} <1 \text{ if } \sigma<1 \\ <0 \text{ if } \sigma>1 \end{cases}. \tag{44}$$

In the the Appendix we show that Assumption 2 is a sufficient condition for (43) to be satisfied. ■

Proposition 4 establishes that, also with a minimum old-age care requirement, under complementarity the savings rate increases during a transition path where capital grows. Now, this is the combined result of the future-requirement and intergenerational distribution effects.

Imposing $\kappa_{t+1} = \kappa_t = \bar{\kappa}_{ss}$ in (34), the steady-state level of capital per worker must satisfy

$$\bar{\kappa}_{ss} = \Gamma\left(\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\right)^{\frac{1}{1-\alpha}}\frac{1}{\ell\left(\bar{\kappa}_{ss}\right)^{\frac{\alpha}{1-\alpha}}}\kappa_{ss}^{\text{canonical}}. \tag{45}$$

Comparing (45) to (41), since Γstrictly exceeds one when $\bar{h} > 0$, we thus conclude that $\bar{\kappa}_{ss} > \kappa_{ss} > \kappa_{ss}^{\text{canonical}}$; the long-run level of capital per worker is higher when there is a positive minimum requirement of old-age care, which drives capital further above the level attained in the canonical model. The reason is the minimum-requirement effect, which prompts households to save more during the transition in response to the continuous increase of the price of old-age care.

4.4 The Savings Multiplier

Steady state capital is affected by the intergenerational distribution and the minimum requirement effects. We now investigate how the same effects come into play when exogenous shocks affect the economy. This sheds further light on transitional dynamics in the model, and also allows us to introduce *the savings multiplier*. For this purpose, in this subsection we consider variations in the productivity level B, and we focus on the case of complementarity, $\sigma < 1$, which seems the most interesting scenario for discussing Chinese household's saving behavior in the model.[26] The effects of exogenous shocks on income per capita may, as we will see in this subsection and the next, differ substantially from those predicted by the canonical model. For expositional clarity, we start out without the minimum requirement effect, before we extend the analysis to include this.

Zero Requirement. In the canonical model, an exogenous increase in productivity increases the long-run level of (log) capital per worker in (42) by

$$\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} = \frac{1}{B\left(1-\alpha\right)}. \tag{46}$$

[26] However, all of the equations to follow are identical also in the case of $\sigma > 1$, the only difference being in the qualitative strength of the effects. As will be easily understood below, all savings multipliers which exceed one when $\sigma < 1$, falls short of one when $\sigma > 1$. Thus shocks that are magnified with complementarity, are instead dampened with substitutability.

Now consider our model without minimum requirement. With $\bar{h} = 0$, the steady-state capital per worker is κ_{ss} defined in (41), and the impact of the productivity shock is determined by

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B} = \underbrace{\frac{1}{1-m_1(\kappa_{ss})}}_{\text{Savings Multiplier}}\left(\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} + m_1(\kappa_{ss})\frac{\ell'_B(\kappa_{ss})}{\ell'_\kappa(\kappa_{ss})\,\kappa_{ss}}\right), \tag{47}$$

The crucial element in (47) is *the savings multiplier*, where m_1 is already defined in (40). Focusing on the case of complementarity, m_1 is strictly positive, and is less than unity in view of the stability of the steady state.[27] Since $0 < m_1 < 1$, the savings multiplier in (47) is strictly higher than unity. Combining this result with $\ell'_\kappa < 0$ and $\ell'_B < 0$,[28] we conclude that the impact of a productivity shock on steady-state capital per worker is stronger than that predicted by the canonical model. There are two reasons for this, both related to the intergenerational distribution effect. The first reason, which is the result of the intergenerational distribution effect in the static part of the model, is represented by the term $m_1\frac{\ell'_B}{\ell'_\kappa\kappa} > 0$. The productivity increase pushes labor into care and out of generic production, increasing the wage further as compared to the canonical model, shifting income distribution in favor of the young. This means that the initial increase in the savings rate as a result of better productivity is higher than in the canonical model. The second reason, which is the result of the intergenerational distribution effect in the dynamic part of the model, is represented by the savings multiplier; the term $\frac{1}{1-m_1} > 1$. In our model, as the capital stock starts to grow, this further pushes labor out of generic production and into care, increasing the wage even further, thus magnifying the initial increase in savings. The implication is that a higher productivity increases the capital stock and wages by more than in the canonical model. As we will see below, the savings multiplier is also part of the explanation why low population growth and one-child policies may have such a massive impact on savings and capital accumulation.

Positive Requirement. To see how the old-age requirement $\bar{h} > 0$ modifies the savings multiplier, we again investigate the response of the steady-state capital stock to increased productivity, which from (45) is now given by

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B} = \underbrace{\frac{1}{1-[m_1(\bar{\kappa}_{ss})+m_2(\bar{\kappa}_{ss})]}}_{\text{Savings multiplier}}\left[\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} + \frac{(m_1(\bar{\kappa}_{ss})+m_2(\bar{\kappa}_{ss}))\,\ell'_B(\bar{\kappa}_{ss})}{\ell'_\kappa(\bar{\kappa}_{ss})\,\bar{\kappa}_{ss}}\right], \tag{48}$$

where m_2 is defined in (44). Focusing again on the case of complementarity, the term $m_1 + m_2$ is strictly positive, and is less than unity in view of the stability of the steady state.[29] Since $0 < m_1 + m_2 < 1$, the savings multiplier in (48) is strictly higher than unity.

[27] Under complementarity, m_1 is positive because $\ell'_\kappa < 0$ – see expression (36) – and is strictly less than unity in view of the stability condition proven in Proposition 3. Under substitutability, instead, expression (36) implies $\ell'_\kappa > 0$ and therefore $m_1 < 0$.

[28] Under complementarity, $\ell'_\kappa < 0$ follows from (36) whereas $\ell'_B < 0$ is established in Proposition 2.

[29] Under complementarity, both m_1 and m_2 are positive because $\ell'_\kappa < 0$ – see expression (36) – and $m_1 + m_2$ is strictly less than unity in view of the stability condition (43) proven in Proposition 4. Under substitutability, instead, expression (36) implies $\ell'_\kappa > 0$ and therefore $m_1 + m_2 < 0$, which yields a savings multiplier below unity.

Compared to the case with $\bar{h} = 0$ in (47), the effect of increased productivity on steady-state capital now involves two additional effects strengthening the impact of productivity on steady state capital. These are identified by the two appearances of the term m_2 in (48). First, in the static equilibrium of the model, the higher wage now also means higher cost of old-age minimum requirement of care, implying an additional increase in savings compared to in the case above. Second, the savings multiplier increases, strengthening the feedback of capital on capital growth: the increase in the capital stock makes the wage rise over time, increasing the cost of the future minimum requirement, in turn increasing savings and the capital stock even more as compared to the case with $\bar{h} = 0$. Thus, both the static and dynamic effects generated by the old-age requirement effect reinforce the steady-state response of capital to increased productivity.

Clearly an important channel of growth in China has been (transfer of labor from low productivity to) high productivity in the manufacturing sector. The analysis, so far, indicates that the effect of higher productivity may have been magnified by the savings multiplier. Many would argue that in addition to this, a defining characteristic of Chinese policy compared to other countries has been the one-child policy. Thus, we now investigate the effects on savings, capital accumulation and growth of such policies.

4.5 Population Growth and One-Child Policies

The one-child policy has main effects on the Chinese economy not only by lowering population growth, but also by transforming society to one where agents need to purchase more care in the market. In particular, parents with one girl are strongly affected, since when married, girls traditionally becomes part of the family of the husband. But parents of boys are also affected, because the one-child policy has implied rising sex ratios, increasing the probability that their boy may end up as unmarried. (In addition, of course, general modernization may also weaken the tradition of children to support their parents with old-age care.) In this subsection, thus, we study how the one-child policy may affect the economy through lower population growth, and increased need to rely on the market to provide old-age care.

As is well known from the canonical model, a lower growth rate of population increases the steady-state level of capital per worker: from (42), we find

$$\frac{\mathrm{d}\log \kappa_{ss}^{\text{canonical}}}{-\mathrm{d}n} = \frac{1}{(1+n)(1-\alpha)} > 0. \tag{49}$$

In contrast, from (45), and taking into account (29), we find that the effect in our extended model with minimum requirement, $\bar{h} > 0$, is given by

$$\frac{\mathrm{d}\log \bar{\kappa}_{ss}}{-\mathrm{d}n} = \frac{1}{1-[m_1+m_2]}\left[\frac{\mathrm{d}\log \kappa_{ss}^{\text{canonical}}}{-\mathrm{d}n} + \frac{\ell'_n}{(-\ell'_\kappa)\bar{\kappa}_{ss}}(m_1+m_2) + \frac{\ell}{(1+n)(-\ell'_\kappa)\bar{\kappa}_{ss}}m_2\right], \tag{50}$$

where we suppress the argument $\bar{\kappa}_{ss}$ to simplify the notation.[30] Assuming again complemen-

[30] In (50), the terms $m_1, m_2, \ell, \ell'_n, \ell'_\kappa$ are all evaluated in the steady state $\bar{\kappa}_{ss}$. Also, in deriving (50), we exploit the fact that $\frac{d\Gamma}{dn} = -\Gamma' \frac{\bar{h}}{(1+n)\ell}$ from expression (32). See the Appendix for a full derivation.

tarity, $\sigma < 1$, the multiplier $\frac{1}{1-m_1-m_2}$ is positive and higher than unity. There are, thus, five reasons the relative increase in capital with a lower growth of population is higher as compared to in the canonical model. The first two effects arise because of the intergenerational distribution effect and the old-age requirement effect, which through the savings multiplier ensures that capital growth has a stronger positive feedback on itself. These two effects are represented by m_1 and m_2 in the savings multiplier. The third and fourth effects are represented by the presence of m_1 and m_2 in the term $\frac{\ell'_n}{(-\ell'_\kappa)\bar{\kappa}_{ss}}(m_1+m_2)$ in (50). They represent the effects on the labor share in the static model. With lower population growth there are fewer young relative to old agents at each point in time. This pull workers out of generic production and into care, increasing the wage. The increased wage increases the aggregate savings rate through both the intergenerational distribution effect and the old-age requirement effect. The fifth effect is represented by the term $\frac{\ell}{(1+n)(-\ell'_\kappa)\bar{\kappa}_{ss}}m_2$ in (50). At each point in time, there is a higher fraction of old-age to young-age agents. Even for a fixed labor allocation this increases the wage. Through the old-age requirement effect, this increases the savings rate of the young in the static model even more, stimulating capital accumulation further. In total, this means that the effect of population growth in the present model may be substantially magnified compared to standard OLG models.

As we discussed above, another potential effect of one-child policies may be that, by lowering the number of young relative to the old, less care will be provided inside the family, and more care has to be purchased in the market. In the model, this can be represented by an increased amount of care that each agent must purchase as old, i.e. a higher $\bar{h}$. Obviously, this draws resources out of generic sector production and into the production of care. The effect on steady-state capital is found by (45) to be

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} = \frac{\frac{\ell'_{\bar{h}}}{\ell'_\kappa\bar{\kappa}_{ss}}(m_1+m_2) - \frac{\ell}{\bar{h}\ell'_\kappa\bar{\kappa}_{ss}}m_2}{1-(m_1+m_2)}. \tag{51}$$

A higher minimum requirement of care has a direct positive effect on savings, represented by the term $-\frac{\ell}{h\ell'_\kappa\bar{\kappa}_{ss}}m_2$. In addition, the demand for labor increases, pushing the wage up. This shifts income distribution in favor of the young, and also makes care more expensive. For both reasons, savings increase, represented by the term $\frac{\ell'_{\bar{h}}}{\ell'_\kappa\bar{\kappa}_{ss}}(m_1+m_2)$. Thus, through three channels, a higher minimum requirement increases savings in the static model. Stimulated by the savings multiplier, steady-state capital increases by more than the immediate effect on savings and capital accumulation. As a consequence, the increased need for market based care may have a strong positive impact on capital accumulation.

5 Extensions

In this section we first extend the model to study social security in the form of a pay as you go system where the government provides old-age care. We then consider the possibility of dynamic inefficiency. We lastly extend the model to study endogenous growth dynamics. To

put the main focus on the new effects our extensions introduce, throughout this section we for simplicity focus on the case where $\bar{h} = 0$.

5.1 Introducing the Welfare State

The savings motive in the economy comes partly from the need to pay for future old-age care, since the one-child policy and the missing welfare system implies that there are few other options. As argued by e.g. Oksanen (2010, p. 14), even in areas where the one-child policy has been less strictly enforced it has main implications: "Although this policy has been more relaxed in rural areas, the migration of descendants to cities has meant that many elderly people in rural areas are left without sufficient family support. ... .as the state induced the decline in fertility by regulation, the state must take responsibility, both financially and otherwise, for support for the elderly who are left with only narrow or no family based security". By extending our model, we can study some potential effects of the introduction of such a welfare state. In this subsection we consider the consequences of adopting a pay as you go scheme where the young pay a proportional tax τ_t on their income so as to finance free care g_t to the old living in the same period.[31] A balanced budget then requires that $\tau_t w_t (1+n) = p_t g_t$, which from (12) is equivalent to

$$\tau_t \eta (1+n) = g_t \tag{52}$$

While (22) is unaffected by the tax, we show in the Appendix that (24) is now modified to

$$p_t = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma}{\sigma-1}} \left[\frac{(1-\ell_t)(1-\alpha)}{\ell_t - (1-\alpha)(1-\tau_t)}\right]^{\frac{1}{1-\sigma}} \tag{53}$$

In the static part of the model, it can thus easily be verified that a higher tax rate reduces employment in the generic sector (and increases employment in the production of care services). A further implication of this, in the static model, is that the decreased generic sector employment increases the wage.

In the Appendix we show that capital accumulation is now given by

$$\kappa_{t+1} = (1-\tau_t)\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^{\alpha}\ell_t^{-\alpha}\left[1 + \frac{(1-\alpha)\tau_{t+1}}{\alpha(1+\beta)\ell_{t+1}}\right]^{-1} \tag{54}$$

The accumulation is affected through decreased savings as current wage income is now taxed. This is captured by the term $1-\tau_t$, and is the same as in the canonical model. In addition, in the present model the tax also affects capital accumulation through the terms $\ell_t = \ell(\kappa_t, \tau_t)$ and $\ell_{t+1} = \ell(\kappa_{t+1}, \tau_{t+1})$, that now depend on the tax rates.

The total effect of taxes on the steady-state level of capital is found by imposing $\kappa_{t+1} = \kappa_t = \kappa_{ss}$, $\tau_{t+1} = \tau_t = \tau$, and taking into account that generic sector employment is decreasing in the

[31] Whether the tax income is used to provide care, or is transferred lump-sum from the young to the old, has no bearing our results (as long as the publicly provided care falls short of what old agents would purchase by themselves in absence of publicly provided care).

tax rate through the term $\ell'_\tau \equiv \mathrm{d}\ell(\kappa,\tau)/\mathrm{d}\tau$ (for a given capital stock). We then find from (54) and (53) that κ_{ss} decreases when a tax financed transfer to the old is introduced.

$$\frac{\mathrm{d}\kappa_{ss}}{\mathrm{d}\tau} < 0$$

The proof of the sign is given in the Appendix. A higher tax rate and publicly provided old-age care redistributes income from the young to the old, and decreases the incentives to save. The higher labor demand in the care sector, which increases the wage for the young, cannot be sufficiently strong to turn this around. Thus, introducing the welfare state in China may remove parts in the engine that has fueled massive capital accumulation, and may have a strong negative impact on Chinese growth.

As is well known, however, a lower capital stock in an OLG framework need not reduce welfare. Thus, we next discuss dynamic inefficiency.

5.2 Dynamic Inefficiency and Golden Rule

Dynamic inefficiency is characterized by a situation where the steady state capital stock is so high that it is possible to consume part of it with no generation becoming worse off. As is well known, this is the case if the capital stock is above the what is often termed the "golden rule" capital stock. The golden rule capital stock, κ^*, is found by letting a social planer optimize with respect to κ, ℓ, d, h, and c, finding $(\kappa^*, \ell^*, d^*, h^*, c^*)$ such that the consumer in each generation gets equal and maximum utility. We can, as all generations are considered equal from the social planer's point of view, suppress time subscripts. This implies maximizing

$$U \equiv u(c) + \beta v(d,h),$$

under the constraints

$$\begin{aligned} x &= B\kappa^\alpha \ell^{1-\alpha} = c + \frac{d}{1+n} + \kappa(1+n), \\ 1 &= \ell + \frac{h}{\eta(1+n)}. \end{aligned}$$

The first constraint states that production in one period should pay for consumption of the young, c, and of the old, d. In addition, the capital labor ratio in each period should be preserved for the next period. The second constraint limits the labor use to the available labor force.

The solution for κ can be found immediately. Optimality requires that, given the optimal choice of all other variables, there should be no gain from changing the value of a subset of variables. In particular, given (ℓ^*, d^*, h^*), there should be no scope for increasing c by altering κ. Hence in optimum

$$\frac{\partial x}{\partial \kappa} = (1+n) \Rightarrow \kappa^* = \ell^* \left(\frac{B\alpha}{1+n}\right)^{\frac{1}{1-\alpha}},$$

where $\ell^* < 1$ is the optimal generic sector employment fraction. The implications for the golden rule capital ratio of introducing the old-age care sector is then immediate by comparing κ^* with that in the canonical version of the model, which we term $\kappa_{ss}^{*\text{canonical}}$. In the canonical model $\ell^* = 1$, hence

$$\kappa^* = \ell^* \left(\frac{B\alpha}{1+n} \right)^{\frac{1}{1-\alpha}} = \ell^* \kappa^{*\text{canonical}} < \kappa^{*\text{canonical}}.$$

Therefore, as compared to the canonical model, our model with the service sector for old-age care increases the potential relevance of dynamic inefficiency for two reasons. First, from (41) we know that the care sector generates an income distribution effect that leads to a steady state capital stock higher in our model than in the canonical model. Second, as the service sector reduces the labor available for the generic sector, this lowers the golden rule capital stock. Thus the actual capital stock in our model is higher than in the canonical model, while the golden rule capital stock is lower. For parameter spaces where the canonical model is efficient, our model could very well be inefficient, and moreover for parameter spaces where the canonical model is inefficient, our model is even further away from efficiency.

5.3 Endogenous Growth

In this subsection, we study the model with learning by doing. We show that, under complementarity, accumulation is self-reinforcing because capital growth induces positive feedback effects on saving rates, and thereby subsequent accumulation. In contrast, under substitutability accumulation is self-balancing, as capital growth induces negative feedback effects on saving rates.

Substituting the learning-by-doing specification (11) in (17) and (33), the linear AK model yields the accumulation law

$$\frac{\kappa_{t+1}}{\kappa_t} = \frac{A\beta\,(1-\alpha)}{(1+\beta)\,(1+n)} \frac{1}{\ell\,(\kappa_t)}. \tag{55}$$

Whether capital accumulation accelerates or converges to a stable pace depends on the co-movements of capital and employment shares. Recalling Proposition 1, we have $\ell'_\kappa < 0$ under complementarity, and $\ell'_\kappa > 0$ under substitutability. We now discuss the consequences of these processes for endogenous long-run growth.

Complementarity: Self-Reinforcing Accumulation and Traps

Considering first the case of complementarity, we may observe (aside from the very special case of permanent steady state discussed below) two types of growth paths:

(i) *Self-Reinforcing Accumulation.* Capital per worker and the price of health care grow forever. During the transition, the employment share of the generic-good sector declines and the saving rate grows. In the long run, the economy converges asymptotically to the equilibrium featuring

$$\lim_{t\to\infty} \frac{\kappa_{t+1}}{\kappa_t} = \frac{A\beta}{(1+\beta)\,(1+n)} > 1 \text{ and } \lim_{t\to\infty} \ell\,(\kappa_t) = 1-\alpha. \tag{56}$$

(ii) *Self-Reinforcing Decumulation.* Capital per worker, the price of health care and the saving rate decline over time while the employment share of the generic-good sector grows. In the long run, the economy converges asymptotically to the equilibrium featuring

$$\lim_{t\to\infty} \kappa_t = 0 \text{ and } \lim_{t\to\infty} \ell(\kappa_t) = 1. \tag{57}$$

Self-reinforcing accumulation results from the fact that, under complementarity, capital accumulation induces positive feedback effects on the savings rate. An initial increase in capital per worker drives up the health-care price and reduces the employment share of the generic sector: the intergenerational distribution effect then implies a higher saving rate, and thereby further capital accumulation. Symmetrically, an initial decline in capital per worker results in self-reinforcing decumulation via lower saving rates. Depending on initial endowments, the economy may undertake a permanent accumulation path, or remain trapped in a permanent decumulation path. The next Proposition defines the critical level of capital per worker at time zero, which acts as a threshold between the accumulation and decumulation outcomes.

Proposition 5 *(AK model under complementarity) If* $1-\alpha < \frac{(1+\beta)(1+n)}{A\beta} < 1$*, there exists a finite critical level* $\tilde{\kappa} > 0$ *satisfying*

$$\ell(\tilde{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)},$$

and acting as a separating threshold: if $\kappa_0 > \tilde{\kappa}$ *(*$\kappa_0 < \tilde{\kappa}$*), the economy follows the self-reinforcing accumulation (decumulation) path forever. If* $\frac{(1+\beta)(1+n)}{A\beta} < 1-\alpha < 1$*, the economy follows the self-reinforcing accumulation path for any* $\kappa_0 > 0$*. If* $1-\alpha < 1 < \frac{(1+\beta)(1+n)}{A\beta}$*, the economy follows the self-reinforcing decumulation path for any* $\kappa_0 > 0$*.*

Proof. See Appendix. ■

The intuition behind Proposition 5 can be seen as follows. When $\sigma < 1$, the equilibrium employment share $\ell(\kappa_t)$ is negatively related to κ_t. Therefore, if the initial stock is sufficiently high to satisfy $\kappa_0 > \tilde{\kappa}$, we have

$$\ell(\kappa_0) < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)},$$

the economy exhibits positive capital growth in the first period, $\kappa_1 > \kappa_0$, the generic-sector employment share declines, $\ell(\kappa_1) < \ell(\kappa_0)$, and the growth rate of capital in the subsequent period is even higher, $\kappa_2/\kappa_1 > \kappa_1/\kappa_0$. This mechanism arises in all subsequent periods and drives the economy towards the asymptotic equilibrium described in expression (56) above.[32] Symmetrically, if the initial stock is relatively low, $\kappa_0 < \tilde{\kappa}$, the same self-reinforcing mechanism works in the opposite direction: generic-sector employment is initially so high that savings are

[32] Expressions (56) represent an asymptotic equilibrium that is never reached in finite time. The proof follows from our previous analysis in Figure 2: as κ grows forever, the curve $\Phi(\ell,\kappa)$ permanently shifts upward and the resulting equilibrium share $\ell(\kappa)$ approaches $1-\alpha$ only asymptotically because $\lim_{\ell\to 1-\alpha^+}\Psi(\ell) = +\infty$.

discouraged and capital per worker declines, implying further increase in $\ell(\kappa)$ and therefore permanent decumulation. In the very special case $\kappa_0 = \tilde{\kappa}$, there is a permanent steady-state equilibrium: capital per worker and employment shares are constant forever. In this situation, however, any small perturbation affecting capital per-worker would drive the economy towards self-reinforcing accumulation or decumulation: see Figure 3, diagram (a), for a graphical description of this result.

Proposition 5 also shows that, if preference and technology parameters do not satisfy $1-\alpha < \frac{(1+\beta)(1+n)}{A\beta} < 1$, only one of the two paths survives: there is either self-reinforcing accumulation or self-reinforcing decumulation because the critical threshold on capital per worker $\tilde{\kappa}$ cannot be positive and finite.[33]

Substitutability: Self-Balancing Accumulation and Stagnation

Considering next the case of substitutability, the analysis of equation (55) is modified by the fact that $\ell'_\kappa > 0$. When $\sigma > 1$, the generic-sector employment share increases with capital because old agents respond to higher health-care prices by spending a higher fraction of income on generic goods. This implies that, contrary to the case of complementarity, an initial increase in capital generates *negative feedback effects* on savings through the intergenerational distribution effect: higher generic-sector employment reduces the total income share of young agents and, hence, the economy's saving rate. The consequences for economic growth are summarized in the following proposition:

Proposition 6 *(AK model under substitutability) If* $1-\alpha < \frac{(1+\beta)(1+n)}{A\beta} < 1$*, there exists a finite critical level* $\hat{\kappa} > 0$ *satisfying*

$$\ell(\hat{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)},$$

and representing a global attractor: if $\kappa_0 < \hat{\kappa}$ *(*$\kappa_0 > \hat{\kappa}$*), the economy follows a self-balancing accumulation (decumulation) path during the transition, and converges from below (above) to the stationary long-run equilibrium featuring* $\lim_{t\to\infty} \kappa_t = \hat{\kappa}$ *and* $\lim_{t\to\infty} \ell(\kappa_t) = \ell(\hat{\kappa})$*. If* $\frac{(1+\beta)(1+n)}{A\beta} < 1-\alpha < 1$*, the economy exhibits positive growth of* κ_t *forever and* $\lim_{t\to\infty} \ell(\kappa_t) = 1$*. If* $1-\alpha < 1 < \frac{(1+\beta)(1+n)}{A\beta}$*, the economy exhibits negative growth of* κ_t *forever and* $\lim_{t\to\infty} \ell(\kappa_t) = 1-\alpha$*.*

Proof. See Appendix. ■

The first result established in Proposition 6 may be restated as follows: when generic consumption and old-specific goods are strict substitutes, then if there exists a steady-state level of capital per worker that is compatible with positive production in both sectors, *the linear AK model behaves similarly to a neoclassical model.* Starting from relatively low capital, capital per worker grows over time but at decreasing rates, until the economy reaches a stable steady

[33] See the proof of Proposition 5 for details.

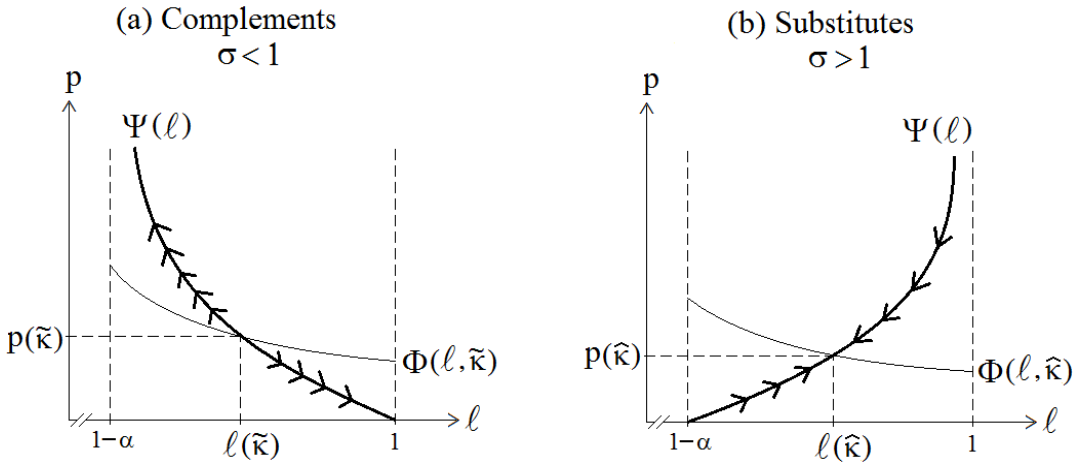

Figure 3: Dynamics of the AK model ($\bar{h} = 0$). The steady state of capital per worker is a separating threshold under complementarity, a global attractor under substitutability.

state representing the long-run equilibrium. However, this result is not due to decreasing returns to capital in production: differently from the neoclassical model, the convergence towards $\hat{\kappa}$ is determined by the reaction of sectoral employment shares to capital accumulation. Since capital growth increases employment in the generic sector, accumulation under substitutability is self-balancing. This conclusion is opposite to that obtained under complementarity, where accumulation is self-reinforcing. Figure 3, diagram (b), describes this result in graphical terms.

Proposition 6 also establishes the conditions under which substitutability admits permanent positive accumulation: when $\frac{(1+\beta)(1+n)}{A\beta} < 1-\alpha < 1$, there is no finite steady state $\hat{\kappa}$ and the economy grows forever. However, the transitional dynamics of employment shares, saving rates and growth rates are qualitatively opposite to the case of complementarity: workers flow to the generic sector, the saving rate declines and growth decelerates.

Propositions 5 and 6 show that the elasticity of substitution between generic goods and health care bears fundamental implications for economic growth. On the one hand, the separating threshold level $\tilde{\kappa}$ that arises under complementarity recalls several conclusions of the literature on poverty traps in endogenous growth models – such as those going back to Azariadis and Drazen (1990) – but in our model this hinges on the degree of substitutability between generic consumption goods and old-specific consumption goods. Under substitutability, on the other hand, the linear AK model of endogenous growth, by generating a stable long-run equilibrium with stationary capital per worker $\hat{\kappa}$, shares fundamental properties with the neoclassical model. This is, to the best of our knowledge, a novel result. The fact that the steady-state levels of capital per worker, $\tilde{\kappa}$ and $\hat{\kappa}$, have opposite characteristics is conceptually linked to the results of Peretto and Valente (2011), who study the existence of pseudo-Malthusian equilibria in a

growth model with endogenous technological progress.[34]

6 Concluding Remarks

Motivated by the dismantling of the social benefits related to state owned enterprises, and the adoption of the one-child policy, this paper has explored some possible general equilibrium implications for savings and capital accumulation. We have argued that savings and capital accumulation has stimulated further savings and capital accumulation, and we have introduced what we term a savings multiplier. In the case of limited substitutability between manufacturing goods and old-age care, we have seen that our approach, although highly simplified, implies a structural transformation with many of the characteristics observed in China since 1978. In particular, growth with increasing savings and investment rates, wage growth (in recent years) exceeding GDP growth, declining share of employment in manufacturing sectors, a widening of the income gap between those retired and those working, and an increasing share of consumption expenditures allocated to purchasing care services. Thus, maybe paradoxically, a regime that has dismantled the cradle-to-grave social benefits provided through the state owned enterprises, and at the same time not allowed citizens to produce children to provide their increased need for old-age care, has put the economy on a path where savings and capital accumulation has skyrocketed.

Growth in China may not only be unsustainable due to the political economy considerations elaborated by Acemoglu and Robinson (2012), but, as we have seen by extending our model, also because the introduction of a welfare system currently discussed in China, may remove parts of the engine that has produced the high required savings for a growth strategy based on capital accumulation.

References

Acemoglu, Daron and James A. Robinson (2012) *Why Nations Fail: The Origins of Power, Prosperity, and Poverty*, New York: Crown Publishers.

Acemoglu, Daron, Amy Finkelstein and Matthew J. Notowidigdo (2013) "Income and Health Spending: Evidence from Oil Price Shocks," *Review of Economics and Statistics*, forthcoming.

Azariadis,Costas and Allan Drazen (1990) "Threshold Externalities in Economic Development," *Quarterly Journal of Economics*, 105, 501-526.

[34] Peretto and Valente (2011) show that population size – somewhat similarly to the variable "capital per worker" in our model – exhibits an unstable steady state when labor and land are complements, and a stable steady state when inputs are substitutes. Besides the totally different aim of the analysis, the implications of these results for economic growth are opposite in the two models: in our framework, growing capital is necessary for output growth whereas stationary population in Peretto and Valente (2011) induces permanent economic growth in income per capita generated by innovations.

Barnett, Steven and Ray Brooks (2010) "China: Does Government Health and Education Spending Boost Consumption?," IMF Working Paper 10/16.

Blanchard, Olivier and Francesco Giavazzi (2006) "Rebalancing Growth in China: A Three-Handed Approach," *China & The World Economy*, 14, 1-20.

Chamon, Marcos D. and Eswar S. Prasad (2010) "Why Are Saving Rates of Urban Households in China Rising?," *American Economic Journal: Macroeconomics*, 2, 93-130.

Chou, Win Lin and Zijun Wang (2009) "Regional Inequality in China's Health Care Expenditures," *Health Economics*, 18, 137-146.

Diamond, Peter (1965) "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55, 1126-1150.

Eggleston, Karen (2012) "Health Care for 1.3 Billion: An Overview of China's Health System," Asia Health Policy Program working paper # 28, Stanford University.

Galor, Oded and Harl E. Ryder (1989) "Existence, Uniqueness, and Stability of Equilibrium in an Overlapping-Generations Model with Productive Capital," *Journal of Economic Theory*, 49, 360-375.

Galor, Oded (1992) "A Two-Sector Overlapping-Generations Model: A Global Characterization of the Dynamical System," *Econometrica*, 60, 1351-1386.

James, Estelle (2002) "How Can China Solve it's Old-Age Security Problem? The Interaction between Pension, State Enterprise and Financial Market Reform," *Journal of Pension Economics & Finance*, 1, 53-75.

Kraay, Aart (2000) "Household Saving in China," *World Bank Economic Review*, 14, 545-570.

Li, Hongbin, Lei Li, Binzhen Wu and Yanyan Xiong (2012) "The End of Cheap Chinese Labor," *Journal of Economic Perspectives*, 26 (4), 57-74.

Litao, Zhao and Sheng Sixin (2009) "Old Age Care in China," EAI Background Brief No. 432.

Ma, Guonan and Wang Yi (2010) "China's High Saving Rate: Myth and Reality," BIS Working Papers No 312.

Meng, Xin (2012) "Labor Market Outcomes and Reforms in China," *Journal of Economic Perspectives*, 26 (4), 75-102.

Modigliani, Franco and Shi Larry Cao (2004) "The Chinese Saving Puzzle and the Life-Cycle Hypothesis," *Journal of Economic Literature*, 42, 145-170.

Oksanen, Heikki (2010) "The Chinese Pension System - First Results on Assessing the Reform Options," Economic Papers 412, European Commission, Brussels.

Peretto, Pietro and Simone Valente (2011) "Growth on a Finite Planet: Resources, Technology and Population in the Long Run," Department of Economics Working Paper, Duke University.

Romer, Paul M. (1989) "Capital Accumulation in the Theory of Long-Run Growth," in *Modern Business Cycle Theory,* ed. R.J. Barro, New York, Basil Blackwell.

Song, Zheng and Dennis T. Yang (2010) "Life Cycle Earnings and the Household Puzzle in a Fast-Growing Economy," Working Paper, Chinese University of Hong Kong.

Song, Zheng, Kjetil Storesletten and Fabrizio Zilibotti (2011) "Growing Like China," *American Economic Review*, 101, 202-241.

Zhong, Hai (2011) "The Impact of Population Aging on Income Inequality in Developing Countries: Evidence from Rural China," *China Economic Review*, 22, 98-107.

Wei, Shang-Jin and Xiaobo Zhang (2011) "The Competitive Saving Motive: Evidence from Rising Sex Ratios and Savings Rates in China," *Journal of Political Economy*, 119, 511-564.

Xu, Chenggang (2011) "The Fundamental Institutions of China's Reforms and Development," *Journal of Economic Literature*, 49, 1076-1151.

Yang, Dennis T. (2012) "Aggregate Savings and External Imbalances in China," *Journal of Economic Perspectives*, 26 (4), 125-146.

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A Appendix

Consumption levels: derivation of (18)-(19). The household maximizes (2) subject to (4)-(5). The Lagrangean at time t reads

$$\mathcal{L} \equiv u\left(c_t\right) + \beta v\left(d_{t+1}, h_{t+1} - \bar{h}\right) + \lambda_1\left(w_t - s_t - c_t\right) + \lambda_2\left(s_t R_{t+1} - d_{t+1} - p_{t+1}^h h_{t+1}\right)$$

where λ_1 and λ_2 are the multipliers. The first-order conditions with respect to $(c_t, d_{t+1}, h_{t+1}, s_t)$ are

$$\begin{aligned} u_{c_t}\left(c_t\right) &= \lambda_1, && \text{(A.1)} \\ \beta v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) &= \lambda_2, && \text{(A.2)} \\ \beta v_{h_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) &= \lambda_2 p_{t+1}, && \text{(A.3)} \\ \lambda_1 &= \lambda_2 R_{t+1}. && \text{(A.4)} \end{aligned}$$

Combining (A.1) with (A.2) and (A.2) with (A.3), we respectively obtain

$$\begin{aligned} u_{c_t}\left(c_t\right) &= \beta R_{t+1} v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right), && \text{(A.5)} \\ v_{h_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) &= p_{t+1} v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right), && \text{(A.6)} \end{aligned}$$

where (A.5) is the Keynes-Ramsey rule for the generic good, and (A.6) equates the relative price of health care services to the marginal rate of substitution with second-period consumption. Exploiting the assumed utility functions (6)-(7), conditions (A.5)-(A.6) respectively read

$$\begin{aligned} d_{t+1} &= c_t \beta R_{t+1} - \left(h_{t+1} - \bar{h}\right) \frac{1-\gamma}{\gamma} \left(\frac{d_{t+1}}{h_{t+1} - \bar{h}}\right)^{\frac{1}{\sigma}}, && \text{(A.7)} \\ p_{t+1} &= \frac{1-\gamma}{\gamma} \left(\frac{d_{t+1}}{h_{t+1} - \bar{h}}\right)^{\frac{1}{\sigma}}. && \text{(A.8)} \end{aligned}$$

Substituting (A.8) in (A.7) gives

$$d_{t+1} + p_{t+1}\left(h_{t+1} - \bar{h}\right) = c_t \beta R_{t+1}. \tag{A.9}$$

Substituting (A.9) in the second-period budget constraint (5) and then using the first-period budget constraint (4) to eliminate savings, we obtain expression (18) in the text. Next, substitute the market clearing condition $K_{t+1} = N_t^y s_t$ into (5) to obtain

$$R_{t+1} K_{t+1} = N_t^y \left(d_{t+1} + p_{t+1} h_{t+1}\right). \tag{A.10}$$

Given the market clearing condition $N_t^o h_t = H_t$, the zero-profit condition for the health care sector reads

$$p_t N_t^o h_t = w_t \left(1 - \ell_t\right) N_t^y. \tag{A.11}$$

Substituting (A.11) for period $t+1$ into (A.10), we obtain

$$N_{t+1}^{o} d_{t+1} = R_{t+1} K_{t+1} - w_{t+1} \left(1 - \ell_{t+1}\right) N_{t+1}^{y}. \tag{A.12}$$

From the profit maximizing conditions (16) and (15), respectively, we have

$$R_t K_t = B\alpha \left[N_t^y \left(\kappa_t\right)^{\alpha} \left(\ell_t\right)^{1-\alpha} \right], \tag{A.13}$$

$$w_t \left(1 - \ell_t\right) N_t^y = B\left(1-\alpha\right) \frac{1-\ell_t}{\ell_t} \left[N_t^y \left(\kappa_t\right)^{\alpha} \left(\ell_t\right)^{1-\alpha} \right]. \tag{A.14}$$

Setting (A.12) at period t and substituting (A.13) and (A.14), we obtain equation (19) in the text. Note that result (19) implies a restriction: second-period generic consumption is positive if and only if $\ell_t > 1 - \alpha$.

Consumer problem: derivation of (20). Setting expression (A.8) at time t, raising both sides to the power of σ, and dividing both sides by p_t, we obtain expression (20) in the text.

Goods Market: derivation of (24). Starting from (20), multiply both d_t and $\left(h_t - \bar{h}\right)$ by old population N_t^o, and substitute the old agents' constraint $N_t^o d_t = N_t^o s_{t-1} R_t - p_t H_t$, to obtain

$$p_t^{\sigma-1} = \left(\frac{1-\gamma}{\gamma}\right)^{\sigma} \frac{N_t^o s_{t-1} R_t - p_t H_t}{p_t H_t - p_t N_t^o \bar{h}}. \tag{A.15}$$

Substituting capital income with the profit-maximizing condition $N_t^o s_{t-1} R_t = \alpha X_t$, we get

$$p_t^{\sigma-1} = \left(\frac{1-\gamma}{\gamma}\right)^{\sigma} \frac{\alpha X_t - p_t H_t}{p_t H_t - p_t N_t^o \bar{h}}. \tag{A.16}$$

Recalling that constant returns to scale in both production sectors imply the zero profit conditions

$$\begin{array}{l} X_t = N_t^y \left(w_t \ell_t + R_t \kappa_t\right), \\ p_t H_t = N_t^y w_t \left(1 - \ell_t\right), \\ \text{and } \frac{R_t \kappa_t}{w_t} = \frac{\alpha}{1-\alpha} \ell_t, \end{array}$$

expression (A.16) reduces to

$$p_t^{\sigma-1} = \left(\frac{1-\gamma}{\gamma}\right)^{\sigma} \frac{1}{1-\alpha} \frac{N_t^y w_t}{p_t} \frac{\ell_t - (1-\alpha)}{H_t - N_t^o \bar{h}}. \tag{A.17}$$

From (8) and the definition of $\ell^{\max}$, we have $H_t - N_t^o \bar{h} = \eta \left(\ell^{\max} - \ell_t\right) N_t^y$. Plugging this result into (A.17), and substituting $\frac{w_t}{\eta p_t} = 1$ from (12), we obtain expression (24) in the text.

Existence and uniqueness of the fixed point (25). The functions $\Phi\left(\ell_t, \kappa_t\right)$ defined in (21) exhibit the following properties:

$$\begin{array}{l} \lim_{\ell_t \to 1-\alpha} \Phi\left(\ell_t, \kappa_t\right) = \left\{ \begin{array}{ll} \left(B/\eta\right)\left(1-\alpha\right)^{1-\alpha} \left(\kappa_t\right)^{\alpha} & \text{(Neoclassical)} \\ \left(A/\eta\right)\kappa_t & \text{(Linear AK)} \end{array} \right\} \\ \lim_{\ell_t \to \ell^{\max}} \Phi\left(\ell_t, \kappa_t\right) = \left\{ \begin{array}{ll} \left(B/\eta\right)\left(1-\alpha\right)\left(\kappa_t/\ell^{\max}\right)^{\alpha} & \text{(Neoclassical)} \\ \left(A/\eta\right)\left(1-\alpha\right)\left(\kappa_t/\ell^{\max}\right) & \text{(Linear AK)} \end{array} \right\}, \end{array} \tag{A.18}$$

with derivatives

$$\Phi_{\ell_t} \equiv \frac{\partial \Phi(\ell_t,\kappa_t)}{\partial \ell_t} = \left\{ \begin{array}{ll} -\alpha \frac{\Phi(\ell_t,\kappa_t)}{\ell_t} < 0 & \text{(Neoclassical)} \\ -\frac{\Phi(\ell_t,\kappa_t)}{\ell_t} < 0 & \text{(Linear AK)} \end{array} \right\} \quad \text{and} \quad \Phi_{\ell_t\ell_t} \equiv \frac{\partial^2 \Phi(\ell_t,\kappa_t)}{\partial \ell_t^2} > 0. \tag{A.19}$$

The elasticity of $\Phi(\ell_t, \kappa_t)$ in the two cases is

$$\frac{\Phi_{\ell_t}\ell_t}{\Phi} = \left\{ \begin{array}{ll} -\alpha & \text{(Neoclassical)} \\ -1 & \text{(Linear AK)} \end{array} \right\} \tag{A.20}$$

The function defined in (24), instead, exhibits

$$\begin{aligned} \lim_{\ell_t \to 1-\alpha} \Psi(\ell_t) = & \quad \left\{ \infty \quad \text{if } \sigma < 1; \quad 0 \quad \text{if } \sigma > 1 \right\}, \\ \lim_{\ell_t \to \ell^{\max}} \Psi(\ell_t) = & \quad \left\{ 0 \quad \text{if } \sigma < 1; \quad \infty \quad \text{if } \sigma > 1 \right\}, \end{aligned} \tag{A.21}$$

with

$$\Psi'(\ell_t) \equiv \frac{\partial \Psi(\ell_t)}{\partial \ell_t} = \frac{\Psi(\ell_t)}{\sigma - 1} \frac{\ell^{\max} - (1-\alpha)}{(\ell^{\max} - \ell_t)[\ell_t - (1-\alpha)]} \left\{ \begin{array}{ll} < 0 & \text{if } \sigma < 1 \\ > 0 & \text{if } \sigma > 1 \end{array} \right\}. \tag{A.22}$$

The elasticity is therefore

$$\frac{\Psi'(\ell_t)\ell_t}{\Psi(\ell_t)} = -\frac{1}{1-\sigma} \underbrace{\frac{\ell^{\max} - (1-\alpha)}{\ell^{\max} - \ell_t}}_{>1} \underbrace{\frac{\ell_t}{\ell_t - (1-\alpha)}}_{>1}. \tag{A.23}$$

Under substitutability, existence and uniqueness of the fixed point (25) are guaranteed by the derivatives (A.19)-(A.22) along with the limits (A.18) and (A.21). Under complementarity, expression (A.20) implies $\Phi_{\ell_t}\ell_t/\Phi > -1$ whereas expression (A.23) implies $\Psi'(\ell_t)\ell_t/\Psi(\ell_t) < -1$. These values of elasticities imply existence and uniqueness of the fixed point (25) even with $\sigma < 1$ despite the fact that both $\Phi(\ell_t, \kappa_t)$ and $\Psi(\ell_t)$ are strictly decreasing. For future reference, note that the limiting properties of $\Phi(\ell_t, \kappa_t)$ and $\Psi(\ell_t)$ described in (A.18) and (A.21) imply

$$\lim_{\kappa \to 0} \ell(\kappa) = \left\{ \begin{array}{ll} \ell^{\max} & \text{if } \sigma < 1 \\ 1-\alpha & \text{if } \sigma > 1 \end{array} \right\} \text{ and } \lim_{\kappa \to \infty} \ell(\kappa) = \left\{ \begin{array}{ll} 1-\alpha & \text{if } \sigma < 1 \\ \ell^{\max} & \text{if } \sigma > 1 \end{array} \right\}. \tag{A.24}$$

Neoclassical growth: elasticity of $\ell(\kappa)$, derivation of (36). Totally differentiating the fixed-point condition $\Psi(\ell(\kappa)) = \Phi(\ell(\kappa), \kappa)$, we obtain

$$\ell'_\kappa(\kappa) = \frac{\Phi_\kappa(\ell(\kappa), \kappa)}{\Psi'(\ell(\kappa)) - \Phi_\ell(\ell(\kappa), \kappa)}. \tag{A.25}$$

In the Neoclassical case, function $\Phi(\ell(\kappa), \kappa)$ exhibits the partial derivatives

$$\Phi_\kappa(\ell, \kappa) = \alpha\Phi(\ell, \kappa)/\kappa \text{ and } \Phi_\ell(\ell, \kappa) = -\alpha\Phi(\ell, \kappa)/\ell. \tag{A.26}$$

Substituting (A.26) and the equilibrium condition $\Psi(\ell(\kappa)) = \Phi(\ell(\kappa), \kappa)$ in (A.25), we obtain

$$\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)} = \frac{\alpha}{\alpha + \frac{\Psi'(\ell(\kappa))\ell(\kappa)}{\Psi(\ell(\kappa))}}. \tag{A.27}$$

Result (A.27) establishes a clear link between the elasticity of the generic-sector employment share $\ell(\kappa)$ to the capital stock κ and the elasticity of the price of health care $\Psi(\ell(\kappa))$ to the generic-sector employment share $\ell(\kappa)$. In particular, substituting (A.23) in (A.27), we have

$$\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)} = \frac{1}{1-\frac{1}{1-\sigma}\frac{1}{\alpha}\left\{\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}\frac{\ell(\kappa)}{\ell(\kappa)-(1-\alpha)}\right\}}, \tag{A.28}$$

where the term in curly brackets equals $Q_1 \equiv \frac{\ell_t}{\ell_t-(1-\alpha)}\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell_t}$ in expression (36). The fact that $Q_1 > 1$ directly follows from the equilibrium requirement $1-\alpha < \ell_t < \ell^{\max}$, and it implies that

$$\begin{array}{ll} \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1 > 1 & \text{if } 0<\sigma<1, \\ \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1 < 0 & \text{if } \sigma>1. \end{array} \tag{A.29}$$

Results (A.29) imply the signs reported in expression (36) in the text.

Proposition 3: Existence, Uniqueness and Stability. To prove existence, consider equation (38) and substitute $p(\kappa_t) = \Psi(\ell(\kappa_t))$ from (26), obtaining

$$\kappa_{t+1} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi(\ell(\kappa_t)). \tag{A.30}$$

The right-hand side of (A.30) is strictly increasing in κ_t: differentiation with respect to κ_t yields

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi'(\ell(\kappa_t))\,\ell'_\kappa(\kappa_t) > 0, \tag{A.31}$$

where the positive sign derives from the fact that both $\Psi'(\ell_t)$ and $\ell'_\kappa(\kappa_t)$ are negative (positive) under complementarity (substitutability). From (A.22), (A.21) and (A.24), we have

$$\lim_{\kappa\to 0}\Psi'(\ell(\kappa)) = \infty \text{ and } \lim_{\kappa\to\infty}\Psi'(\ell(\kappa)) = 0 \tag{A.32}$$

under both complementarity and substitutability. Results (A.31) and (A.32) imply existence of at least one steady state $\kappa_{ss} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi(\ell(\kappa_{ss}))$. Moreover, if the elasticity condition (40) is valid for any κ, i.e.

$$-\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha} < 1, \tag{A.33}$$

then the steady state $\kappa = \kappa_{ss}$ is unique and globally stable. Under substitutability, inequality (A.33) is necessarily satisfied: when $\sigma > 1$, expression (36) implies $\ell'_\kappa(\kappa) > 0$ and therefore a strictly negative left hand side in (A.33). To study the case of complementarity, substitute the elasticity $\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}$ by means of expression (A.28), and rearrange terms, to rewrite condition (A.33) as

$$1 - \frac{(1-\sigma)(1-\ell(\kappa))(\ell(\kappa)-(1-\alpha))}{\ell(\kappa)} > \alpha. \tag{A.34}$$

Condition (A.34) implies a more restrictive requirement on parameters the lower is the left hand side. In this respect, the left hand side of (A.34) is strictly increasing in σ so that, all else equal, it reaches its smallest value when $\sigma = 0$. Letting $\sigma = 0$, the stability condition becomes

$$\left(\ell(\kappa) - \frac{1}{2}\right)^2 > \alpha - \frac{3}{4}, \tag{A.35}$$

which is surely satisfied when $\alpha < 3/4$. Therefore, a generously sufficient, not necessary condition for stability and uniqueness under complementarity is $\alpha < 3/4$. Given uniqueness and stability, as κ_t converges to κ_{ss} in the long run, both the price of health care $p_t = p(\kappa_t)$ and the employment share $\ell_t = \ell(\kappa_t)$ converge to constant levels. Results (A.31) and (A.32) guarantee that the transitional dynamics of κ_t are monotonic. The transitional dynamics of $p(\kappa_t)$ and $\ell_t = \ell(\kappa_t)$ then follow directly from Proposition 1. ■

Proposition 4: Uniqueness and Stability. The existence of the steady state $\bar{\kappa}_{ss}$ is proved in Appendix B along with the discussion of possible multiple steady states. Given existence, the steady state $\kappa = \bar{\kappa}_{ss}$ is unique and globally stable if the elasticity condition (43) is valid for any κ, i.e.

$$m_1(\kappa) + m_2(\kappa) < 1, \tag{A.36}$$

where

$$m_1(\kappa) \equiv -\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha}, \tag{A.37}$$

$$m_2(\kappa) \equiv -\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa)}\frac{1}{1-\alpha}. \tag{A.38}$$

Expression (A.37) follows from generalizing the definition of m_1 given in (40) whereas (A.38) follows from generalizing the definition of m_2 given in (44). The inequalities appearing in (44) are part of the following proof. First, consider substitutability. When $\sigma > 1$, both $m_1(\kappa)$ and $m_2(\kappa)$ are strictly negative because expression (36) implies $\ell'_\kappa(\kappa) > 0$ and expression (32) implies $\Gamma' > 0$. Therefore, condition (A.36) is necessarily satisfied when $\sigma > 1$. To study the case of complementarity, note that (32) implies

$$\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa)} = \frac{\frac{(1-\alpha)}{\alpha(1+\beta)\eta(1+n)}\frac{\bar{h}}{\ell(\kappa)}}{1-\frac{(1-\alpha)}{\alpha(1+\beta)\eta(1+n)}\frac{\bar{h}}{\ell(\kappa)}} = \frac{\frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}}{\ell(\kappa)-\frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}}, \tag{A.39}$$

where the last term follows from substituting $\bar{h} = \eta(1+n)(1-\ell^{\max})$ by definition (13). Defining the convenient parameter

$$q_1 \equiv \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)} > 0, \tag{A.40}$$

we can substitute (A.39) in (A.38) and rewrite the stability condition (A.36) as

$$-\frac{\ell'(\kappa)\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha} - \frac{\ell'(\kappa)\kappa}{\ell(\kappa)}\frac{q_1}{\ell(\kappa)-q_1}\frac{1}{1-\alpha} < 1. \tag{A.41}$$

Fro future reference, note that parameter q_1 is always strictly less than $1-\alpha$.[35] This implies that $\ell(\kappa) > q_1$ holds in any interior equilibrium:

$$q_1 < 1-\alpha < \ell(\kappa). \tag{A.42}$$

[35] Because any interior equilibrium satisfies $(1-\alpha) < \ell(\kappa_t) < \ell^{\max}$, it is necessarily true that $\ell^{\max} > 1-\alpha-\alpha\beta$. Consequently, the factor $\frac{(1-\ell^{\max})}{\alpha(1+\beta)}$ is strictly less than unity and this, in turn, implies that $q_1 \equiv (1-\alpha)\frac{(1-\ell^{\max})}{\alpha(1+\beta)}$ is strictly less than $(1-\alpha)$.

Going back to (A.41), substituting $\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}$ by means of (A.28), the stability condition reduces to

$$\frac{1-\alpha}{\alpha(1-\sigma)}\left[\frac{\ell(\kappa)-q_1}{\ell(\kappa)-(1-\alpha)}\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}\right]>1. \tag{A.43}$$

From result (A.42), the term in square brackets in (A.43) is strictly greater than unity.[36] Therefore, a sufficient but not necessary condition for satisfying (A.43) is that $\frac{1-\alpha}{\alpha(1-\sigma)}>1$, which is equivalent to Assumption 2 in the text. The conclusion is that, when $\sigma<1$, satisfying Assumption 2 is sufficient to guarantee stability and uniqueness of the steady state. Also note that the stability condition under complementarity guarantees $m_2(\kappa)<1$, as reported in expression (44).[37]

Derivation of (46)-(47) . Expression (46) directly follows from log-differentiating (42) with respect to B. In (41), instead, log-differentiation with respect to B yields

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B}=-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\kappa_{ss})\kappa_{ss}}{\ell(\kappa_{ss})}\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B}+\frac{\ell'_B(\kappa_{ss})}{\ell(\kappa_{ss})}\right]+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B}, \tag{A.44}$$

where the term in square brackets is the chain derivative $\frac{\mathrm{d}\log\ell(\kappa_{ss})}{\mathrm{d}B}$, with $\ell'_B(\kappa_{ss})$ representing the static derivative $\frac{\mathrm{d}\ell(\kappa;B)}{\mathrm{d}B}$ defined in Proposition 2, evaluated in the steady state κ_{ss}. Substituting the definition of $m_1\equiv-\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\cdot\kappa}{\ell}$ from (40) into (A.44), and rearranging terms, we obtain (47).

Derivation of (48). From definition (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}B}\log\Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right)=-\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B},$$

where both Γ and Γ' in the right hand side are evaluated in $\left(\bar{h}/\ell(\bar{\kappa}_{ss})\right)$. Therefore, log-differentiating (45) with respect to B yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}=-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B}-\frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B}+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B},$$

where we can substitute $\frac{\mathrm{d}\log\ell(\kappa)}{\mathrm{d}B}=\frac{\ell'_\kappa\kappa}{\ell}\frac{\mathrm{d}\log\kappa}{\mathrm{d}B}+\frac{\ell'_B}{\ell}$ to obtain

$$\begin{aligned}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}=&-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}+\frac{\ell'_B(\kappa_{ss})}{\ell(\bar{\kappa}_{ss})}\right]+\\&-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}+\frac{\ell'_B(\kappa_{ss})}{\ell(\bar{\kappa}_{ss})}\right]+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B},\end{aligned} \tag{A.45}$$

where $\ell'_B(\bar{\kappa}_{ss})$ is the static derivative $\frac{\mathrm{d}\ell(\kappa;B)}{\mathrm{d}B}$ defined in Proposition 2, evaluated in the steady state $\bar{\kappa}_{ss}$. Recalling (40) and (40), the definitions $m_1\equiv-\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2\equiv-\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.45) reduces to equation (48) in the text.

Derivation of (50). From definition (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}n}\log\Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right)=-\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\left[(1+n)\ell(\bar{\kappa}_{ss})\right]}{\mathrm{d}n},$$

[36] The fact that $q_1<1-\alpha$ implies $\frac{\ell(\kappa)-q_1}{\ell(\kappa)-(1-\alpha)}>1$. Also, the equilibrium restriction $\ell(\kappa)>(1-\alpha)$ implies $\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}>1$.

[37] Since condition (A.43) is equivalent to (A.36), satisfying (A.43) implies $m_2(\kappa)<1-m_1(\kappa)$, where $m_1(\kappa)>0$ under complementarity. Therefore, $m_2(\kappa)<1$.

where both Γ and Γ' in the right hand side are evaluated in $\left(\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right)$. Therefore, log-differentiating (45) with respect to n yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}=-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[(1+n)\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}n}-\frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}n}+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}n}.$$

Substituting in the above expression the chain derivatives

$$\begin{array}{l}\frac{\mathrm{d}\log[(1+n)\ell(\bar{\kappa}_{ss})]}{\mathrm{d}n}=\frac{1}{1+n}+\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}n},\\ \frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}n}=\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}+\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n},\end{array}$$

we obtain

$$\begin{array}{l}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}=-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}\right]-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}\right]+\\ -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right]-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right]+\\ -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{1}{1+n}\right]+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}n}.\end{array}\tag{A.46}$$

Recalling (40) and (40), the definitions $m_1\equiv-\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2\equiv-\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.46) reduces to

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}=\frac{1}{1-(m_1+m_2)}\left[(m_1+m_2)\frac{\ell'_n}{\ell'_\kappa\kappa}+\frac{m_2}{1+n}\frac{\ell}{\ell'_\kappa\kappa}+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}n}\right],$$

where we can invert the sign of the variation $\mathrm{d}n$ to $-\mathrm{d}n$, and rearrange terms, to obtain equation (50) in the text.

Derivation of (51). From definition (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{h}}\log\Gamma\left(\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\right)=\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}\bar{h}},$$

where both Γ and Γ' in the right hand side are evaluated in $\left(\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right)$. Therefore, log-differentiating (45) with respect to $\bar{h}$ (recalling that $\mathrm{d}\kappa_{ss}^{\text{canonical}}/\mathrm{d}\bar{h}=0$) yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}}=\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}\bar{h}}-\frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}\bar{h}}.$$

Substituting in the above expression the chain derivatives

$$\begin{array}{l}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}}=\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}+\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}},\\ \frac{\mathrm{d}\log\left[\bar{h}/\ell(\bar{\kappa}_{ss})\right]}{\mathrm{d}\bar{h}}=\frac{1}{\bar{h}}-\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}},\end{array}$$

we obtain

$$\begin{array}{l}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}}=\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{1}{\bar{h}}-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}+\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}}\right]+\\ -\frac{\alpha}{1-\alpha}\left[\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}+\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right].\end{array}\tag{A.47}$$

Recalling (40) and (40), the definitions $m_1\equiv-\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2\equiv-\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.47) reduces to (51).

Introducing the welfare state: derivation of (53). We assume that the government taxes young agents' income at rate τ_t, and uses these revenues to pay for an amount g_t of care

(at the regular price p_t) under balanced budget, as established in condition (52). The private budget constraints thus read

$$c_t = w_t(1-\tau_t) - s_t, \tag{A.48}$$

$$s_t R_{t+1} = d_{t+1} + p_{t+1}(h_{t+1} - g_{t+1}). \tag{A.49}$$

Since labor is supplied inelastically, the equilibrium in the labor market is not directly affected by the tax: from (12), (15) and (16), we obtain

$$p_t = (B/\eta)(1-\alpha)(\kappa_t/\ell_t)^{\alpha} \equiv \Phi(\ell_t, \kappa_t). \tag{A.50}$$

which is the same relationship derived in the model without taxes – see expressions (21)-(22). From the modified household problem with constraints (A.48)-(A.49), utility maximization requires

$$c_t = \tfrac{1}{1+\beta}\left[w_t(1-\tau_t) - \tfrac{p_{t+1}}{R_{t+1}}\left(\bar{h} - g_{t+1}\right)\right] \text{ and } s_t = \tfrac{1}{1+\beta}\left[\beta w_t(1-\tau_t) + \tfrac{p_{t+1}}{R_{t+1}}\left(\bar{h} - g_{t+1}\right)\right] \tag{A.51}$$

and second-period consumption equals

$$d_t = (1+n)\left[\ell_t - (1-\alpha)(1-\tau_t)\right] a_t^{1-\alpha}(\kappa_t/\ell_t)^{\alpha}, \tag{A.52}$$

Result (19) implies that second-period consumption is positive only if $\ell_t > (1-\alpha)(1-\tau_t)$, which will always turn out to be the case in equilibrium. Utility maximization also determines the relative demand for health care services:

$$\frac{p_t}{d_t}\left(h_t - \bar{h}\right) = \left(\frac{1-\gamma}{\gamma}\right)^{\sigma} p_t^{1-\sigma}. \tag{A.53}$$

By inserting for d_t from (A.52) and p_t from (21) in the left hand side of (A.53), as well as $h_t = \eta(1+n)(1-\ell_t)$ and $\bar{h} = \eta(1+n)(1-\ell^{\max})$, we get

$$p_t = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma}{\sigma-1}} \left[\frac{(\ell^{\max} - \ell_t)(1-\alpha)}{\ell_t - (1-\alpha)(1-\tau_t)}\right]^{\frac{1}{1-\sigma}} \equiv \Psi(\ell_t; \tau_t), \tag{A.54}$$

which is expression (53) in the text.

Introducing the welfare state: static equilibrium. Combining (A.54) and (21), the equilibrium employment share in the generic sector is the fixed point

$$\ell_t = \ell(\kappa_t; \tau_t) \equiv \arg\operatorname{solve}_{\{\ell_t\}}\left[\Phi(\ell_t, \kappa_t) = \Psi(\ell_t; \tau_t)\right] \tag{A.55}$$

Note that (A.54) implies that $\Psi(\ell_t; \tau_t)$ is decreasing (increasing) in τ_t when $\sigma < 1$ ($\sigma > 1$). Therefore, recalling the graphical analysis of Figure 2, with the curve $\Psi(\ell_t)$ replaced by $\Psi(\ell_t; \tau_t)$, an increase in τ implies a leftward shift in the $\Psi(\ell_t; \tau_t)$ under both complementarity and substitutability. Consequently, the static effect of an increase in τ *for given* κ on sectoral labor shares is

$$\ell'_{\tau} \equiv \frac{\mathrm{d}\ell(\kappa;\tau)}{\mathrm{d}\tau} < 0. \tag{A.56}$$

Introducing the welfare state: derivation of (54). Following the same steps as in equation (30), the workers' share of output

$$\frac{w_t N_t^y (1-\tau_t)}{Y_t} = \frac{(1-\tau_t)(1-\alpha)\frac{x_t}{\ell_t}N_t^y}{x_t\left(\frac{1-\alpha}{\ell_t}+\alpha\right)N_t^y} = \frac{(1-\alpha)(1-\tau_t)}{1-\alpha(1-\ell_t)}, \tag{A.57}$$

Combining (A.57) with the saving function in (A.51), and substituting p_{t+1} and R_{t+1} by means of (15) and (16), we obtain

$$\frac{s_t}{w_t(1-\tau_t)} = \frac{\beta}{1+\beta}\left(1+\frac{1}{\eta}\frac{1-\alpha}{\alpha}\frac{\kappa_{t+1}}{\ell_{t+1}}\frac{\left(\bar{h}-g_{t+1}\right)}{\beta w_t(1-\tau_t)}\right). \tag{A.58}$$

Substituting $s_t = (1+n)\kappa_{t+1}$ and inserting $\bar{h} = \eta(1+n)(1-\ell^{\max})$ from (13), as well as substituting g_{t+1} and w_t by means of (52) and (15), repsectively, we get

$$\kappa_{t+1} = (1-\tau_t)\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^{\alpha}\ell_t^{-\alpha}\left[1-\frac{(1-\alpha)(1-\ell^{\max}-\tau_{t+1})}{\alpha(1+\beta)\ell_{t+1}}\right]^{-1}. \tag{A.59}$$

Setting $\bar{h}=0$ implies $\ell^{\max}=1$, so that equation (A.59) reduces to expression (54) in the text.

Introducing the welfare state: long-run equilibrium

Under the assumption ($\bar{h}=0, \ell^{\max}=1$), setting a constant tax rate $\tau_t=\tau$ in each t, and substituting the equilibrium condition (A.55) into (54), we have

$$\kappa_{t+1}\left[1+\frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{t+1};\tau)}\right] = \tilde{B}\kappa_t^{\alpha}\ell(\kappa_t;\tau)^{-\alpha}(1-\tau), \tag{A.60}$$

where we have defined $\tilde{B}\equiv\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}$. The dynamic stability of (A.60) requires that $\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t}\frac{\kappa_t}{\kappa_{t+1}}$ evaluated in the steady state κ_{ss} is less than unity, i.e.[38]

$$-\frac{1}{1-\alpha}\left(\alpha-\frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right)\frac{\ell'_{\kappa_{ss}}\kappa_{ss}}{\ell(\kappa_{ss};\tau)} \equiv m_\tau(\kappa_{ss}) < 1, \tag{A.61}$$

where we have defined $\tilde{Q}(\kappa_{ss};\tau)\equiv\frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{ss};\tau)}$. Note that setting $\tau=0$, the term $\tilde{Q}(\kappa_{ss};\tau)$ reduces to zero and the term $m_\tau(\kappa_{ss})$ reduces to $m_1(\kappa_{ss})$ defined in (40). Therefore, condition (A.61) is the stability condition (40) generalized to the model with welfare state. The steady state of equation (A.60) is represented by

$$\kappa_{ss} = \tilde{B}^{\frac{1}{1-\alpha}}(1-\tau)^{\frac{1}{1-\alpha}}\left[1+\tilde{Q}(\kappa_{ss};\tau)\right]^{-\frac{1}{1-\alpha}}\ell(\kappa_{ss};\tau)^{-\frac{\alpha}{1-\alpha}}. \tag{A.62}$$

Log-differentiating (A.62) with respect to τ we have

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} = -\frac{1}{1-\alpha}\left[\frac{1}{1-\tau}+\frac{\mathrm{d}\log\left[1+\tilde{Q}(\kappa_{ss};\tau)\right]}{\mathrm{d}\tau}+\alpha\frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}\right]. \tag{A.63}$$

[38] Log-differentiating the left hand side of (A.60) yields $\frac{1}{\kappa_{t+1}} - \frac{\frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{t+1};\tau)}}{1+\frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{t+1};\tau)}}\frac{\ell'_{\kappa_{t+1}}}{\ell(\kappa_{t+1};\tau)}$ whereas log-differentiating the right hand side gives $\alpha\frac{1}{\kappa_t}-\alpha\frac{\ell'_{\kappa_t}}{\ell(\kappa_t;\tau)}$. Taking the ratio and imposing the steady state $\kappa_{t+1}=\kappa_t=\kappa_{ss}$, the stability condition $\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t}\frac{\kappa_t}{\kappa_{t+1}}<1$ reduces to expression (A.61).

From the definition of $\tilde{Q}(\kappa_{ss};\tau)$, we can substitute $\frac{\mathrm{d}\log\left[1+\tilde{Q}(\kappa_{ss};\tau)\right]}{\mathrm{d}\tau} = \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\left(1-\frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}\right)$ in the above expression to obtain

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} = -\frac{1}{1-\alpha}\left[\frac{1}{1-\tau}+\frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}+\left(\alpha-\frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right)\frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}\right].$$

Further substituting the chain derivative $\frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau} = \frac{\ell'_{\kappa_{ss}}\kappa_{ss}}{\ell}\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau}+\frac{\ell'_\tau}{\ell}$, we have

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} = -\frac{1}{1-\alpha}\frac{1}{1-m_\tau(\kappa_{ss})}\left\{\left[\frac{1}{1-\tau}+\left(\alpha-\frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right)\frac{\ell'_\tau(\kappa_{ss};\tau)}{\ell(\kappa_{ss};\tau)}\right]+\frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right\}. \tag{A.64}$$

where $\ell'_\tau \equiv \frac{\mathrm{d}\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}$ is the derivative for given κ defined in (A.56) above, evaluated in κ_{ss}.

We next need to prove that a higher tax implies a lower steady state capital stock:

Proof. In order to prove that $\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} < 0$ we start by deriving $\ell'_\tau(\kappa_{ss};\tau)$. By implicit differentiation the definition of $\ell(\kappa_{ss};\tau)$ from (A.55) we get

$$\ell'_\tau(\kappa_{ss};\tau) = \frac{-\alpha\ell\left(1-\ell\right)\left(1-\alpha\right)}{\left(\tau+\alpha-\alpha\tau\right)\ell-\alpha\left(1-\ell\right)\left(\ell-\left(1-\tau\right)\left(1-\alpha\right)\right)\left(1-\sigma\right)} < 0$$

where the sign follows from the fact that, as $\ell > (1-\alpha)(1-\tau)$, the denominator is increasing in σ, and that when $\sigma = 0$ it can be written

$$D \equiv \alpha\ell^2 + (1-\alpha)\ell\tau + (1-\alpha)\alpha(1-\tau)(1-\ell) > 0$$

If we can show that the curly bracket, $\{\dots\}$ in (A.64), cannot become negative we have proven that $\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} < 0$. First we note that

$$\alpha > \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}$$

is a necessary condition for the curly bracket in (A.64) to be negative. Then as $\tilde{Q}(\kappa_{ss};\tau)/\left(1+\tilde{Q}(\kappa_{ss};\tau)\right)$ is ewerywhere decreasing in β and as the absolute value of ℓ'_τ is everywhere decreasing in σ, the curly bracket in (A.64) cannot become negative for any β and σ unless it can become negative when β takes its largest allowed value of unity and σ takes its smallest allowed value of zero. We further note that a necessary condition for the curly bracket in (A.64) to be negative is that the bracket $[\dots]$ in (A.64) is negative. Inserting for $\beta = 1$ and $\sigma = 0$ we get

$$[\dots] = \frac{2\alpha^2\ell^3 + 3\left(1-\alpha\right)\alpha\ell 2 + \tau\left(1-\alpha\right)^2\left(1+2\tau\ell-\tau-\ell\right)}{D\left(1-\tau\right)\left(2\ell\alpha+\left(1-\alpha\right)\tau\right)} > 0$$

Both the numerator and the denominator are positive.[39] Hence we have proven that $[\dots]$ is positive for all τ, α and ℓ. Therefore $\{\dots\}$,the curly bracket in (A.64), is also always positive and thus $\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau}$ is always negative. ■

[39] $(1+2\tau\ell-\tau-\ell)$ is a hyperbolic paraboloid and can be written as $(2(\tau-1/2)(\ell-1/2)+1/2)$ which cannot become negative for any ℓ and τ between zero and one.

Proof of Proposition 5. From (55), we have

$$\frac{\kappa_{t+1}}{\kappa_t} \gtreqless 1 \quad \Longleftrightarrow \quad \frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)} \gtreqless \ell\left(\kappa_t\right), \tag{A.65}$$

which determines whether accumulation is positive, and

$$\frac{\mathrm{d}\frac{\kappa_{t+1}}{\kappa_t}}{\mathrm{d}\ell\left(\kappa_t\right)} < 0 \quad \text{and} \quad \frac{\mathrm{d}\frac{\kappa_{t+1}}{\kappa_t}}{\mathrm{d}\kappa_t} = \frac{\mathrm{d}\frac{\kappa_{t+1}}{\kappa_t}}{\mathrm{d}\ell\left(\kappa_t\right)}\ell'_{\kappa_t}, \tag{A.66}$$

which determines whether accumulation accelerates or decelerates. The sign of ℓ'_{κ_t} depends on the elasticity of substitution. From Proposition 1, we have $\ell'_{\kappa} < 0$ when $\sigma < 1$. Therefore, (A.66) implies that $\frac{\kappa_{t+1}}{\kappa_t}$ increases over time if $\kappa_{t+1}/\kappa_t > 1$, and decreases over time if $\kappa_{t+1}/\kappa_t < 1$. Depending on parameter values, we can distinguish among three cases.

(i) Suppose that there exists a finite critical level $\tilde{\kappa} > 0$ satisfying $\ell\left(\tilde{\kappa}\right) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ in an interior equilibrium. Existence requires that parameters satisfy $1-\alpha < \ell\left(\tilde{\kappa}\right) < 1$, and this condition implies

$$1-\alpha < \frac{\left(1+\beta\right)\left(1+n\right)}{A\beta} \quad \text{and} \quad \frac{\left(1+\beta\right)\left(1+n\right)}{A\beta} < 1. \tag{A.67}$$

In this case, it is possible that a finite positive initial endowment κ_0 is above or below $\tilde{\kappa}$. From (A.65), and the fact that $\ell'_{\kappa} < 0$, this yields three scenarios at time zero:

$$\begin{array}{l} \text{(i.a) If } \kappa_0 = \tilde{\kappa} \text{ then } \ell\left(\kappa_0\right) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 = \kappa_0 \\ \text{(i.b) If } \kappa_0 > \tilde{\kappa} \text{ then } \ell\left(\kappa_0\right) < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 > \kappa_0 \\ \text{(i.c) If } \kappa_0 < \tilde{\kappa} \text{ then } \ell\left(\kappa_0\right) > \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 < \kappa_0 \end{array} \tag{A.68}$$

Since $\ell'_{\kappa} < 0$, scenario (i.a) implies constant $\kappa_t = \tilde{\kappa}$ forever; scenario (i.b) implies $\kappa_{t+1} > \kappa_t$ forever, with $\ell\left(\kappa_{t+1}\right) < \ell\left(\kappa_t\right)$ forever and the limiting results (56); scenario (i.c) implies $\kappa_{t+1} < \kappa_t$ forever, with $\ell\left(\kappa_{t+1}\right) > \ell\left(\kappa_t\right)$ forever and the limiting results (57).

(ii) Suppose that parameters satisfy $\frac{(1+\beta)(1+n)}{A\beta} < 1-\alpha < 1$. In this case, we have

$$1 < \frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)} < \frac{A\beta}{\left(1+\beta\right)\left(1+n\right)} \tag{A.69}$$

and a hypothetical critical level $\tilde{\kappa}$ implying $\ell\left(\tilde{\kappa}\right) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the first inequality in (A.69) would imply $\ell\left(\tilde{\kappa}\right) > 1$, which is not feasible. This is equivalent to say that $\ell\left(\tilde{\kappa}\right)$ is unfeasibly high – that is, $\tilde{\kappa}$ is unfeasibly low: given $\ell'_{\kappa} < 0$, any finite endowment $\kappa_0 > 0$ must be associated with an interior equilibrium

$$\ell\left(\kappa_0\right) < \frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}$$

which implies self-reinforcing accumulation as in scenario (i.b) in expression (A.68).

(iii) Suppose that parameters satisfy $1-\alpha<1<\frac{(1+\beta)(1+n)}{A\beta}$. In this case, we have

$$\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}<\frac{A\beta}{\left(1+\beta\right)\left(1+n\right)}<1 \tag{A.70}$$

and a hypothetical critical level $\tilde{\kappa}$ implying $\ell\left(\tilde{\kappa}\right)=\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the last inequality in (A.70) would imply $\ell\left(\tilde{\kappa}\right)<1-\alpha$. This is equivalent to say that $\ell\left(\tilde{\kappa}\right)$ is unfeasibly low – that is, $\tilde{\kappa}$ is unfeasibly high: given $\ell_{\kappa}^{\prime}<0$, any finite endowment $\kappa_0>0$ must be associated with an interior equilibrium

$$\ell\left(\kappa_0\right)>\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}$$

which implies self-reinforcing decumulation as in scenario (i.c) in expression (A.68). ■

AK model: Proof of Proposition 6. From Proposition 1, we have $\ell_{\kappa}^{\prime}>0$ when $\sigma>1$. Therefore, (A.66) implies that $\frac{\kappa_{t+1}}{\kappa_t}$ declines over time if $\kappa_{t+1}/\kappa_t>1$, and increases over time if $\kappa_{t+1}/\kappa_t<1$. Depending on parameter values, we can distinguish among three cases.

(I) Suppose that there exists a finite critical level $\hat{\kappa}>0$ satisfying $\ell\left(\hat{\kappa}\right)=\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ in an interior equilibrium. Existence requires that parameters satisfy $1-\alpha<\ell\left(\hat{\kappa}\right)<1$, and this condition implies

$$1-\alpha<\frac{\left(1+\beta\right)\left(1+n\right)}{A\beta}\quad\text{and}\quad\frac{\left(1+\beta\right)\left(1+n\right)}{A\beta}<1. \tag{A.71}$$

In this case, it is possible that a finite positive initial endowment κ_0 is above or below $\hat{\kappa}$. From (A.65), and the fact that $\ell_{\kappa}^{\prime}>0$, this yields three scenarios at time zero:

$$\begin{array}{l}\text{(I.a) If } \kappa_0=\hat{\kappa} \text{ then } \ell\left(\kappa_0\right)=\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1=\kappa_0\\ \text{(I.b) If } \kappa_0<\hat{\kappa} \text{ then } \ell\left(\kappa_0\right)<\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1>\kappa_0\\ \text{(I.c) If } \kappa_0>\hat{\kappa} \text{ then } \ell\left(\kappa_0\right)>\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1<\kappa_0\end{array} \tag{A.72}$$

Since $\ell_{\kappa}^{\prime}>0$, scenario (I.a) implies constant $\kappa_t=\hat{\kappa}$ forever; scenario (I.b) implies $\kappa_{t+1}>\kappa_t$ and $\ell\left(\kappa_{t+1}\right)>\ell\left(\kappa_t\right)$ during the transition, and the asymptotic steady state $\lim_{t\to\infty}\kappa_t=\hat{\kappa}$ and $\lim_{t\to\infty}\ell\left(\kappa_t\right)=\ell\left(\hat{\kappa}\right)$; scenario (I.c) implies $\kappa_{t+1}<\kappa_t$ and $\ell\left(\kappa_{t+1}\right)<\ell\left(\kappa_t\right)$ during the transition, and the asymptotic steady state $\lim_{t\to\infty}\kappa_t=\hat{\kappa}$ and $\lim_{t\to\infty}\ell\left(\kappa_t\right)=\ell\left(\hat{\kappa}\right)$.

(II) Suppose that parameters satisfy $\frac{(1+\beta)(1+n)}{A\beta}<1-\alpha<1$. In this case, we have

$$1<\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}<\frac{A\beta}{\left(1+\beta\right)\left(1+n\right)} \tag{A.73}$$

and a hypothetical critical level $\hat{\kappa}$ implying $\ell\left(\hat{\kappa}\right)=\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the first inequality in (A.73) would imply $\ell\left(\hat{\kappa}\right)>1$, which is not feasible. This is equivalent to say that $\ell\left(\hat{\kappa}\right)$ is unfeasibly high – that is, $\hat{\kappa}$ is unfeasibly high: given $\ell_{\kappa}^{\prime}>0$, any finite endowment $\kappa_0>0$ must be associated with an interior equilibrium

$$\ell\left(\kappa_0\right)<\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}$$

which implies permanent (but decelerating) positive accumulation as in scenario (I.b) in expression (A.72).

(III) Suppose that parameters satisfy $1-\alpha<1<\frac{(1+\beta)(1+n)}{A\beta}$. In this case, we have

$$\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}<\frac{A\beta}{\left(1+\beta\right)\left(1+n\right)}<1 \tag{A.74}$$

and a hypothetical critical level $\hat{\kappa}$ implying $\ell\left(\hat{\kappa}\right)=\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the last inequality in (A.74) would imply $\ell\left(\hat{\kappa}\right)<1-\alpha$. This is equivalent to say that $\ell\left(\hat{\kappa}\right)$ is unfeasibly low – that is, $\tilde{\kappa}$ is unfeasibly low: given $\ell_{\kappa}^{\prime}>0$, any finite endowment $\kappa_0>0$ must be associated with an interior equilibrium

$$\ell\left(\kappa_0\right)>\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}$$

which implies permanent (but decelerating) decumulation as in scenario (I.c) in expression (A.72).

B Further Details

Proposition 4: Further Details on Existence and Uniqueness of the steady state. To prove existence, we transform equation (35) into an equivalent dynamic law that maps $\ell\left(\kappa_t\right)$ into $\ell\left(\kappa_{t+1}\right)$. Starting from expression (35), substitute $p_t=\Phi\left(\ell_t,\kappa_t\right)=\left(B/\eta\right)\left(1-\alpha\right)\left(\kappa_t/\ell_t\right)^{\alpha}$ from (22) to write

$$\frac{\kappa_{t+1}}{\ell\left(\kappa_{t+1}\right)}\left[\ell\left(\kappa_{t+1}\right)-\frac{\left(1-\alpha\right)\left(1-\ell^{\max}\right)}{\alpha\left(1+\beta\right)}\right]=\frac{\eta\beta}{\left(1+\beta\right)\left(1+n\right)}p_t. \tag{B.1}$$

Imposing the equilibrium condition $p_t=p\left(\kappa_t\right)\equiv\Psi\left(\ell\left(\kappa_t\right)\right)$ from (26), we have

$$\frac{\kappa_{t+1}}{\ell\left(\kappa_{t+1}\right)}\left[\ell\left(\kappa_{t+1}\right)-\frac{\left(1-\alpha\right)\left(1-\ell^{\max}\right)}{\alpha\left(1+\beta\right)}\right]=\frac{\eta\beta}{\left(1+\beta\right)\left(1+n\right)}\Psi\left(\ell\left(\kappa_t\right)\right). \tag{B.2}$$

Also notice that, setting (22) at time $t+1$ and solving for the input ratio, we have $\frac{\kappa_{t+1}}{\ell_{t+1}}=\left[\frac{p_{t+1}}{(B/\eta)(1-\alpha)}\right]^{\frac{1}{\alpha}}$. Plugging this result in (B.2), and imposing the static equilibrium condition $p_{t+1}=p\left(\kappa_{t+1}\right)\equiv\Psi\left(\ell\left(\kappa_{t+1}\right)\right)$ from (26), we obtain the *dynamic law*

$$\Psi\left(\ell\left(\kappa_{t+1}\right)\right)\left(\ell\left(\kappa_{t+1}\right)-q_1\right)^{\alpha}=q_2\Psi\left(\ell\left(\kappa_t\right)\right)^{\alpha}, \tag{B.3}$$

where we again have used that the definition of q_1 and also have defined the convenient variable q_2:

$$q_1\equiv\tfrac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}>0\quad\text{and}\quad q_2\equiv\tfrac{B}{\eta}\left(1-\alpha\right)\left[\tfrac{\eta\beta}{(1+\beta)(1+n)}\right]^{\alpha}>0. \tag{B.4}$$

Expression (B.3) fully characterizes the dynamics of capital per worker. Dynamics are well defined only if both sides are strictly positive, which requires $\ell\left(\kappa_{t+1}\right)>q_1$ in each period $t+1$:

this inequality is always satisfied as shown in (A.42). In (B.3), the steady state condition $\kappa_{t+1} = \kappa_t = \bar{\kappa}_{ss}$ is satisfied when

$$\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \underbrace{\left[q_2\left(\ell\left(\bar{\kappa}_{ss}\right) - q_1\right)^{-\alpha}\right]^{\frac{1}{1-\alpha}}}_{\Omega(\ell(\bar{\kappa}_{ss}))}. \tag{B.5}$$

For future reference, we define the elasticities of $\Psi(.)$ and $\Omega(.)$ with respect to $\bar{\kappa}_{ss}$ as

$$\epsilon_1 \equiv \frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} \frac{\bar{\kappa}_{ss}}{\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)} = \frac{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)} \ell'_{\kappa}\left(\bar{\kappa}_{ss}\right) \bar{\kappa}_{ss}, \tag{B.6}$$

$$\epsilon_2 \equiv \frac{\mathrm{d}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} \frac{\bar{\kappa}_{ss}}{\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)} = -\frac{\frac{\alpha}{1-\alpha}}{\ell\left(\bar{\kappa}_{ss}\right) - q_1} \ell'_{\kappa}\left(\bar{\kappa}_{ss}\right) \bar{\kappa}_{ss}. \tag{B.7}$$

The remainder of the proof studies separately the two cases of substitutability and complementarity.

Substitutability. When $\sigma > 1$, results (A.22) imply that

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} = \underbrace{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}_{\text{positive}} \underbrace{\ell'_{\kappa}\left(\bar{\kappa}_{ss}\right)}_{\text{positive}} > 0 \text{ and } \ell'\left(\bar{\kappa}_{ss}\right) > 0. \tag{B.8}$$

Result (B.8) implies that, given the definitions in (B.5), function $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ is strictly increasing in $\bar{\kappa}_{ss}$ whereas $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ is strictly decreasing in $\bar{\kappa}_{ss}$. Using the limiting properties (A.21) and (A.24), we also have

$$\begin{array}{ll} \lim_{\bar{\kappa}_{ss}\to 0} \Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = 0, & \lim_{\bar{\kappa}_{ss}\to 0} \Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(1-\alpha-q_1\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0, \\ \lim_{\bar{\kappa}_{ss}\to \infty} \Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \infty, & \lim_{\bar{\kappa}_{ss}\to \infty} \Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(\ell^{\max}-q_1\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0. \end{array} \tag{B.9}$$

These results imply that, under substitutability, there exists a unique steady state $\bar{\kappa}_{ss}$ satisfying condition (B.5).

Complementarity. When $\sigma < 1$, results (A.22) imply that

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} = \underbrace{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}_{\text{negative}} \underbrace{\ell'_{\kappa}\left(\bar{\kappa}_{ss}\right)}_{\text{negative}} > 0 \text{ and } \ell'_{\kappa}\left(\bar{\kappa}_{ss}\right) < 0. \tag{B.10}$$

Result (B.10) implies that, given the definitions in (B.5), both $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ and $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ in (B.5) are strictly increasing in $\bar{\kappa}_{ss}$. Using the limiting properties (A.21) and (A.24), we also have

$$\begin{array}{ll} \lim_{\bar{\kappa}_{ss}\to 0} \Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = 0, & \lim_{\bar{\kappa}_{ss}\to 0} \Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(\ell^{\max}-q_1\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0, \\ \lim_{\bar{\kappa}_{ss}\to \infty} \Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \infty, & \lim_{\bar{\kappa}_{ss}\to \infty} \Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(1-\alpha-q_1\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0. \end{array} \tag{B.11}$$

The limits in (B.11) imply that *there always exists at least one steady state* $\bar{\kappa}_{ss(1)}$ *satisfying condition (B.5) in which* $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ *cuts* $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ *from below*: this steady state therefore satisfies

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss(1)}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss(1)}} > \frac{\mathrm{d}\Omega\left(\ell\left(\bar{\kappa}_{ss(1)}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss(1)}}. \tag{B.12}$$

Considering stability, equation (B.3) implies that any steady state $\bar{\kappa}_{ss(i)}$ is stable if

$$\frac{\Psi'\left(\ell\left(\bar{\kappa}_{ss(i)}\right)\right)}{\Psi\left(\ell\left(\bar{\kappa}_{ss(i)}\right)\right)} < -\frac{\alpha}{1-\alpha} \frac{1}{\ell\left(\bar{\kappa}_{ss(i)}\right) - q_1}. \tag{B.13}$$

It is easily shown that (B.12) implies that the steady state $\bar{\kappa}_{ss(1)}$ satisfies the stability condition (B.13). Hence, under complementarity, there always exist a stable steady state $\bar{\kappa}_{ss(1)}$. In order to assess the uniqueness of the steady state, re-write the steady-state condition (B.5) in explicit form by substituting $\Psi(\cdot)$ with (24), obtaining

$$\ell(\bar{\kappa}_{ss}) - q_1 = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma(1-\alpha)}{\alpha(1-\sigma)}} q_2^{\frac{1}{\alpha}} \left[\frac{\ell(\bar{\kappa}_{ss}) - (1-\alpha)}{(1-\alpha)(\ell^{\max} - \ell(\bar{\kappa}_{ss}))}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}}. \tag{B.14}$$

Hence, defining $q_3 \equiv \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma(1-\alpha)}{\alpha(1-\sigma)}} q_2^{\frac{1}{\alpha}}$, the steady-state condition reads

$$\ell(\bar{\kappa}_{ss}) = \digamma(\ell(\bar{\kappa}_{ss})) \quad \text{where} \quad \digamma(\ell(\bar{\kappa}_{ss})) \equiv q_1 + q_3 \left[\frac{\ell(\bar{\kappa}_{ss}) - (1-\alpha)}{(1-\alpha)(\ell^{\max} - \ell(\bar{\kappa}_{ss}))}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}}. \tag{B.15}$$

In general, the function $\digamma(\ell)$ is strictly increasing and exhibits the following properties

$$\lim_{\ell \to 1-\alpha} \digamma(\ell) = q_1 < \ell \text{ and } \lim_{\ell \to \ell^{\max}} \digamma(\ell) = \infty, \tag{B.16}$$

$$\digamma'(\ell) = \frac{q_3}{\alpha(1-\sigma)} \left[\frac{\ell - (1-\alpha)}{(1-\alpha)(\ell^{\max} - \ell)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}-1} \frac{\ell^{\max} - (1-\alpha)}{(\ell^{\max} - \ell)^2} > 0. \tag{B.17}$$

Evaluating $\digamma'(\ell)$ in a steady state $\ell(\bar{\kappa}_{ss}) = \digamma(\ell(\bar{\kappa}_{ss}))$, we have

$$\digamma'(\ell(\bar{\kappa}_{ss})) = \frac{1-\alpha}{\alpha(1-\sigma)} \underbrace{\frac{\ell(\bar{\kappa}_{ss}) - q_1}{\ell(\bar{\kappa}_{ss}) - (1-\alpha)}}_{>1} \underbrace{\frac{\ell^{\max} - (1-\alpha)}{\ell^{\max} - \ell(\bar{\kappa}_{ss})}}_{>1}. \tag{B.18}$$

Comparing (B.18) with (A.43), it is evident that the stability condition (A.43) is equivalent to $\digamma'(\ell(\bar{\kappa}_{ss})) > 1$. In graphical terms, this means that a stable steady state is a an intersection $\ell = \digamma(\ell)$ in which the function $\digamma(\ell)$ cuts the 45-degree line $\ell = \ell$ from below – e.g., like the steady state shown in Figure 4, graph (a). Properties (B.16)-(B.17) thus confirm the existence of at least one stable steady state. Concerning the uniqueness of the steady state, we must consider two sub-cases, depending on whther the parameter values imply $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$ or $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$.

Subcase I. When $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, expression (B.17) implies $\digamma''(\ell) > 0$ for all $\ell \in (1-\alpha, \ell^{\max})$, so that $\digamma(\ell)$ is strictly increasing and strictly convex for all $\ell \in (1-\alpha, \ell^{\max})$. This means that there is a unique steady state $\ell(\bar{\kappa}_{ss}) = \digamma(\ell(\bar{\kappa}_{ss}))$, as shown in Figure 4, graph (a). Moreover, $\ell(\bar{\kappa}_{ss})$ is stable, as is immediately evident from (B.18): when $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, all the three terms at the right hand side are strictly greater than unity, implying $\digamma'(\ell(\bar{\kappa}_{ss})) > 1$. Recalling that $\ell'_{\kappa}(\kappa) < 0$ under complementarity, an initial condition $\kappa(0) < \bar{\kappa}_{ss}$ implies positive accumulation and declining employment in generic production: the economy starts from an initial level $\ell_0 = \ell(\kappa(0))$ and then declines towards $\ell_{ss} = \ell(\bar{\kappa}_{ss})$ as shown in Figure 4, graph (a).

Subcase II. When $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$, we have $\lim_{\ell \to 1-\alpha} \digamma'(\ell) = \infty$ and $\lim_{\ell \to \ell^{\max}} \digamma'(\ell) = \infty$. Expression (B.17) implies that $\digamma(\ell)$ is initially concave and then convex: from

$$\frac{\digamma''(\ell)}{\digamma'(\ell)} = \frac{1}{\ell^{\max} - \ell} \left\{2 - \left(1 - \frac{1-\alpha}{\alpha(1-\sigma)}\right) \frac{\ell^{\max} - (1-\alpha)}{\ell - (1-\alpha)}\right\}, \tag{B.19}$$

there exists a point of inflection

$$\tilde{\ell} \equiv (1-\alpha) + \frac{1}{2}\left(1 - \frac{1-\alpha}{\alpha(1-\sigma)}\right)\left[\ell^{\max} - (1-\alpha)\right]$$

such that $F''\left(\tilde{\ell}\right) = 0$, with $F''(\ell)$ is negative for $\ell < \tilde{\ell}$ and positive $\tilde{\ell} > \ell$. This implies that, in the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$, we may have in principle two possible outcomes: a unique stable steady state or three steady states, as shown in Figure 4, graphs (b) and (c). When there are three steady states, $\bar{\kappa}_{ss(1)} < \bar{\kappa}_{ss(2)} < \bar{\kappa}_{ss(3)}$, the middle steady state $\bar{\kappa}_{ss(2)}$ is unstable because $F'\left(\ell\left(\bar{\kappa}_{ss(2)}\right)\right) < 1$, whereas $\bar{\kappa}_{ss(1)}$ and $\bar{\kappa}_{ss(3)}$ are both stable. This scenario is thus characterized by

$$F(\ell_{ss3}) < F(\ell_{ss2}) < F(\ell_{ss1}), \tag{B.20}$$

$$F'(\ell_{ss3}) < 1, \quad F'(\ell_{ss2}) > 1, \quad F(\ell_{ss1}) < 1, \tag{B.21}$$

where $\ell^i_{ss} \equiv \ell\left(\bar{\kappa}_{ss(i)}\right)$. Recalling that $\ell'_\kappa(\kappa) < 0$ under complementarity, an initial condition $\kappa(0) < \bar{\kappa}_{ss(1)}$ implies positive accumulation and declining employment in generic production: the economy starts from an initial level $\ell_0 = \ell(\kappa(0))$ and then declines towards $\ell^1_{ss} = \ell\left(\bar{\kappa}_{ss(1)}\right)$ as shown in Figure 4, graph (d).

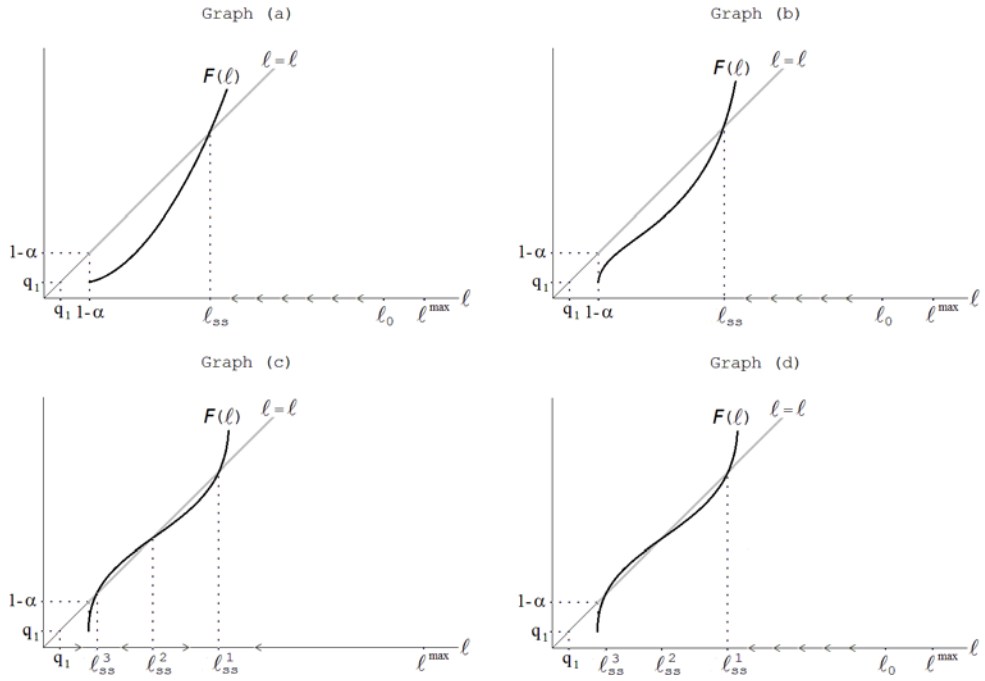

Figure 4: Existence and uniqueness of steady states under complementarity. Graph (a): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$ features a unique stable steady state. Graph (b): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$ when the steady state is unique. Graphs (c)-(d): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$ when the steady states are three – the middle one being unstable.

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